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The Gift of
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ELECTROMAGNETIC THEORY OF LIGHT

ELECTROMAGNETIC THEORY OF LIGHT

BY

CHARLES EMERSON CURRY, PH.D.

PART I.

London
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PREFACE.

IN the following work I have endeavoured to account for the manifold phenomena of light as electromagnetic phenomena, deriving the same from the fundamental differential equations for electromagnetic disturbances. I have treated in Part I. the more familiar phenomena that can be explained by Maxwell's theory, and have reserved for Part II. those for which his theory fails to offer a satisfactory explanation.

In the treatment of the subject-matter, I have laid more stress on a rigorous development of the fundamental laws of optics than on the derivation of the many consequences or secondary laws, that can be deduced from the former by familiar principles, and have little to do with our conception of the nature of light; for the consequences or secondary laws that can thus be deduced I refer the reader to the various text-books on optics, in which the same are most extensively treated. I have also omitted a description of all experiments on the subject-matter treated and have referred to empirical facts only where a comparison with the theoretical results has seemed of interest.

At the beginning of each chapter I have endeavoured to give a brief historical sketch of the subject-matter treated; and each chapter has been developed as independently of the preceding ones, as the treatment of the subject has allowed. Examples pertaining to the matter treated in the text have been added at the end of each chapter; these have been of great service to me in the general treatment of the principles set forth in the text, and I hope they may prove as useful to the reader.

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The spherical waves and the so-called primary and secondary waves, which have been so extensively treated in the first four chapters, are perhaps only of theoretical interest. One of my chief reasons for the elaborate treatment of this peculiar class of waves has been to indicate another fertile field of research offered by Maxwell's equations. For those interested only in the more familiar phenomena of electromagnetic wave-motion those portions of the text can be omitted.

In the treatment of the familiar problems on optics I have made free use of all sources with which I am acquainted, but in particular of Preston's "Theory of Light," Helmholtz's "Vorlesungen über die Electromagnetische Theorie des Lichts," Volkmann's "Vorlesungen über die Theorie des Lichtes," and Drude's "Lehrbuch der Optik."

I have to return my best thanks to Prof Dr. K. Fischer, Munich, for many valuable suggestions, as well as for a most careful revision of the proofs.

C. E. CURRY.

MUNICH, *January*, 1905.

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ERRATA.

Page 48, line 20, *for* $a_3\gamma$ *read* a_3a.
Page 146, line 22, *for* Zone *read* Zonal Element.
Page 284, line 5, *for* $\cos^2\phi$ *read* $\cos 2\phi$.
Page 344, line 6, *for* shorter *read* longer.
Page 344, lines 15 and 17, *for* longer *read* shorter.

CHAPTER I.

INTRODUCTION.

Finite Velocity of Light.—Until Roemer's discovery of the finite velocity of propagation of light from his observations of the satellites of Jupiter, the many theories and speculations offered for the explanation of its manifold phenomena were of a most varied and even extravagant nature. The further discovery of the aberration of light by Bradley some fifty years later not only confirmed the great truth revealed by his Danish predecessor, but showed that the light of the fixed stars travelled with the same velocity as that reflected from the sun. Finally, about the middle of the last century the ingenious methods devised by Fizeau and Foucault for the direct determination of the velocity of light within a room left no further doubt as to its finite velocity. The discoveries of Roemer and Bradley not only gave us another example of the continuity of nature, but they opened up a new era in the history of optics.

Two Modes of Transmission.—If we accept the velocity of light as finite and the phenomena of vision as a manifestation of mechanical energy transmitted from the luminous object to the retina of the eye, we can evidently conceive only two modes of its transmission: either by material particles or corpuscles, which are projected at high velocities (that of light) from the luminous body, strike the retina of the eye and impart their kinetic energy to it; or by means of a medium, as a fluid, which carries the energy, imparted to it by the luminous body in the form of waves or oscillations, from one particle to the next, until that motion finally reaches the observer, and is transmitted to the retina of his eye in the form of a similar oscillation, which calls forth the phenomenon of vision; this latter mode of transmission is characterised by an entire absence of any passage of material particles between the luminous object and the observer. The former mode of transmission forms the fundament of the so-called "corpuscular" or "emission" theories of light, the latter mode that of the wave theory.

Emission Theory.—The emission theories are embodied in the theory first propounded by Newton and modified by him and others to surmount the many difficulties encountered in a satisfactory explanation of empirical laws. Most formidable difficulties are met here at the very outset, among others the assumption of such enormous velocities as that of light for material particles; for particles travelling at such high velocities would impart an enormous momentum to the object they strike, and thus set it in motion; but observations have failed to detect any such motion,* even when the supposed particles are brought to a focus on the given object by means of a lens or mirror. Moreover, although the law of reflection is evidently the same for elastic particles and beams of light, it is easy to show that the former in passing from one medium into another obey quite different laws from those for the refraction of light: according to the former the velocity of propagation increases with the density of the medium, a law which is in direct disagreement with all empirical laws of light.

Its Modifications.—In order to make the material particles behave according to the empirical laws of refraction, it was found necessary not only to endow them with many new properties but actually to assume first the presence of an intervening medium capable of being set into an oscillatory motion and then certain reciprocal actions between that medium—its oscillations or waves—and the particles themselves. The result of these many modifications was that Newton's emission theory finally assumed all the aspects of the wave theory proper; in its ultimate form it was, in fact, known as Newton's wave theory of light; it differs from the wave theory proper only in the assumption of the presence of the material particles themselves and the laws regulating the action between the same and the waves of the medium. Such a complication of ideas, especially where nothing is to be gained, alone justifies us in abandoning Newton's wave theory and accepting in its place the simpler one, the wave theory proper.

Wave Theory.—Huygens must undoubtedly be regarded as the founder of the wave theory proper; not alone because he was the first to state it in explicit form, but because he was able to offer a satisfactory explanation for the greater part of the phenomena then known to the world, namely those of reflection, refraction, and double refraction (in crystals). The difficulty Huygens encountered in attempting to explain the rectilinear propagation of light and the presence of shadows

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* See, however, P. Lebedew: "Untersuchungen ueber die Drückkraefte des Lichts," Drudi's *Annalen* 6, 1901, vol. 11, pp. 433-458.

(cf. Chapter V.) alone accounts for the little recognition accorded his theory when first stated and its entire neglect for almost a century; it first received attention upon Young's discovery of the principle of interference and Fresnel's confirmation of the same by experiment. These phenomena of interference dealt perhaps the last blow to the emission theory, since the presence of an intervening medium capable of being set into an oscillatory motion then became not only the essential but the predominating feature in every theory of light. Finally, the last formidable difficulty besetting the wave theory, the explanation of the phenomenon of polarisation, was removed by Fresnel's assumption that the light waves were not longitudinal like those of sound, as had hitherto been supposed, but transverse, that is, that the vibrations took place at right angles to their direction of propagation; the rectilinear propagation of light and the presence of shadows soon after found a satisfactory explanation (cf. Chapter V.).

The Ether.—The medium assumed for the propagation of light is termed "ether." Since ether evidently pervades not only terrestial but interstellar space, it cannot be identical with our atmosphere. Moreover, we must assume that ether pervades all transparent bodies, but, as the behaviour of light in such bodies is different from its behaviour in the air, that the ether pervading the former is different from that of the air; that is, that the properties to be assigned the ether differ for different bodies or media. These properties are evidently determined by certain unknown actions (resistances) between the material particles of the given body and the particles (elements) of the ether pervading that body; they thus differ for different bodies. Consequently, opaque bodies could also be conceived as permeated by ether, and thus ether itself as pervading not only all space, both terrestial and interstellar, but all bodies; that is, it may be regarded as a continuous medium.

The Elastic Solid Ether.—Many ethers have already been offered for the explanation of the phenomena of light, and many more could readily be conceived that might give similar satisfaction. This freedom of choice is due chiefly to our ignorance of the properties of the ether sought, and our consequent inability to form any concrete conception of it.* We have observed above that the phenomena of interference and polarisation can alone be explained satisfactorily by an ether that is capable of transmitting transverse oscillations; one of the first properties to be demanded of a luminiferous ether is, therefore, that it is capable of being set into transverse vibrations, and of transmitting those vibrations further. Such ethers resemble now the solid rather than the fluid, since the former alone is capable of being set into transverse vibrations; hence the termination "elastic solid ether," that particular ether which possesses not only the required property of transmitting transverse oscillations, but other properties common to the solid.* The elastic solid theory of light was soon universally accepted; and it remained the accepted theory till almost the end of the last century, until Hertz's great discoveries, which revealed the striking similarity between the electric waves and those of light, suggested certain modifications in the constitution of the elastic solid ether; we could thus designate this new luminiferous ether as the "electromagnetic ether." The necessary modifications to be made in the elastic solid ether were naturally such that the phenomena of light already explained by it were readily and similarly deduced from the electromagnetic ether. Those text-books that treat light from the elastic solid standpoint have not, therefore, become entirely obsolete; on the other hand, they may be used to great advantage by the student, and will often be referred to in the present treatise. For the fundamental differences between the elastic solid and the electromagnetic ethers, I refer the student to Section 1 of my *Theory of Electricity and Magnetism*.

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* Cf. Curry: *Theory of Electricity and Magnetism*, p. 4.

Maxwell's Ether.—We shall accept Maxwell's equations as our definition of the electromagnetic or luminiferous ether, and, as in my *Theory of Electricity and Magnetism*, I shall leave it to the student to form any conception of the ether thus defined, that is consistent with the different properties of these equations. There are, indeed, other ethers defined by other systems of equations; some may explain certain phenomena, or even groups of phenomena, as satisfactorily as Maxwell's equations do, others are more general and thus allow greater freedom in the choice of the properties that may be assigned them, but none have stood the severe test of twenty-five years or more that Maxwell's have. This alone surely justifies us in accepting Maxwell's ether as the seat and transmitter of not only the electric and magnetic energy, but that of light.

Helmholtz's Ether.—Helmholtz's equations of electricity and magnetism define an ether that is more general than Maxwell's, but at the same time includes the same as particular case ($\epsilon = 0$, k arbitrary †); it also includes other particular ethers, which differ essentially from Maxwell's; many of these have been more or less extensively investigated with regard not only to the electric phenomena, but to those of light. Tumlirz makes use of such a particular system of equations

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* Cf. Curry: *Theory of Electricity and Magnetism*, p. 6.

† Cf. *Ibid.*, p. 356.

($k=0$) in his book on the electromagnetic theory of light,* but the value of his book is so greatly impaired by an unfortunate choice of surface conditions, that it is impossible, before examining his equations in detail, to pass judgment on their real value or ability to explain the phenomena treated. Helmholtz's ether is more general than Maxwell's, chiefly in that it is capable of transmitting not only transverse, but longitudinal oscillations; the latter are represented by a certain function ϕ† that appears in Helmholtz's equations. A peculiarity of Helmholtz's ether is that its property of being able to transmit longitudinal oscillations is quite independent of certain other of its properties and *vice versa*, provided the given medium be homogeneous, that is, its medium-constants ϵ and κ constants, in which case it should be possible to eliminate this property or function ϕ from Helmholtz's equations; this has, in fact, been accomplished by Boltzmann by means of certain substitutions.‡ On the other hand, the possibility of eliminating ϕ revealed its independence to the other functions. Helmholtz's ether could thus be conceived here as defined by two independent systems of equations, the one representing its longitudinal oscillations and the other its other properties. On making Boltzmann's substitutions we find that Helmholtz's equations reduce to Maxwell's.§ Since now, as we know, Maxwell's ether is capable of transmitting only transverse oscillations—we confine this statement to the ordinary waves, whose intensity varies inversely as the square of the distance from the source—we can thus modify our present conception of Helmholtz's ether, and conceive it as defined by two independent systems of equations, the one representing its longitudinal and the other its transverse oscillations. But, as all attempts to explain the phenomena of light by longitudinal oscillations have proved fruitless, whereas the assumption that light is a manifestation of transverse oscillations has become empirical now-a-days, only the latter oscillations would concern us here; that is, only that system of equations, which represent the transverse oscillations, need be examined. On account of the identity between these equations and Maxwell's, it is therefore immaterial, as far as the derivation and explanation of the phenomena of light are concerned, whether we regard Maxwell's or Helmholtz's ether as the seat and transmitter of light, and treat the different phenomena according to the equations of the former or those of the latter; but, for brevity, we shall employ the more familiar equations of Maxwell.

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* *Die elektromagnetische Theorie des Lichtes*, Leipzig, 1883.

† Cf. Curry: *Theory of Electricity and Magnetism*, § xl.

‡ Cf. *Ibid.*, p. 401.

§ Cf. *Ibid.*, p. 402.

Fundamental Equations.—The phenomena of light with which we are acquainted are confined almost exclusively to the transparent bodies, as air, glass, crystals, etc.—the behaviour of light on the surface of opaque bodies, as the metals, is perhaps the only exception of any importance. Since now transparent bodies are bad conductors, we shall thus have to seek the phenomena of light, with the above exception, in Maxwell's equations for insulators and dielectrics; these are

$$\left.\begin{aligned} \frac{4\pi}{v_0}\frac{dX}{dt} &= \frac{d\beta}{dz} - \frac{d\gamma}{dy} \\ \frac{4\pi}{v_0}\frac{dY}{dt} &= \frac{d\gamma}{dx} - \frac{d\alpha}{dz} \\ \frac{4\pi}{v_0}\frac{dZ}{dt} &= \frac{d\alpha}{dy} - \frac{d\beta}{dx} \end{aligned}\right\}, \qquad (1)^*$$

and

$$\left.\begin{aligned} \frac{4\pi}{v_0}\frac{da}{dt} &= \frac{dR}{dy} - \frac{dQ}{dz} \\ \frac{4\pi}{v_0}\frac{db}{dt} &= \frac{dP}{dz} - \frac{dR}{dx} \\ \frac{4\pi}{v_0}\frac{dc}{dt} &= \frac{dQ}{dx} - \frac{dP}{dy} \end{aligned}\right\}, \qquad (2)\dagger$$

where P, Q, R and α, β, γ denote the components of the electric and magnetic forces respectively along the x, y, z axes, X, Y, Z and a, b, c their respective moments, and v_0 the velocity of propagation of electromagnetic disturbances (light) in any standard medium as air.

Isotropic bodies are thereby characterised that the magnitude of the electric moment (displacement) is independent of the direction of the force acting; we can thus write

$$X = \frac{D}{4\pi}P, \quad Y = \frac{D}{4\pi}Q, \quad Z = \frac{D}{4\pi}R, \qquad (3)\ddagger$$

where D denotes the electric inductive capacity of the given medium. The analogous relations between the components of the magnetic moment and those of the magnetic force, namely

$$a = \frac{M}{4\pi}\alpha, \quad b = \frac{M}{4\pi}\beta, \quad c = \frac{M}{4\pi}\gamma, \qquad (4)\S$$

where M denotes the magnetic inductive capacity or the magnetic permeability of the medium, are assumed to hold for *all* media, as no

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[*] Cf. Curry: *Theory of Electricity and Magnetism*, formulae (9, ii.) and § xliii.

[†] Cf. *Ibid.*, formulae (10, ii.) and § xliii.

[‡] Cf. *Ibid.*, § xiv.

[§] Cf. *Ibid.*, § xxv.

appreciable variation with regard to direction has yet been detected in the value of this quantity M in one and the same medium.

Maxwell's Equations for Isotropic Dielectrics.—By the relations (3) and (4) Maxwell's equations can be written as follows for isotropic media:

$$\left.\begin{aligned}\frac{D}{v_0}\frac{dP}{dt}&=\frac{d\beta}{dz}-\frac{d\gamma}{dy}\\ \frac{D}{v_0}\frac{dQ}{dt}&=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\\ \frac{D}{v_0}\frac{dR}{dt}&=\frac{d\alpha}{dy}-\frac{d\beta}{dx}\end{aligned}\right\}, \dots\dots\dots\dots(5)^*$$

and

$$\left.\begin{aligned}\frac{M}{v_0}\frac{d\alpha}{dt}&=\frac{dR}{dy}-\frac{dQ}{dz}\\ \frac{M}{v_0}\frac{d\beta}{dt}&=\frac{dP}{dz}-\frac{dR}{dx}\\ \frac{M}{v_0}\frac{d\gamma}{dt}&=\frac{dQ}{dx}-\frac{dP}{dy}\end{aligned}\right\} \dots\dots\dots\dots(6)^*$$

Aeolotropic Media.—Media, in which the electric moment varies with the direction of the electric force, are called *anisotropic or aeolotropic*; the only such, with which we are familiar, are certain crystals, as the Iceland spar. In such media there are, in general, three directions, and these are at right angles to each other, along each of which the electric inductive capacity becomes either a maximum or a minimum; these directions are known as the *principal* axes of the crystal. If we choose these axes as coordinate-axes and denote the values of D along the same by D_1, D_2, and D_3, we must evidently replace the above relations (3) by the following:

$$X=\frac{D_1}{4\pi}P,\ Y=\frac{D_2}{4\pi}Q,\ Z=\frac{D_3}{4\pi}R; \dots\dots\dots\dots(7)$$

by which Maxwell's equations (1) can be written

$$\left.\begin{aligned}\frac{D_1}{v_0}\frac{dP}{dt}&=\frac{d\beta}{dz}-\frac{d\gamma}{dy}\\ \frac{D_2}{v_0}\frac{dQ}{dt}&=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\\ \frac{D_3}{v_0}\frac{dR}{dt}&=\frac{d\alpha}{dy}-\frac{d\beta}{dx}\end{aligned}\right\} \dots\dots\dots\dots(8)$$

Equations (2) evidently retain their above form, formulae (6).

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* Cf. *Theory of Electricity and Magnetism*, formulae (9) and (10), p. 34.

It is not, however, always convenient to choose the principal axes of the crystal as coordinate-axes (cf. Chapter VIII.); in which case X, Y, Z of formulae (1) are assumed * to be given by the expressions

$$\left.\begin{aligned} X &= \frac{1}{4\pi}(D_{11}P + D_{12}Q + D_{13}R) \\ Y &= \frac{1}{4\pi}(D_{12}P + D_{22}Q + D_{23}R) \\ Z &= \frac{1}{4\pi}(D_{13}P + D_{23}Q + D_{33}R) \end{aligned}\right\}, \qquad (9)\dagger$$

where these D's are the following functions of D_1, D_2, D_3, the values of D along the principal axes of the crystal, and the cosines of the angles between the principal and the coordinate-axes:

$$D_{11} = D_1 \cos^2(x', x) + D_2 \cos^2(x', y) + D_3 \cos^2(x', z),$$

$$D_{12} = D_1 \cos(x', x)\cos(y', x) + D_2 \cos(x', y)\cos(y', y) + D_3 \cos(x', z)\cos(y', z),$$

etc.,

where x, y, z and x', y', z' denote the coordinate and principal axes respectively. For $x = x'$, $y = y'$, and $z = z'$, it is evident that

$$D_{12} = D_{13} = D_{23} = 0,$$

and $$D_{11} = D_1, \quad D_{22} = D_2, \quad D_{33} = D_3;$$

by which the general formulae for aeolotropic media reduce to formulae (8).

Non-homogeneous Media.—With the exception of our atmosphere, there are few non-homogeneous media within which phenomena of light have been observed; the treatment of the behaviour of light in such media would, however, meet with no serious difficulties; the phenomena of refraction, absorption, etc., of our atmosphere can be quite simply deduced, only an exacter knowledge of the law of variation of its density would be desirable; its introduction, on the other hand, offers no difficulty.

Transition Films.—The assumption that adjacent media are separated by transition films, within which all quantities are supposed to vary rapidly but continuously as we pass through the films, assuming on any surface the values in the respective medium,‡ suggests a kind of non-homogeneity; on the other hand, it does away with all discontinuities and thus permits an integration throughout entire space. The importance of these films in the theory of electricity and magnetism urges as simple a treatment of them as possible; the simplest and most natural assumption concerning their constitution is that, aside from the rapid and continuous change of all quantities as we pass through the films, Maxwell's equations (1) and (2) hold at every point of the same. The surface conditions derived therefrom are then not only consistent with one another but they suffice for the determination of the quantities sought.

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* Cf. *Theory of Electricity and Magnetism*, § xliii., pp. 435-437.

† Cf. *Ibid.*, § xliii., formulae (25).

‡ Cf. *Ibid.*, § v.

Surface Conditions.—If we choose the normal to the given dividing surface as x-axis, we find, on integrating Maxwell's equations (1) and (2) throughout any film, the surface conditions

$$\left.\begin{array}{c} \frac{d}{dt}(X_1 - X_0) = 0 \\ \gamma_1 = \gamma_0,\ \beta_1 = \beta_0 \end{array}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

and

$$\left.\begin{array}{c} \frac{d}{dt}(a_1 - a_0) = 0 \\ R_1 = R_0,\ Q_1 = Q_0 \end{array}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

where the indices 0 and 1 refer to the two adjacent media. For the derivation of these equations cf. § v. of my *Theory of Electricity and Magnetism.*

Homogeneous Isotropic Dielectrics.—Let us examine the electromagnetic state of an homogeneous isotropic dielectric; it is given by the above equations (5) and (6), or, if we replace the electric and magnetic forces by their moments (cf. formulae (3) and (4)), by

$$\left.\begin{array}{c} \frac{M}{v_0}\frac{dX}{dt} = \frac{db}{dz} - \frac{dc}{dy} \\ \frac{M}{v_0}\frac{dY}{dt} = \frac{dc}{dx} - \frac{da}{dz} \\ \frac{M}{v_0}\frac{dZ}{dt} = \frac{da}{dy} - \frac{db}{dx} \end{array}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

and

$$\left.\begin{array}{c} \frac{D}{v_0}\frac{da}{dt} = \frac{dZ}{dy} - \frac{dY}{dz} \\ \frac{D}{v_0}\frac{db}{dt} = \frac{dX}{dz} - \frac{dZ}{dx} \\ \frac{D}{v_0}\frac{dc}{dt} = \frac{dY}{dx} - \frac{dX}{dy} \end{array}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

It is quite immaterial which of these two systems of equations (5) and (6) or (12) and (13) we employ in examining the state of the given medium; the former contains the forces acting in the medium, by which, when determined, its state is indirectly given (cf. formulae (3) and (4)), whereas the latter contains the moments themselves, and thus defines its state directly. We shall, however, follow Maxwell's example here and make use of equations (12) and (13).

The Electric Moments.—To determine the electric state of the given medium, we must evidently eliminate the magnetic moments from formulae (12) and (13); for this purpose differentiate the first equation of formulae (12) with regard to t, and we have, on replacing $\frac{db}{dt}$ and $\frac{dc}{dt}$ by their values from formulae (13),

$$\frac{M}{v_0}\frac{d^2X}{dt^2}=\frac{v_0}{D}\left[\frac{d}{dz}\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)-\frac{d}{dy}\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\right],$$

or
$$\frac{DM}{v_0^2}\frac{d^2X}{dt^2}=\nabla^2X-\frac{d\epsilon}{dx},$$

where
$$\nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}, \qquad\text{(14)}$$

and
$$\epsilon=\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}; \qquad\text{(15)}$$

ϵ denotes the density of the real electricity.*

Similarly, we find for the other component-moments

$$\frac{DM}{v_0^2}\frac{d^2Y}{dt^2}=\nabla^2Y-\frac{d\epsilon}{dy}$$

$$\frac{DM}{v_0^2}\frac{d^2Z}{dt^2}=\nabla^2Z-\frac{d\epsilon}{dz}.$$

In the given case—for dielectrics—the quantity ϵ is independent of the time t, but may be a function of the coordinates x, y, z. To confirm this, differentiate formulae (12), the first with regard to x, the second to y, and the third to z, add, and we have

$$\frac{M}{v_0}\frac{d}{dt}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)=0,$$

or by formula (15)

$$\frac{d\epsilon}{dt}=0, \quad\text{hence } \epsilon=f(x, y, z).$$

The integration of the above equations can thus be performed without any regard to the value of ϵ at the given point; that is, the accumulation of electricity within a dielectric will evidently have no effect whatever on the passage of rapid oscillations, as those of light or the Hertzian waves through it; its presence can thus be thoroughly ignored, that is, we may put $\epsilon=0$ † in all such cases.

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* Cf. Curry: *Theory of Electricity and Magnetism*, § vi., p. 46.

† Cf. also *Ibid.*, p. 64.

We may thus replace the above equations by the simpler ones

$$\frac{DM}{v_0^2}\frac{d^2X}{dt^2}=\nabla^2X,\quad \frac{DM}{v_0^2}\frac{d^2Y}{dt^2}=\nabla^2Y,\quad \frac{DM}{v_0^2}\frac{d^2Z}{dt^2}=\nabla^2Z, \ldots\ldots\ldots(16)$$

The particular integrals of these equations representing plane-waves have been examined in Chapter IV. of my *Theory of Electricity and Magnetism*, to which I refer the student here; their velocity of propagation is $\frac{v_0}{\sqrt{DM}}$. The Hertzian waves are also treated in the same chapter (cf. also Ex. 1, Chapter II.).

The Magnetic Moments.—Similarly, we can determine the magnetic state of the given dielectric, on eliminating the electric moments from formulae (12) and (13); to accomplish this, we proceed exactly as in the preceding case, and we find

$$\frac{DM}{v_0^2}\frac{d^2a}{dt^2}=\nabla^2a,\quad \frac{DM}{v_0^2}\frac{d^2b}{dt^2}=\nabla^2b,\quad \frac{DM}{v_0^2}\frac{d^2c}{dt^2}=\nabla^2c. \ldots\ldots\ldots(17)$$

The similarity between these equations and those above (16) for the electric moments—this similarity could have been anticipated from the analogous parts played by the electric and magnetic moments in the fundamental equations—renders an examination of only the one system necessary, provided, of course, the given solutions or expressions hold for both systems or one and the same system of electromagnetic disturbances; but this would not, in general, be the case, since certain relations exist between the electric and magnetic moments (cf. formulae (12) and (13) and Chapter II.), among others that expressing the empirical law that the electric and magnetic moments take place at right angles to each other, the proof of which follows on pp. 14-15.

Plane-Waves.—The most general expression for plane-wave motion is

$$y=\phi\,(x\pm vt), \ldots\ldots\ldots\ldots\ldots\ldots(18)$$

where y denotes the displacement of any vibrating particle from its position of rest, x its distance from any given point in the direction of propagation of the wave, v the velocity of propagation, t the time, and ϕ an arbitrary function. The most familiar such functions ϕ are

$$\left.\begin{aligned} y&=a\sin n\left(t\pm\frac{x}{v}\right)\\ y&=a\cos n\left(t\pm\frac{x}{v}\right)\end{aligned}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

or

$$y=ae^{in\left(t\pm\frac{x}{v}\right)}, \ldots\ldots\ldots\ldots\ldots\ldots(20)$$

where e denotes the so-called "base" of the natural system of logarithms and i the imaginary unit, $\sqrt{-1}$.

The expression

$$y = a \sin n\left(t - \frac{x}{v}\right), \qquad\qquad (21)$$

evidently represents a simple plane-wave of amplitude a propagated along the x-axis in a sine curve with the velocity v. Since y remains unaltered, when t is increased by $\frac{2\pi}{n}$, it follows that the periodic time $T = \frac{2\pi}{n}$; the wave-length λ is thus

$$\lambda = vT = \frac{2\pi v}{n}, \quad \text{hence} \quad n = \frac{2\pi v}{\lambda}.$$

We can, therefore, write the expression (21)

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right); \qquad\qquad (22)$$

the angle $\frac{2\pi}{\lambda}(vt - x) = 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$ is called the "phase" of the wave.

The other expressions (19) and (20) can be similarly interpreted.

The Intensity of Plane Waves.—We define the intensity of an electric or magnetic oscillation or beam of light at any point as the average kinetic energy I of the vibrating particle or particles at that point. To determine the average kinetic energy of the plane-wave, represented by formula (22), at any point, differentiate formula (22) with regard to t, and we have

$$\frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x),$$

and hence its kinetic energy at any time t

$$\frac{m}{2}\left(\frac{dy}{dt}\right)^2 = \frac{2m\pi^2 v^2}{\lambda^2} a^2 \cos^2 \frac{2\pi}{\lambda}(vt - x),$$

where m denotes the mass of the given particle; its *average* kinetic energy I is, therefore,

$$I = \frac{1}{T}\int_0^T \frac{m}{2}\left(\frac{dy}{dt}\right)^2 dt = \frac{m\pi^2 v^2}{\lambda^2 T} a^2 \int_0^T 2\cos^2 \frac{2\pi}{\lambda}(vt - x)\,dt$$

$$= \frac{m\pi^2 v^2}{\lambda^2 T} a^2 \int_0^T \left[1 + \cos \frac{4\pi}{\lambda}(vt - x)\right] dt = \frac{m\pi^2 v^2 a^2}{\lambda^2} = \frac{m\pi^2 a^2}{T^2}, \qquad (23)$$

The intensity of an ordinary* electromagnetic or luminous plane-wave thus varies directly as the square of its amplitude and inversely as the square of its wave-length or period of oscillation. This law is not restricted alone to plane-waves, but it also holds for spherical waves (cf. ex. 3 at end of Chapter). In photometry, where we compare sources of light of the same period of oscillation, we thus have the following simple relation between their intensities:

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* Cf. p. 5.

$$I_1 : I_2 = a_1^2 : a_2^2; \qquad \text{..............................(24)}$$

that is, the intensity of the one is to that of the other as the squares of their respective amplitudes.

We cannot easily compare the intensities of waves or beams of light of different wave-length or colour, since they produce quite different impressions on the retina of the eye; this is, of course, due to the fact that the expression for the intensity contains the wave-length (cf. formula (23)). We encounter, in fact, this same difficulty to a less degree in most photometric measurements, where the given sources are assumed to emit waves of the same wave-length.

Principle of Superposition.—The principle of the superposition of disturbances or waves is recognised as empirical in the theory of light, since all problems treated according to the same agree with observation and experiment. We can state the principle of superposition as follows: When two or more disturbances are simultaneously brought to act on one and the same particle of a medium, the resultant disturbance is determined by the direct superposition of the single disturbances (cf. also Chapter IV.).

Doctrine of Interference. Simple and Compound Waves.—The doctrine of interference is only another form of or sequel to the principle of superposition. The acceptance of some such principle is evidently indispensable in the treatment of most problems on light; it must, indeed, be employed at the very outset in the examination of the particular integrals or solutions of our fundamental equations. With the exception of the phenomena of interference proper (cf. Chapter IV.), the only other simple particular integrals of these equations, (16) and (17), that would concern us here, are those that represent the so-called "stationary" waves. We shall now find it convenient to distinguish between the "simple" waves, represented by such simple or fundamental particular integrals as (19) and (20), and the "compound" wave, the resultant obtained according to the principle of superposition, of two or more such simple waves; the stationary waves belong to the latter class, as we shall see directly.

Stationary Plane-Waves.—The "stationary" waves are so termed because they have apparently no velocity of propagation, their crests and troughs remaining stationary with regard to their direction of propagation. The stationary plane-wave must, therefore, be represented by some such function as

$$y=a\sin\frac{2\pi}{\lambda}vt\cos\frac{2\pi}{\lambda}x, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

which can also be written

$$y=\frac{a}{2}\sin\frac{2\pi}{\lambda}(vt-x)+\frac{a}{2}\sin\frac{2\pi}{\lambda}(vt+x);$$

that is, according to the principle of superposition, we could thus conceive this stationary plane-wave as the resultant of two simple plane-waves of one and the same amplitude $\frac{a}{2}$ and wave-length λ, the one advancing with the velocity v along the x-axis, and the other with the same velocity in the opposite direction. It is evident that the given expression (25) is also a particular integral of our differential equations (16) and (17).

Other compound waves, the resultants of simple waves of different amplitude and phase, are treated in Chapter IV. on interference.

The Electric and Magnetic Oscillations take place at ⊥ to each other.—Lastly, let us return to the proof of the law stated on p. 11, namely, that the electric and magnetic oscillations take place at right angles to each other, restricting ourselves thereby, as above, to plane-wave motion. Take, for example, the plane-wave

$$y=a\sin\frac{2\pi}{\lambda}(vt-x), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

and let y correspond to the electric moment Y of formulae (12) and (13); the other two moments, X and Z, vanish, and formulae (13) reduce to

$$\frac{D}{v_0}\frac{da}{dt}=-\frac{dY}{dz}=0,$$

$$\frac{D}{v_0}\frac{db}{dt}=0 \text{ and } \frac{D}{v_0}\frac{dc}{dt}=\frac{dY}{dx},$$

hence $$a=b=0 \quad \text{(cf. p. 10)},$$

and $$c=\frac{v_0}{D}\int\frac{dY}{dx}dt;$$

replace here Y by its value (26), and we find

$$c=-\frac{v_0}{D}\int a\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)dt=\frac{v_0}{D}\frac{a}{v}\sin\frac{2\pi}{\lambda}(vt-x)$$

or, since $v=\frac{v_0}{\sqrt{DM}}$ (cf. p. 11),

$$c=a\sqrt{\frac{M}{D}}\sin\frac{2\pi}{\lambda}(vt-x);$$

that is, the (resultant) magnetic oscillation accompanying the given electric one takes place parallel to the z-axis, and hence at right angles to the given electric oscillation. Since these two oscillations otherwise differ from one another only in amplitude—their wave-lengths and phases are the same—it follows that the crests and troughs of the one wave will coincide, with regard to direction of propagation, with those of the other; that is, the electric and magnetic moments will attain maxima simultaneously and periodically at any given point. These oscillations are represented graphically in the annexed figure.

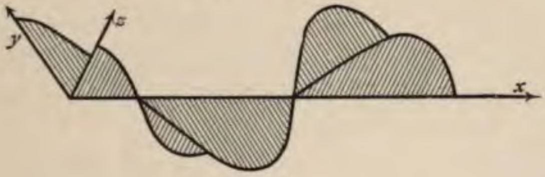

FIG. 1.

Relative Position of Crests of Electric and Magnetic Plane-Waves.—Although the above law is quite general—for its proof see Chapters II. and III.*—we cannot conclude that the relative position of the crests or troughs of all electric waves and the magnetic ones accompanying the same is always that just deduced; for example, the stationary electromagnetic plane-wave behaves quite differently in this respect. Let us examine it briefly; take, for example, the stationary plane-wave

$$y=a\sin\frac{2\pi}{\lambda}vt\cos\frac{2\pi}{\lambda}x,$$

formula (25), as our electric wave; we have then

$$X=Z=0 \text{ and } y=Y,$$

and formulae (13) reduce to

$$a=b=0$$

and

$$c=\frac{v_0}{D}\int\frac{dY}{dx}dt=\frac{v_0}{D}\int a\frac{2\pi}{\lambda}\sin\frac{2\pi}{\lambda}vt\sin\frac{2\pi}{\lambda}x\,dt$$

$$=\frac{v_0}{D}\frac{a}{v}\cos\frac{2\pi}{\lambda}vt\sin\frac{2\pi}{\lambda}x=a\sqrt{\frac{M}{D}}\cos\frac{2\pi}{\lambda}vt\sin\frac{2\pi}{\lambda}x.$$

It is evident from these expressions for Y and c that

(1) for $$x=0 \text{ and } t=\frac{T}{4},$$

$$Y=a \text{ and } c=0,$$

---

* Cf. also Maxwell: *Electricity and Magnetism*, §§ 790-791, vol. ii., pp. 339-401 (second edition).

or the electric oscillation has a crest (maximum), where the amplitude of the magnetic oscillation vanishes,

(2) for $$x=\frac{\lambda}{2} \text{ and } t=\frac{T}{4},$$

$$Y=-a \text{ and } c=0,$$

or the electric oscillation has a trough (minimum), where the amplitude of the magnetic oscillation vanishes, and, lastly,

(3) for $$x=\frac{\lambda}{4} \text{ and } t=\frac{T}{4},$$

$$Y=0 \text{ and } c=0,$$

or the amplitude of the electric oscillation vanishes, as the magnetic oscillation is passing through its (initial) position of rest from a trough to a crest (cf. the annexed figures).

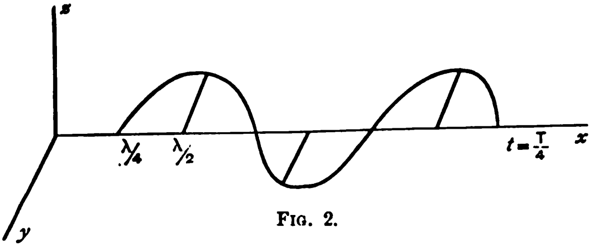

FIG. 2.

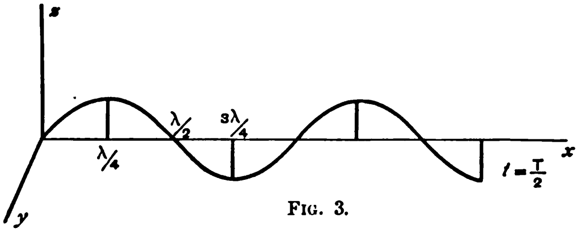

FIG. 3.

Similary we find that for $t=\frac{T}{2}$ and $x=\frac{\lambda}{4}, \frac{\lambda}{2}$, and $\frac{3\lambda}{4}$,

$$y=0, \quad c=a,$$
$$y=0, \quad c=0,$$
and $$y=0, \quad c=-a \text{ respectively,}$$

or the magnetic oscillation has a crest, where the amplitude of the electric oscillation vanishes, the amplitude of the magnetic oscillation vanishes, as the electric oscillation is passing through its (initial) position of rest from a crest to a trough, and the magnetic oscillation has a trough, where the amplitude of the electric oscillation vanishes. The given waves are represented graphically in the above figures.

Spherical Waves.—Waves that diverge radially from a common centre or source at finite distance are termed "spherical"; this is evidently both the more general and the commoner form of radiation, the plane-wave being only a particular case of it, that, namely, where the common centre of the advancing wave-fronts lies at infinite distance.

The general equation of wave-motion is

$$\frac{d^2\phi}{dt^2}=v^2\nabla^2\phi, \quad \text{.............................}(27)$$

where v denotes the velocity of propagation of the waves represented by the function ϕ. For plane-wave motion propagated along the x-axis this equation evidently assumes the simpler form

$$\frac{d^2\phi}{dt^2}=v^2\frac{d^2\phi}{dx^2}. \quad \text{.............................}(28)$$

Purely Spherical Waves.—To obtain the particular form assumed by equation (27) for spherical wave-motion, we shall make use of that property, by which the simplest kind of spherical waves is characterised; these would evidently be waves that diverge radially with one and the same intensity in all directions from a common source; and they would thus possess the common property that their wave-function, ϕ of formula (27), be a function of r alone, the distance of any wave-front from its source, and not of the coordinates x, y, z singly; let us term such waves "purely" spherical waves. We may thus express $\nabla^2\phi$ here as a function of r and t, on the assumption that ϕ itself be a function of these variables only. By the analytic relation

$$r^2=x^2+y^2+z^2,$$

we thus have

$$\frac{d\phi}{dx}=\frac{d\phi}{dr}\frac{dr}{dx}=\frac{d\phi}{dr}\frac{x}{r},$$

$$\frac{d^2\phi}{dx^2}=\frac{d}{dx}\left(\frac{1}{r}\frac{d\phi}{dr}\right)x+\frac{1}{r}\frac{d\phi}{dr}=\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\frac{x^2}{r}+\frac{1}{r}\frac{d\phi}{dr}$$

$$=\frac{x^2}{r^2}\frac{d^2\phi}{dr^2}-\frac{x^2}{r^3}\frac{d\phi}{dr}+\frac{1}{r}\frac{d\phi}{dr},$$

and similar expressions for

$$\frac{d^2\phi}{dy^2} \text{ and } \frac{d^2\phi}{dz^2};$$

hence

$$\nabla^2\phi=\frac{x^2+y^2+z^2}{r^2}\frac{d^2\phi}{dr^2}-\frac{x^2+y^2+z^2}{r^3}\frac{d\phi}{dr}+\frac{3}{r}\frac{d\phi}{dr}$$

$$=\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}.$$

Equation (27) can thus be written here

$$\frac{d^2\phi}{dt^2}=v^2\left(\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}\right).$$

On replacing here ϕ by the new variable

$$\psi=r\phi, \quad\text{......................................}(29)$$

we have

$$\frac{1}{r}\frac{d^2\psi}{dt^2}=\frac{v^2}{r}\frac{d^2\psi}{dr^2},$$

which for $r \gtrless 0$ reduces to

$$\frac{d^2\psi}{dt^2}=v^2\frac{d^2\psi}{dr^2}. \quad\text{..................................}(30)$$

This equation is similar in form to those that represent plane-waves; the only difference is that the radius-vector r takes the place here of one of the coordinates x, y, z (cf. formula (28)). The solution of the latter was

$$\phi=f(x \pm vt).$$

The corresponding solution of the given equation (30) for purely spherical waves would thus be

$$\psi=f(r \pm vt),$$

or, by formula (29),

$$\phi=\frac{1}{r}f(r \pm vt), \quad\text{................................}(31)$$

where f is an arbitrary function of $(r \pm vt)$.

The function $\phi=\frac{1}{r}f(r-vt)$ represents a system of spherical waves diverging radially with uniform velocity, and one and the same intensity and phase in all directions from a common centre $r=0$; their phases remain the same, as they advance, but their amplitudes decrease inversely as the distance r, since ϕ decreases as r increases (cf. also pp. 72-74 of my *Theory of Electricity and Magnetism*). Hence the empirical law that the intensity of a (purely) spherical wave varies inversely as the square of the distance from its source.

On the other hand, the function $\phi=\frac{1}{r}f(r+vt)$ would represent a system of spherical waves, converging radially with uniform velocity and one and the same intensity and phase in all directions towards a common centre; their phases remain the same, as they advance, but their amplitudes increase inversely as the distance r.

The Point $r=0$.—The determination of the behaviour of the given spherical waves at the point $r=0$ would require special investigations and these would naturally have to be of a purely mathematical—

character—since the above equation (30) does not, as we have observed above, necessarily hold at that point. The purely mathematical treatment of such physical problems seems to me, however, to be seldom justified, and we surely cannot be surprised when it leads to unsatisfactory or even absurd results; for what is to be understood by the presence of a system of waves or the occurrence of a natural phenomenon in a mathematical point? It is, therefore, customary to exclude such points from the region treated, as we shall do later in Chapter V.

The Derivatives of ϕ as Integrals.—We have examined above two classes of particular integrals of our equation of wave-motion (27), the plane-waves and spherical waves (31) of the simplest kind; the latter are of particular interest, since the derivatives of any such integral or function ϕ with regard to x, y, z are also particular integrals of our equation of wave-motion, provided of course the same can be formed and physically interpreted. This follows, since our equation of wave-motion (27) is both homogeneous and linear, and its co-efficient v^2 a constant. These derivatives of ϕ form a new and interesting class of particular integrals of our equation of wave-motion; each such integral is a compound* integral (cf. p. 13), that is, it consists of two or more terms—not necessary integrals themselves—which are thereby characterised that they contain the different powers of $\frac{1}{r}$ as factors. Let the following examples serve as illustrations of this new class of integrals.

The Particular Integral $\frac{d\phi}{dx}$.

$$\frac{d\phi}{dx}=\frac{d}{dx}\left[\frac{1}{r}f(r\pm vt)\right]=\frac{x}{r^2}\frac{df}{dr}-\frac{x}{r^3}f,$$

or, since $x=r\cos\alpha$, where α denotes the angle between the radius-vector r and the x-axis,

$$\frac{d\phi}{dx}=\left(\frac{1}{r}\frac{df}{dr}-\frac{1}{r^2}f\right)\cos\alpha. \qquad (32)$$

This integral is a function not only of r and $(r\pm vt)$ but also of the angle α. For $\alpha=\frac{\pi}{2}$ the expression for $\frac{d\phi}{dx}$ vanishes, whereas for $\alpha>\frac{\pi}{2}$ it reverses its sign; it thus follows that there will be no disturbance throughout the yz-plane, and that on the one side of this neutral plane the oscillations will take place in an opposite direction to those on the other. The amplitude, and hence the intensity, of the oscillations will thus decrease, as we recede over any sphere, $r = \text{const.}$, from the x-axis towards the neutral (yz) plane, where both vanish.

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* The word "compound" is used here in a somewhat different sense from that employed on p. 13; but, as in any given case, the meaning intended is apparent, we shall make no further discrimination between the two

Since the expression (32) consists of two terms, we can conceive the given wave as the resultant of the two waves represented by those terms. We should observe, however, that, although the expression itself is a particular integral of our equation of wave-motion (27), it does not necessarily follow that its single terms are also, or, in general, that every compound wave is the resultant of two or more *waves*, that is, waves in the sense that the functions by which those waves are represented, be particular integrals of our equation of wave motion (27).

The amplitude of the given compound wave can be determined by the method given in Chapter IV—it is the resultant of the amplitudes of the two waves $\frac{1}{r}\frac{df}{dr}\cos\alpha$ and $-\frac{f}{r^2}\cos\alpha$ determined by that method. Since the amplitude of the latter varies inversely as the square of the distance, and that of the former inversely as the distance itself, it follows that the amplitude of the given wave would approach that of its one simple wave, $-\frac{f}{r^2}\cos\alpha$, near its source and that of its other, $\frac{1}{r}\frac{df}{dr}\cos\alpha$, at greater distances from it; the wave $-\frac{f}{r^2}\cos\alpha$ would thus predominate near the source of disturbance and the wave $\frac{1}{r}\frac{df}{dr}\cos\alpha$ at greater distances from the same.

The particular integrals $\frac{d\phi}{dy}$ and $\frac{d\phi}{dz}$ evidently represent similar waves to that just examined.

The Particular Integral $\frac{d^2\phi}{dx^2}$.

$$\frac{d^2\phi}{dx^2} = \frac{d}{dx}\left(\frac{x}{r^2}\frac{df}{dr} - \frac{x}{r^3}f\right)$$

$$= \frac{1}{r^2}\frac{df}{dr} - \frac{1}{r^3}f + \frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr} - \frac{f}{r^3}\right)\frac{x^2}{r}$$

$$\left.\begin{aligned} &= \frac{1}{r}\frac{d^2f}{dr^2}\cos^2\alpha + \frac{1}{r^2}\frac{df}{dr}(1 - 3\cos^2\alpha) - \frac{1}{r^3}f(1 - 3\cos^2\alpha) \\ \text{or}\quad &= \left(\frac{1}{r^2}\frac{df}{dr} - \frac{f}{r^3}\right) + \left(\frac{1}{r}\frac{d^2f}{dr^2} - \frac{3}{r^2}\frac{df}{dr} + \frac{3f}{r^3}\right)\cos^2\alpha \end{aligned}\right\} \ldots\ldots\text{(33)}$$

As above, we can conceive the compound wave, represented by this integral, as the resultant of the three waves, represented by the three terms in $\frac{1}{r}$, $\frac{1}{r^2}$, and $\frac{1}{r^3}$, of which that integral is composed, and advancing quite independently of one another with intensities that vary inversely as the second, fourth, and sixth powers of the distance.

On the other hand, we could conceive the given compound wave as the resultant of the two waves

$$\frac{1}{r^2}\frac{df}{dr}-\frac{f}{r^3} \text{ and } \left(\frac{1}{r}\frac{d^2f}{dr^2}-\frac{3}{r^2}\frac{df}{dr}-\frac{3f}{r^3}\right)\cos^2\alpha.$$

The former, as function of r alone, represents a purely spherical wave, or one that is emitted radially with one and the same intensity in all directions from the given source. The latter is not a function of r alone, but contains $\cos^2\alpha$ as factor; the disturbance represented by it is thus confined chiefly to the region round the x-axis, diminishing rapidly in intensity, as $\cos^4\alpha$, as we recede along the surface of any sphere with centre at origin from that axis towards the yz-plane, throughout which it vanishes entirely; on either side of this neutral plane the given disturbance is one and the same. The amplitude of the purely spherical wave is the resultant of amplitudes that vary inversely as the square and third powers of the distance, whereas the amplitude of the other component wave is the resultant of amplitudes that vary inversely, not only as the square and third powers of the distance, but as the distance itself. The given compound wave would, therefore, be represented throughout the yz-plane alone by the purely spherical wave, and in the immediate neighbourhood of the x-axis, but especially at greater distances from the source, approximately by the other component wave.

The particular integrals $\frac{d^2\phi}{dy^2}$ and $\frac{d^2\phi}{dz^2}$ represent similar waves.

The Particular Integral $\frac{d^2\phi}{dxdy}$.

$$\frac{d^2\phi}{dxdy}=x\frac{d}{dy}\left(\frac{1}{r^2}\frac{df}{dr}-\frac{f}{r^3}\right)$$

$$=\frac{xy}{r}\frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr}-\frac{f}{r^3}\right)=\frac{xy}{r^3}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3f}{r^2}\right).$$

If we denote the angle between the radius-vector r and the x-axis by α, as above, and that which the xr-plane makes with the xy-plane by θ, as indicated below in figure 4, we have the following relations between these two systems of co-ordinates:

$$x=r\cos\alpha,$$
$$y=r\sin\alpha\cos\theta,$$
$$z=r\sin\alpha\sin\theta;$$

by which the above expression can be written

$$\frac{d^2\phi}{dxdy}=\frac{1}{r}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3}{r^2}f\right)\sin\alpha\cos\alpha\cos\theta.$$

This function evidently represents an even more complicated disturbance or, more strictly, distribution of the amplitudes according to their magnitudes and direction of oscillation over any given sphere, than those already examined. Here not only the *yz*- and *xz*-planes are neutral planes, but the *x*-axis is also a neutral axis, or one along which no disturbance appears, whereas the direction of oscillation is reversed, as we pass through either neutral plane.

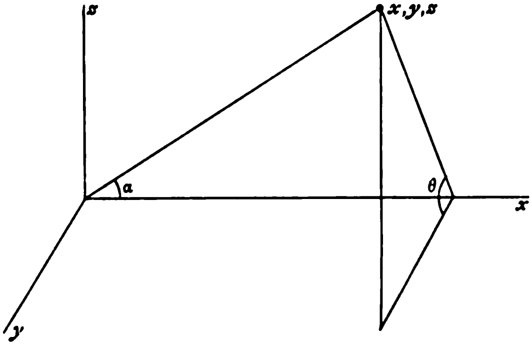

FIG. 4.

The Particular Integral $\frac{d^n\phi}{dx^\lambda dy^\mu dz^\nu}$, $\lambda+\mu+\nu=n$.—It is evident that the higher the derivative of ϕ, the more complicated the disturbance represented by that derivative. Although we cannot enter into the further explicit examination of such particular integrals here, we shall, nevertheless, call attention to some of the properties peculiar to them in general. A glance at the above solutions shows that the n^{th} derivative of ϕ would have the form

$$\left.\begin{aligned}&P_0(\cos\alpha,\ \sin\alpha\cos\theta,\ \sin\alpha\sin\theta)\,\frac{1}{r^{n+1}}f\\&+P_1(\cos\alpha,\ \sin\alpha\cos\theta,\ \sin\alpha\sin\theta)\,\frac{1}{r^n}\frac{df}{dr}\\&+\ldots P_k(\cos\alpha,\ \sin\alpha\cos\theta,\ \sin\alpha\sin\theta)\,\frac{1}{r^{n-k+1}}\frac{d^kf}{dr^k}\\&+\ldots P_n(\cos\alpha,\ \sin\alpha\cos\theta,\ \sin\alpha\sin\theta)\,\frac{1}{r}\frac{d^nf}{dr^n}\end{aligned}\right\},\ \ldots\ldots\ldots\ldots(34)$$

where P_k, the coefficient of the $k+1^{\text{st}}$ term of this series, denotes a function of the n^{th} degree in $\cos\alpha$, $\sin\alpha\cos\theta$, $\sin\alpha\sin\theta$; any given coefficient P_k is evidently a function not only of the integer k but also of the n derivatives taken or the number of differentiations of ϕ with regard to x, y, and z respectively.

The form of any coefficient P_k determines the law of distribution of the amplitudes * of the wave k over any given sphere, and the other factors the law of variation of those amplitudes along any vector. The total resultant disturbance at any point would thus be determined not only by the various coefficients P_k but by the factors $\frac{1}{r^{n-k+1}}$ and $\frac{d^kf}{dr^k}-k=0, 1, 2 \ldots n$; its actual determination would require, however, considerable calculation, especially for large values of n, as we shall see in the following chapter. Near its source the given disturbance would be represented approximately by the first term or terms of the series (34) and at greater distances from it by the last term, $\frac{P_n}{r}\frac{d^nf}{dr^n}$, or terms of the same.

We have just observed that the form assumed by any coefficient P_k depends upon the number of differentiations of ϕ with regard to x, y, and z respectively; the number of such coefficients P_k of the n^{th} degree for any given k evidently increases rapidly as n increases. Each and every such coefficient represents a given particular law of distribution of the amplitudes *; other laws than those determined by any such group—k given—could be expressed by the sums of these different coefficients; let us denote such a sum of P_k's by ΣP_k. It is now shown in the theory of spherical harmonics that any given law of distribution over any finite plane can be represented by a similar coefficient P_k or ΣP_k; hence it follows that any pencil of plane-waves, whose amplitudes shall be distributed over any plane pierced by that pencil according to any given law, however complicated, could be represented analytically by some given series (34) or sum of such series, provided that pencil be conceived as the residual—represented by $\frac{P_n}{r}\frac{d^nf}{dr^n}$ or $\Sigma\frac{P_n}{r}\frac{d^nf}{dr^n}$—of the spherical waves, represented by that series or sum of series, at a very great (infinite) distance from their source.

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* For brevity, we use the expression "distribution of the amplitudes" instead of "distribution of the amplitudes according to their magnitude and direction of oscillation."

EXAMPLES.

1. Show that the expression $\frac{a}{r}\sin\frac{2\pi}{\lambda}(vt-r)$

represents a system of purely spherical waves of amplitude $\frac{a}{r}$ and wave length λ diverging from the common source $r=0$ with the velocity of propagation v.

$$\phi_1=\frac{a}{r}\sin n\left(t-\frac{r}{v}\right)$$

is evidently a particular solution of the general function

$$\phi=\frac{1}{r}f(r\pm vt),$$

which represents purely spherical wave-motion.

ϕ_1, therefore, represents a system of purely spherical waves of amplitude $\frac{a}{r}$ diverging from $r=0$ with the velocity v. Since ϕ_1 remains unaltered, when t is increased by $\frac{2\pi}{n}$, it follows that the periodic time $T=\frac{2\pi}{n}$, and hence the wave length $\lambda=vT=\frac{2\pi v}{n}$. ϕ_1 can thus be written

$$\phi_1=\frac{a}{r}\sin\frac{2\pi}{\lambda}(vt-r).$$

2. Show that the following expressions represent possible forms of spherical waves:

$$-\frac{2\pi}{\lambda}\frac{a}{r}\cos\alpha\cos\omega-\frac{a}{r^3}\cos\alpha\sin\omega,$$

$$-\frac{4\pi^2}{\lambda^2}\frac{a}{r}\cos^2\alpha\cos\omega-(3\cos^2\alpha-1)\frac{2\pi}{\lambda}\frac{a}{r^2}\sin\omega+(3\cos^2\alpha-1)\frac{a}{r^3}\cos\omega,$$

$$-\frac{4\pi^2}{\lambda^2}\frac{a}{r}\sin\alpha\cos\alpha\cos\theta\cos\omega-3\frac{a}{r^2}\sin\alpha\cos\alpha\cos\theta\left(\frac{2\pi}{\lambda}\sin\omega-\cos\omega\right),$$

and
$$\frac{8\pi^3}{\lambda^3}\frac{a}{r}\sin^2\alpha\cos\alpha\sin\theta\cos\theta\cos\omega+\frac{24\pi^2}{\lambda^2}\frac{a}{r^2}\sin^2\alpha\cos\alpha\sin\theta\cos\theta\sin\omega$$

$$-\frac{30\pi}{\lambda}\frac{a}{r^3}\sin^2\alpha\cos\alpha\sin\theta\cos\vartheta\cos\omega-15\frac{a}{r^4}\sin^2\alpha\cos\alpha\sin\theta\cos\theta\sin\omega,$$

where
$$\omega=\frac{2\pi}{\lambda}(vt-r),$$

and α and θ are the angles employed on p. 21.

3. The average kinetic energy I of any particle of the simple spherical wave

$$\phi=\frac{a}{r}\sin\frac{2\pi}{\lambda}(vt-r)$$

is
$$\frac{m\pi^2v^2}{\lambda^2}\frac{a^2}{r^2},$$

where m denotes the mass of that particle.

We have
$$\frac{d\phi}{dt}=\frac{2\pi v}{\lambda}\frac{a}{r}\cos\frac{2\pi}{\lambda}(vt-r),$$

and hence
$$I=\frac{1}{T}\int_0^T\frac{m}{2}\left(\frac{d\phi}{dt}\right)^2dt=\frac{m\pi^2v^2}{\lambda^2T}\frac{a^2}{r^2}\int_0^T2\cos^2\frac{2\pi}{\lambda}(vt-r)dt$$

$$=\frac{m\pi^2v^2}{\lambda^2}\frac{a^2}{r^2}.$$

The following relation would, therefore, hold between the intensities of two such simple spherical waves of one and the same wave-length (and phase),

$$I_1 : I_2 = a_1^2 r_2^2 : a_2^2 r_1^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

(Cf. pp. 12-13.)

Cor. 1. For one and the same wave ($a_1 = a_2$) this proportion would assume the familiar form

$$I_1 : I_2 = r_2^2 : r_1^2.$$

(Cf. pp. 12-13 and formula 24.)

Cor. 2. If we make $I_1 = I_2$, which is customary in photometric measurements, proportion (I.) would give

$$a_1 : a_2 = r_1 : r_2,$$

or the amplitude of the one wave is to that of the other as r_1, the distance of the one eye of the observer from the source of the former, is to r_2, the distance of his other eye from the source of the latter; these distances are determined by measurement.

4. The average kinetic energy of any particle of mass m of the spherical wave

$$\phi = \frac{a}{r}\left[\left(\frac{3}{r^2} - \frac{4\pi^2}{\lambda^2}\right)\cos\omega - \frac{6\pi}{\lambda r}\sin\omega\right]\sin\alpha\cos\alpha\cos\theta$$

is

$$\frac{m\pi^2 v^2}{\lambda^2}\,\frac{a^2}{r^2}\left(\frac{16\pi^4}{\lambda^4} + \frac{12\pi^2}{\lambda^2 r^2} + \frac{9}{r^4}\right)\sin^2\alpha\cos^2\alpha\cos^2\theta.$$

We have

$$\frac{d\phi}{dy} = \frac{a}{r}\left[\frac{2\pi v}{\lambda}\left(\frac{4\pi^2}{\lambda^2} - \frac{3}{r^2}\right)\sin\omega - \frac{12\pi^2 v}{\lambda^2 r}\cos\omega\right]\sin\alpha\cos\alpha\cos\theta,$$

hence

$$\begin{aligned}
I &= \frac{2m\pi^2 v^2}{\lambda^2 T}\,\frac{a^2}{r^2}\sin^2\alpha\cos^2\alpha\cos^2\theta\int_0^T\left[\left(\frac{4\pi^2}{\lambda^2} - \frac{3}{r^2}\right)^2\sin^2\omega\right.\\
&\qquad\left. - \frac{12\pi}{\lambda r}\left(\frac{4\pi^2}{\lambda^2} - \frac{3}{r^2}\right)\sin\omega\cos\omega + \frac{36\pi^2}{\lambda^2 r^2}\cos^2\omega\right]dt\\
&= \frac{m\pi v}{\lambda T}\,\frac{a^2}{r^2}\sin^2\alpha\cos^2\alpha\cos^2\theta\,\Bigg|\left[\frac{1}{2}\left(\frac{4\pi^2}{\lambda^2} - \frac{3}{r^2}\right)^2 + \frac{18\pi^2}{\lambda^2 r^2}\right]\omega\\
&\qquad - \left[\frac{1}{2}\left(\frac{4\pi^2}{\lambda^2} - \frac{3}{r^2}\right)^2 - \frac{18\pi^2}{\lambda^2 r^2}\right]\sin\omega\cos\omega + \frac{6\pi}{\lambda r}\left(\frac{4\pi^2}{\lambda^2} - \frac{3}{r^2}\right)\sin^2\omega\,\Bigg|_0^T\\
&= \frac{m\pi^2 v^2}{\lambda^2}\,\frac{a^2}{r^2}\left(\frac{16\pi^4}{\lambda^4} + \frac{12\pi^2}{\lambda^2 r^2} + \frac{9}{r^4}\right)\sin^2\alpha\cos^2\alpha\cos^2\theta.
\end{aligned}$$

The following relation would thus hold between the intensities of two such spherical waves of one and the same wave-length (and phase):

$$I_1 : I_2 = a_1^2 r_2^6(16\pi^4 r_1^4 + 12\pi^2\lambda^2 r_1^2 + 9\lambda^4) : a_2^2 r_1^6(16\pi^4 r_2^4 + 12\pi^2\lambda^2 r_2^2 + 9\lambda^4). \ \ldots\text{(II.)}$$

Cor. 1. If we make $I_1 = I_2$, as in photometric measurements, this proportion would give the following relation between the amplitudes of the two waves:

$$a_1^2 : a_2^2 = r_1^6(16\pi^4 r_2^4 + 12\pi^2\lambda^2 r_2^2 + 9\lambda^4) : r_2^6(16\pi^4 r_1^4 + 12\pi^2\lambda^2 r_1^2 + 9\lambda^4).$$

Cor. 2. If the wave-length λ is small in comparison to the distances r_1 and r_2 of the given sources from the point of observation—this would be the case with light-waves—the last proportion would assume the familiar form:

$$a_1 : a_2 = r_1\, r_2 \quad (I_1 = I_2).$$

5. Show that the functions

$$\phi_1 = \frac{a}{r}\sin\frac{2\pi}{\lambda}vt\cos\frac{2\pi}{\lambda}r$$

and

$$\phi_2 = \frac{a}{r}\sin\frac{2\pi}{\lambda}vt\sin\frac{2\pi}{\lambda}r$$

represent stationary spherical waves.

6. Show that the coefficients P_k of formula (34) assume the following forms:

(*a*) For $n=\lambda=1, \quad \mu=\nu=0,$

$$P_0=-\cos a, \quad P_1=\cos a.$$

(*b*) For $n=\lambda=2, \quad \mu=\nu=0,$

$$P_0=-(1-3\cos^2 a), \quad P_1=1-3\cos^2 a, \quad P_2=\cos^2 a.$$

and (*c*) For $n=2, \quad \lambda=\mu=1, \quad \nu=0,$

$$P_0=3\sin a\cos a\cos\theta=\frac{3}{2}\sin 2a\cos\theta,$$

$$P_1=-\frac{3}{2}\sin 2a\cos\theta, \quad P_2=\frac{1}{2}\sin 2a\cos\theta.$$

7. The particular integral $\frac{d^3\phi}{dx^3}$ has the form

$$\frac{d^3\phi}{dx^3}=\frac{\cos^3 a}{r}\frac{d^3 f}{dr^3}+3(1-2\cos^2 a)\cos a\frac{1}{r^2}\frac{d^2 f}{dr^2}$$

$$-3(3-5\cos^2 a)\cos a\frac{1}{r^3}\frac{df}{dr}+3(3-5\cos^2 a)\cos a\frac{1}{r^4}f.$$

By formula (33) we have

$$\frac{d^3\phi}{dx^3}=\frac{d}{dx}\left(\frac{1}{r^2}\frac{df}{dr}-\frac{f}{r^3}\right)+x^2\frac{d}{dx}\left(\frac{1}{r^3}\frac{d^2f}{dr^2}-\frac{3}{r^4}\frac{df}{dr}+\frac{3f}{r^5}\right)+2x\left(\frac{1}{r^3}\frac{d^2f}{dr^2}-\frac{3}{r^4}\frac{df}{dr}+\frac{3f}{r^5}\right)$$

$$=\frac{x}{r}\frac{d}{dr}\left(\frac{1}{r^2}\frac{df}{dr}-\frac{f}{r^3}\right)+\frac{x^3}{r}\frac{d}{dr}\left(\frac{1}{r^3}\frac{d^2f}{dr^2}-\frac{3}{r^4}\frac{df}{dr}+\frac{3f}{r^5}\right)+2x\left(\frac{1}{r^3}\frac{d^2f}{dr^2}-\frac{3}{r^4}\frac{df}{dr}+\frac{3f}{r^5}\right)$$

$$=\frac{3x}{r}\left(\frac{1}{r^2}\frac{d^2f}{dr^2}-\frac{3}{r^3}\frac{df}{dr}+\frac{3f}{r^4}\right)+\frac{x^3}{r}\left(\frac{1}{r^3}\frac{d^3f}{dr^3}-\frac{6}{r^4}\frac{d^2f}{dr^2}+\frac{15}{r^5}\frac{df}{dr}-\frac{15}{r^6}f\right)$$

$$=3\cos a\left(\frac{1}{r^2}\frac{d^2f}{dr^2}-\frac{3}{r^3}\frac{df}{dr}+\frac{3f}{r^4}\right)+\cos^3 a\left(\frac{1}{r}\frac{d^3f}{dr^3}-\frac{6}{r^2}\frac{d^2f}{dr^2}+\frac{15}{r^3}\frac{df}{dr}-\frac{15}{r^4}f\right).$$

The coefficients P_k of formula (34) thus assume here the form

$$P_0=3(3-5\cos^2 a)\cos a, \quad P_1=-3(3-5\cos^2 a)\cos a,$$

$$P_2=3(1-2\cos^2 a)\cos a, \quad P_3=\cos^3 a.$$

8. Show that the coefficients P_k of formula (34) assume the following forms:

(*a*) For $n=3, \quad \lambda=2, \quad \mu=1, \quad \nu=0.$

$$P_0=3(1-5\cos^2 a)\sin a\cos\theta,$$

$$P_1=-3(1-5\cos^2 a)\sin a\cos\theta,$$

$$P_2=(1-6\cos^2 a)\sin a\cos\theta,$$

$$P_3=\sin a\cos^2 a\cos\theta=\frac{1}{2}\sin a\sin 2a\cos\theta;$$

and (*b*) For $n=3, \quad \lambda=\mu=\nu=1,$

$$P_0=-15\sin^2 a\cos a\sin\theta\cos\theta=-\frac{15}{4}\sin a\sin 2a\sin 2\theta,$$

$$P_1=\frac{15}{4}\sin a\sin 2a\sin 2\theta,$$

$$P_2=-\frac{3}{2}\sin a\sin 2a\sin 2\theta,$$

$$P_3=-\frac{1}{4}\sin a\sin 2a\sin 2\theta.$$

CHAPTER II.

SPHERICAL ELECTROMAGNETIC WAVES: PRIMARY AND SECONDARY WAVES; PECULIAR PROPERTIES OF SECONDARY WAVE; THE ROENTGEN RAYS.

Wave-Functions and Electromagnetic Wave-Functions.—In the preceding chapter we have sought solutions of the equation of wave-motion $\frac{d^2\phi}{dt^2}=v^2\nabla^2\phi$; not only these but all particular solutions of this equation are particular integrals of any or every one of our differential equations (16 and 17, I)* for electromagnetic disturbances in homogeneous dielectrics; but it does not necessarily follow that arbitrarily chosen particular integrals of the former will be particular integrals of our *systems* of equations ((16), I) and ((17), I), that is, that the same will represent an electromagnetic wave, for not only do certain relations hold between the components of the electric moment and others between those of the magnetic moment, but the latter are also always related to the former; and we have, in fact, made use of those very relations to obtain our fundamental equations in the given familiar form; those relations were

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=0 \quad \text{..............................(1)}$$

and

$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}=0 \quad \text{..............................(2)}$$

(cf. p. 10), or there is no accumulation of electricity or magnetism in the given dielectric.

A more convenient System of Differential Equations for Electromagnetic Waves.—It will now be found desirable to have a more convenient form of expression for the electric and magnetic moments than the above given one, the differential equations (16 and 17, I)

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* In referring to formulae of other chapters, we shall insert the chapter directly after the formula in question.

with the six variables X, Y, Z, a, b, c and the conditional relations (1) and (2) between the same; such would be a single system of differential equations with three variables and one conditional relation between those variables (potentials), from which the electric and magnetic moments could readily be deduced (see below). For this purpose we first replace the three variables X, Y, Z by four new ones, U, V, W, and ψ, which shall be determined as functions of the former by the four equations

$$\left.\begin{aligned}\frac{X}{D}&=\frac{dW}{dy}-\frac{dV}{dz}-\frac{1}{4\pi}\frac{d\psi}{dx}\\ \frac{Y}{D}&=\frac{dU}{dz}-\frac{dW}{dx}-\frac{1}{4\pi}\frac{d\psi}{dy}\\ \frac{Z}{D}&=\frac{dV}{dx}-\frac{dU}{dy}-\frac{1}{4\pi}\frac{d\psi}{dz}\end{aligned}\right\}, \qquad\text{(3)}$$

and

$$\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}=0. \qquad\text{(4)}$$

As we are replacing here three variables by four, we can evidently assume any given relation, as (4), between the latter; this is, in fact, necessary, if the new functions shall be determined uniquely.

Differentiate, next, equations (3), the first with regard to x, the second to y, and the third to z, add, and we have

$$\frac{d}{dx}\left(\frac{X}{D}\right)+\frac{d}{dy}\left(\frac{Y}{D}\right)+\frac{d}{dz}\left(\frac{Z}{D}\right)=-\frac{1}{4\pi}\nabla^2\psi,$$

or, since D is constant,

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=-\frac{D}{4\pi}\nabla^2\psi;$$

hence, by formula (1), $\nabla^2\psi=0.$

It thus follows from the well-known theorem from the theory of the potential that ψ also vanishes, namely: "If $\nabla^2\psi=0$ at every point of any region, and ψ vanishes at infinite distance—this is, of course, assumed in all such physical problems, ψ itself then vanishes at every point of that region."

Formulae (3) thus reduce to

$$\left.\begin{aligned}\frac{X}{D}&=\frac{dW}{dy}-\frac{dV}{dz}\\ \frac{Y}{D}&=\frac{dU}{dz}-\frac{dW}{dx}\\ \frac{Z}{D}&=\frac{dV}{dx}-\frac{dU}{dy}\end{aligned}\right\}, \qquad\text{(5)}$$

The electric component moments, X, Y, Z, are replaced here by the three new variables, U, V, W, and the conditional relation (1) between the former is thereby fulfilled, being replaced by a similar relation (4) between the latter.

We next replace X, Y, Z in formulae (13, I) by their values (5) in U, V, W, and we have

$$\frac{1}{v_0}\frac{da}{dt}=\frac{d^2V}{dxdy}-\frac{d^2U}{dy^2}-\frac{d^2U}{dz^2}+\frac{d^2W}{dxdz}=\frac{d}{dx}\left(\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}\right)-\nabla^2U,$$

$$\begin{array}{ll}\text{or, by formula (4),} & \left.\begin{array}{l}\dfrac{1}{v_0}\dfrac{da}{dt}=-\nabla^2U\\[2ex] \dfrac{1}{v_0}\dfrac{db}{dt}=-\nabla^2V\\[2ex] \dfrac{1}{v_0}\dfrac{dc}{dt}=-\nabla^2W\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(6)\\ \text{and similarly} & \end{array}$$

Since the electric moments X, Y, Z are particular integrals of the equation of wave-motion (27, I) (cf. also formulae (16, I)), whereby, however, condition (1) must always be fulfilled, we shall assume that U, V, W are also particular integrals of the equation of wave-motion or so-called "wave-functions," which shall satisfy condition (4); the moments X, Y, Z remain thereby wave-functions, since the derivatives of any wave-function are also particular integrals of the equation of wave-motion itself (cf. Chapter I.). Formulae (6) can then be written

$$\frac{1}{v_0}\frac{da}{dt}=-\frac{1}{v^2}\frac{d^2U}{dt^2},$$

with similar equations for b and c, or integrated,

$$a=-\frac{v_0}{v^2}\frac{dU}{dt}+f(x,y,z),$$

with similar equations for b and c, where $f(x, y, z)$ denotes the initial magnetic component moment along the x-axis at given point (x, y, z). $f(x, y, z)\gtrless 0$[*] would denote that the given medium contained a certain quantity of magnetism that remained constant during the passage of the given waves; since any such function would evidently have no effect on the oscillatory behaviour of the medium (see p. 10), we can therefore put $f(x, y, z)=0$; and we have

$$\begin{array}{ll} & \left.\begin{array}{l}a=-\dfrac{v_0}{v^2}\dfrac{dU}{dt}\\[2ex] b=-\dfrac{v_0}{v^2}\dfrac{dV}{dt}\\[2ex] c=-\dfrac{v_0}{v^2}\dfrac{dW}{dt}\end{array}\right\}.\ \ldots\ldots\ldots\ldots\ldots\ldots(7)\\ \text{and similarly} & \end{array}$$

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[*] Physically speaking, $f(x, y, z)\gtrless 0$ would indicate the presence of foreign bodies, as permanent magnets, in the given dielectric, which has been assumed above to be homogeneous.

To determine the component-moments of the magnetic wave that accompanied any given electric wave, we should, therefore, have to find U, V, W as functions of X, Y, Z by formulae (5), and then replace these auxiliary functions by those values in formulae (7). It is, however, customary to assume U, V, W as given, and to determine X, Y, Z as functions of the former.

Let the following problems serve as illustrations of electromagnetic spherical waves:

Problem 1. Let U, V, W be given by the wave-functions

$$U=0,\quad V=-\frac{d\phi}{dz},\quad W=\frac{d\phi}{dy},^{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where ϕ shall denote any purely spherical wave-function (cf. p. 17). These values evidently satisfy the given condition (4).

We replace U, V, W by these values (8) in formulae (5) and (7), and we have

$$\frac{X}{D}=\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2},\quad \frac{Y}{D}=-\frac{d^2\phi}{dxdy},\quad \frac{Z}{D}=-\frac{d^2\phi}{dxdz},$$

and

$$a=0,\quad b=\frac{v_0}{v^2}\frac{d^2\phi}{dtdz},\quad c=-\frac{v_0}{v^2}\frac{d^2\phi}{dtdy}$$

(cf. also Ex. 1 at end of chapter).

Since we are assuming that ϕ is a purely spherical function, that is, a function of r and t only, the following relations will hold between its derivatives:

$$\frac{d\phi}{dy}=\frac{d\phi}{dr}\frac{y}{r},\quad \frac{d\phi}{dz}=\frac{d\phi}{dr}\frac{z}{r},$$

$$\frac{d^2\phi}{dy^2}=\frac{1}{r}\frac{d\phi}{dr}+y\frac{d}{dy}\left(\frac{1}{r}\frac{d\phi}{dr}\right)=\frac{1}{r}\frac{d\phi}{dr}+y\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\frac{y}{r},$$

similarly

$$\frac{d^2\phi}{dz^2}=\frac{1}{r}\frac{d\phi}{dr}+z\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\frac{z}{r},$$

$$\frac{d^2\phi}{dxdy}=y\frac{d}{dx}\left(\frac{1}{r}\frac{d\phi}{dr}\right)=y\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\frac{x}{r},$$

and similarly

$$\frac{d^2\phi}{dxdz}=z\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\frac{x}{r};$$

by which the above formulae for the electric and magnetic moments can be written

$$\left.\begin{aligned}\frac{X}{D}&=\frac{2}{r}\frac{d\phi}{dr}+\frac{y^2+z^2}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\\ \frac{Y}{D}&=-\frac{xy}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\\ \frac{Z}{D}&=-\frac{xz}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\end{aligned}\right\},\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

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* H. von Helmholtz, *Vorlesungen über die elektromagnetische Theorie des Lichts*, §§ 36 and 37, pp. 125-130.

and
$$\left.\begin{aligned} a&=0\\ b&=\frac{v_0}{v^2}\frac{z}{r}\frac{d^2\phi}{dtdr}\\ c&=-\frac{v_0}{v^2}\frac{y}{r}\frac{d^2\phi}{dtdr}\end{aligned}\right\}\quad\text{.........(10)}$$

Electric and Magnetic Oscillations at ⊥ to each other.—The analytic condition that two moments, forces or vectors, f and h, stand at right angles to each other is

$$\cos(f,\ h)=\cos(f,\ x)\cos(h,\ x)+\cos(f,\ y)\cos(h,\ y)+\cos(f,\ z)\cos(h,\ z)=0,$$

or, if we replace these cosines by the quotients $\frac{f_1}{f}$, $\frac{f_2}{f}$, $\frac{f_3}{f}$ and $\frac{h_1}{h}$, $\frac{h_2}{h}$, $\frac{h_3}{h}$, where f_1, f_2, f_3 and h_1, h_2, h_3 denote the components of f and h respectively along the x, y, z axes respectively,

$$\cos(f,\ h)=\frac{f_1}{f}\frac{h_1}{h}+\frac{f_2}{f}\frac{h_2}{h}+\frac{f_3}{f}\frac{h_3}{h}=0,$$

or
$$f_1h_1+f_2h_2+f_3h_3=0.\quad\text{.........(11)}$$

The following relation evidently holds between the values (9) and (10) for the component electric and magnetic moments X, Y, Z and a, b, c:

$$Xa+Yb+Zc=0,$$

which interpreted according to formula (11) expresses the familiar law: the electric and magnetic oscillations take place at right angles to each other.

Magnetic Oscillations at ⊥ to Direction of Propagation.—Let us first examine the magnetic oscillations (10); we evidently have not only

$$ax+by+cz=0,$$

or, by formula (11), the magnetic oscillations take place at right angles to their radius vector, but, since $a=0$, also

$$by+cz=0,$$

or they take place along the parallel circles intercepted on the spheres with centres at $r=0$ by the planes parallel the yz-plane. Since the resultant magnetic moment is given by

$$\sqrt{b^2+c^2}=\frac{v_0}{v^2}\frac{\sqrt{y^2+z^2}}{r}\frac{d^2\phi}{dtdr}=\frac{v_0}{v^2}\sin\alpha\frac{d^2\phi}{dtdr},$$

where α denotes the latitude of any circle with regard to the x-axis as pole, it follows that the amplitude of the given oscillations will vary as the *sin* α, as we pass along any meridian from either pole, where it vanishes, towards the equator, where it reaches a maximum. The magnetic oscillations could thus be represented mechanically by the periodic oscillations of spherical shells about their (x)-axis.

The Electric Wave as Resultant of two Waves.—As above, we can conceive the electric wave represented by formulae (9) as the resultant of the two waves

$$\frac{X'}{D}=\frac{2}{r}\frac{d\phi}{dr},\quad Y'=Z'=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

and

$$\frac{X''}{D}=\frac{y^2+z^2}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right),\quad \frac{Y''}{D}=-\frac{xy}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right),\quad \frac{Z''}{D}=-\frac{xz}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right).\text{*}\quad(12\text{A})$$

The former oscillations evidently retain one and the same amplitude and direction of oscillation, that parallel the x-axis, over one and the same sphere with centre at $r=0$;* they could thus be represented mechanically by the periodic oscillation of spherical shells with centres at $r=0$ along that axis. The oscillations X'', Y'', Z'' take place along the meridians of the spheres with centres at $r=0$;* this follows from the two analytic relations

$$X''a+Y''b+Z''c=0,$$

or these oscillations take place at right angles to the magnetic ones, that is, to the circles intercepted on the spheres with centres at $r=0$ by the planes parallel the yz-plane, and

$$X''x+Y''y+Z''z=0,$$

or they take place along the surfaces of the spheres with centres at $r=0$. The resultant electric moment X'', Y'', Z'' is evidently

$$\sqrt{X''^2+Y''^2+Z''^2}=\frac{D}{r}\sqrt{y^4+2y^2z^2+z^4+x^2y^2+x^2z^2}\,\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)$$

$$=\frac{D}{r}\sqrt{(x^2+y^2+z^2)(x^2+y^2)}\,\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)$$

$$=D\sqrt{y^2+z^2}\,\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right).$$

The Electric Waves at great distance from Source.—For large values of r the moments X', Y', Z' may be neglected in comparison to the moments X'', Y'', Z'', and thus be rejected; that is, the periodic oscillations parallel to the x-axis are gradually lost sight of, when compared with those along the meridians, as the given disturbance recedes from its source. This becomes evident when we replace the purely spherical function ϕ by its value $\frac{1}{r}f(r \pm vt)$ (cf. formula (31, I)) in formulae (9) and also $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ by the direction cosines α, β, γ. We have then

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* Cf. v. Helmholtz: *Vorlesungen über die elektromagnetische Theorie des Lichts, p. 128.*

$$\left.\begin{aligned}\frac{X}{D}&=\frac{2}{r}\left(\frac{1}{r}\frac{df}{dr}-\frac{1}{r^2}f\right)+(\beta^2+\gamma^2)\left[\frac{d^2}{dr^2}\left(\frac{1}{r}f\right)-\frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}f\right)\right]\\ &=\frac{2}{r^2}\frac{df}{dr}-\frac{2}{r^3}f+(\beta^2+\gamma^2)\left[\frac{1}{r}\frac{d^2f}{dr^2}-\frac{3}{r^2}\frac{df}{dr}+\frac{3}{r^3}f\right]\\ \frac{Y}{D}&=-\alpha\beta\left[\frac{1}{r}\frac{d^2f}{dr^2}-\frac{3}{r^2}\frac{df}{dr}+\frac{3}{r^3}f\right]\\ \frac{Z}{D}&=-\alpha\gamma\left[\frac{1}{r}\frac{d^2f}{dr^2}-\frac{3}{r^2}\frac{df}{dr}+\frac{3}{r^3}f\right]\end{aligned}\right\},\ldots\ldots(13)$$

which for large values of r could evidently be replaced by

$$\frac{X}{D}=\frac{\beta^2+\gamma^2}{r}\frac{d^2f}{dr^2},\quad \frac{Y}{D}=-\frac{\alpha\beta}{r}\frac{d^2f}{dr^2},\quad \frac{Z}{D}=-\frac{\alpha\gamma}{r}\frac{d^2f}{dr^2}. \ \ldots\ldots\ldots(14)$$

The Electric Waves near Source.—Near the source of the disturbance the following formulae would evidently be approximate:

$$\frac{X}{D}=\frac{1-3\alpha^2}{r^3}f,\quad \frac{Y}{D}=-\frac{3\alpha\beta}{r^3}f,\quad \frac{Z}{D}=-\frac{3\alpha\gamma}{r^3}f.\ldots\ldots\ldots\ldots(15)$$

Observe that the electric moment X is here a function of both the moments X' and X'', and is not, as might be supposed, given alone by the latter.

The Magnetic Wave.—On replacing ϕ by its value $\frac{1}{r}f$ and $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ by α, β, γ in formulae (10) for the magnetic moments, we have

$$a=0,$$

$$b=\frac{v_0}{v^2}\gamma\left[\frac{1}{r}\frac{d^2f}{dtdr}-\frac{1}{r^2}\frac{df}{dt}\right],$$

$$c=-\frac{v_0}{v^2}\beta\left[\frac{1}{r}\frac{d^2f}{dtdr}-\frac{1}{r^2}\frac{df}{dt}\right].$$

For large values of r the resultant magnetic moment is thus approximately

$$\sqrt{a^2+b^2+c^2}=\pm\frac{v_0}{v^2}\frac{\sqrt{\beta^2+\gamma^2}}{r}\frac{d^2f}{dtdr};$$

we observe that the electric and magnetic moments are here of the same order of magnitude in $\left(\frac{1}{r}\right)$.

Near the source of the disturbance the resultant magnetic moment would be approximately

$$\sqrt{a^2+b^2+c^2} = \pm\frac{v_0}{v^2}\frac{\sqrt{\beta^2+\gamma^2}}{r^2}\frac{df}{dt},$$

which is of a lower order of magnitude in $\frac{1}{r}$ than the expression for the resultant electric moment. It would thus follow that, as we approached the source of the disturbance, the electric oscillations would become more perceptible than the magnetic ones.

Linearly Polarized Light.—We have seen above that for large values of r the electric moment X' ($Y'=Z'=0$) vanishes in comparison to the electric moment X'', Y'', Z''; as r increases, every element of wave-front approaches more and more that of a plane, until for infinitely large values of r X' vanishes entirely, and the wave-front itself becomes plane. The oscillations X'', Y'', Z'' take place, moreover, at right angles not only to the magnetic ones accompanying them, but also to their direction of propagation, and are propagated according to the law that their amplitudes decrease in magnitude inversely as the distance from their source.* At great distances from their source these waves may, therefore, be identified with those of linearly polarized light, or, conversely, linearly polarized light may be represented by the given system of equations (12A) or (14).

Primary and Secondary Waves.—The electric wave represented by the electric moment X' differs materially from all ordinary electric waves; it appears only in the neighbourhood of its source, within which region its amplitude decreases inversely as the square and third power of the distance (r), whereas the oscillations themselves take place, in general, neither at right angles to nor along their direction of propagation. As I shall henceforth draw a sharp distinction between this kind of wave-motion and the ordinary one, let us term the ordinary electromagnetic waves, or those whose oscillations take place at right angles to their direction of propagation, and whose amplitudes decrease inversely as the distance "primary" and all other electromagnetic waves, or those whose oscillations do not take place at right angles to their direction of propagation, and whose amplitudes vary inversely as the square and higher powers of the distance "secondary." These two definitions of secondary waves, "those whose oscillations do not take place at right angles to their direction of propagation," and "those whose amplitudes vary inversely as the square and higher powers of the distance," will be found to be identical.

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* We are rejecting here the terms in $\frac{1}{r^2}$ and $\frac{1}{r^3}$, which for large values of r approximately vanish, when compared with those in $\frac{1}{r}$.

As an illustration of the resolution of a compound wave into its primary and secondary, take that represented by formulae (13); we conceive the given wave as the resultant of two waves, the primary

$$\frac{X_1}{D}=\frac{\beta^2+\gamma^2}{r}\frac{d^2f}{dr^2},\quad \frac{Y_1}{D}=-\frac{\alpha\beta}{r}\frac{d^2f}{dr^2},\quad \frac{Z_1}{D}=-\frac{\alpha\gamma}{r}\frac{d^2f}{dr^2},$$

and the secondary

$$\frac{X_2}{D}=\frac{2-3(\beta^2+\gamma^2)}{r^2}\frac{df}{dr}-\frac{2-3(\beta^2+\gamma^2)}{r^3}f,$$

$$\frac{Y_2}{D}=\frac{3\alpha\beta}{r^2}\frac{df}{dr}-\frac{3\alpha\beta}{r^3}f,\quad \frac{Z_2}{D}=\frac{3\alpha\gamma}{r^2}\frac{df}{dr}-\frac{3\alpha\gamma}{r^3}f.$$

The primary and secondary waves are, in general, dependent on each other, or the presence of the one demands that of the other, that is, neither can exist alone by itself; this follows, since the expressions for either wave are not, in general, particular integrals of our differential equations, although their sums are such, the compound wave being represented by those sums; this is demonstrated by the given example.

Besides the above class of electromagnetic waves, a primary accompanied by a secondary wave, we have, of course, the simple electromagnetic wave or primary wave, if we may then term it such, that is not accompanied by a secondary wave. Such waves are represented by the simple or purely spherical wave-functions ϕ, and not by their derivatives. An irregular distribution of wave-intensity over any given sphere, with centre at source of disturbance, would, therefore, always indicate the presence of a secondary wave in the given electromagnetic wave and *vice versa*.

Analogy between Primary and Secondary Waves and Primary and Secondary Currents. The Roentgen Rays.—The idea of conceiving a compound electromagnetic wave as composed of two waves, a primary and a secondary one, was suggested by the somewhat analogous behaviour of the primary and secondary currents in current-electricity. As the primary and secondary currents are dependent upon each other, so are the primary and secondary waves; this dependence lies in the one case in the variation in the current-strength of the primary current, in the configuration of the two conductors or circuits and their relative position to each other, and in the other case indirectly in the similarity or relations—the appearance of the same quantities—between the analytical expressions for the two waves. Let us pursue this analogy further. The generation of a secondary current assumes, first of all, the presence of a conductor or circuit in the field for the passage of the same; analogously, the appearance of secondary waves would demand the presence of foreign bodies or media, aside from those within which the given electromagnetic disturbance is generated, in the given, otherwise homogeneous, field; the vacuum tube—the vacuum within the tube, the tube (glass) itself, etc.—employed in the generation of the Roentgen (X) rays may serve as an example of such foreign bodies or media brought into the field. The introduction of the second brass knob of the Hertzian vibrator would, in fact, constitute a field, within which such secondary oscillations might be expected to appear; but these knobs are placed so near to each other—2 to 3 millimetres apart—in the generation of the Hertzian oscillations, that it would be difficult to detect these secondary oscillations except in the neighbourhood of the vibrator (cf. Ex. 12 at end of chapter). On the other hand, could not the insertion of the vacuum tube, employed for the generation of the Roentgen rays, into the field give rise to secondary waves that could easily be detected? The integrals employed for representing given disturbances would naturally have to be supplemented by the corresponding surface conditions. The observed variation in the intensity of the (primary) vibrations emitted, due to the heating of the apparatus, sparking and radiation, would correspond to the variation in the current-strength of the primary current, and thus give rise to secondary oscillations. Henceforth I shall lay no great weight on the given analogy, which is to be regarded merely as a suggestion..

Problem 2. Let the auxiliary functions U, V, W be given by the wave-functions

$$U=-2\frac{d^2\phi}{dydz},\quad V=\frac{d^2\phi}{dxdz},\quad W=\frac{d^2\phi}{dxdy},\ \ldots\ldots\ldots\ldots\ldots(16)$$

where ϕ is a purely spherical wave-function, that is, any spherical wave-function that is a function of r and t only. These functions evidently satisfy the required condition (4).

Replace U, V, W by these values in formulae (5), and we find

$$\frac{X}{D}=\frac{d^3\phi}{dxdy^2}-\frac{d^3\phi}{dxdz^2}=\frac{d}{dx}\left(\frac{d^2\phi}{dy^2}-\frac{d^2\phi}{dz^2}\right),$$

$$\frac{Y}{D}=-2\frac{d^3\phi}{dydz^2}-\frac{d^3\phi}{dx^2dy}=-\frac{d}{dy}\left(2\frac{d^2\phi}{dz^2}+\frac{d^2\phi}{dx^2}\right),$$

$$\frac{Z}{D}=\frac{d^3\phi}{dx^2dz}+2\frac{d^3\phi}{dy^2dz}=\frac{d}{dz}\left(\frac{d^2\phi}{dz^2}+2\frac{d^2\phi}{dy^2}\right),$$

or, since ϕ is a function of r and t only,

$$\begin{aligned}\frac{X}{D}&=\frac{d}{dx}\left[\frac{y^2-z^2}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\right]=\frac{x(y^2-z^2)}{r^2}\left[\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\phi}{dr}\right)-\frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\right],\\ \frac{Y}{D}&=-\frac{d}{dy}\left[\frac{3}{r}\frac{d\phi}{dr}+\frac{x^2+2z^2}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\right]\\ &=-\frac{y}{r}\left[\frac{x^2+2z^2}{r}\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\phi}{dr}\right)+\frac{2x^2+3y^2+z^2}{r^2}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\right],\end{aligned}$$

and, similarly,

$$\frac{Z}{D}=\frac{z}{r}\left[\frac{x^2+2y^2}{r}\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\phi}{dr}\right)+\frac{2x^2+y^2+3z^2}{r^2}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\right].$$

We next replace here the purely spherical wave-function ϕ by its value $\frac{1}{r}f(r\pm vt)$ (cf. formula (31, I.)); for this purpose we build the following expressions:

$$\begin{aligned}\frac{d\phi}{dr}&=\frac{1}{r}\frac{df}{dr}-\frac{1}{r^2}f,\\ \frac{d^2\phi}{dr^2}&=\frac{1}{r}\frac{d^2f}{dr^2}-\frac{2}{r^2}\frac{df}{dr}+\frac{2}{r^3}f,\\ \frac{d^3\phi}{dr^3}&=\frac{1}{r}\frac{d^3f}{dr^3}-\frac{3}{r^2}\frac{df^2}{dr^2}+\frac{6}{r^3}\frac{df}{dr}-\frac{6}{r^4}f,\end{aligned}$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)=\frac{1}{r}\frac{d^2\phi}{dr^2}-\frac{1}{r^2}\frac{d\phi}{dr},$$

hence

$$=\frac{1}{r^2}\frac{d^2f}{dr^2}-\frac{3}{r^3}\frac{df}{dr}+\frac{3}{r^4}f,$$

and

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\phi}{dr}\right)=\frac{1}{r}\frac{d^3\phi}{dr^3}-\frac{2}{r^2}\frac{d^2\phi}{dr^2}+\frac{2}{r^3}\frac{d\phi}{dr},$$

hence

$$=\frac{1}{r^2}\frac{d^3f}{dr^3}-\frac{5}{r^3}\frac{d^2f}{dr^2}+\frac{12}{r^4}\frac{df}{dr}-\frac{12}{r^5}f;$$

by which the above formulae can be written

$$\left.\begin{aligned}\frac{X}{D}&=\frac{x(y^2-z^2)}{r^4}\left[\frac{d^3f}{dr^3}-\frac{6}{r}\frac{d^2f}{dr^2}+\frac{15}{r^2}\frac{df}{dr}-\frac{15}{r^3}f\right]\\ \frac{Y}{D}&=-\frac{y}{r^4}\left[(x^2+2z^2)\frac{d^3f}{dr^3}-\frac{3(x^2-y^2+3z^2)}{r}\frac{d^2f}{dr^2}\right.\\ &\qquad\left.+\frac{3(2x^2-3y^2+7z^2)}{r^2}\left(\frac{df}{dr}-\frac{1}{r}f\right)\right]\\ \text{and, similarly, }\frac{Z}{D}&=\frac{z}{r^4}\left[(x^2+2y^2)\frac{d^3f}{dr^3}-\frac{3(x^2+3y^2-z^2)}{r}\frac{d^2f}{dr^2}\right.\\ &\qquad\left.+\frac{3(2x^2+7y^2-3z^2)}{r^2}\left(\frac{df}{dr}-\frac{1}{r}f\right)\right]\end{aligned}\right\}\dots\dots(17)$$

The Primary and Secondary Electric Waves.—Since f and its derivatives with regard to r are evidently of the same order of magnitude in $\frac{1}{r}$, we can thus conceive the given electric wave as composed of four single waves, whose amplitudes vary as the different powers—the first four—of $\frac{1}{r}$; the secondary wave would, therefore, be here the resultant of three waves, whose amplitudes decrease inversely as the second, third, and fourth powers of the distance. If we exclude the immediate neighbourhood of the source of disturbance from the region in question, we may evidently reject the terms in $\frac{1}{r^3}$ and $\frac{1}{r^4}$ of formulae (17) in comparison to those in $\frac{1}{r^2}$, and the secondary wave would then be given approximately by the latter terms. Since the immediate neighbourhood of any source is seldom accessible to examination, we shall, in general, exclude it from the region in question; the secondary wave would then be represented approximately by the terms of the second order of magnitude in $\frac{1}{r^2}$, and we shall, henceforth, refer to it as thus defined, unless otherwise specified. For the given region the electric moments of formulae (17) can thus be written (approximately)

$$\left.\begin{aligned}
\frac{X}{D}&=\frac{X_1}{D}+\frac{X_2}{D}=\frac{\alpha(\beta^2-\gamma^2)}{r}\frac{d^3f}{dr^3}-\frac{6\alpha(\beta^2-\gamma^2)}{r^2}\frac{d^2f}{dr^2}\\
\frac{Y}{D}&=\frac{Y_1}{D}+\frac{Y_2}{D}=-\frac{\beta(\alpha^2+2\gamma^2)}{r}\frac{d^3f}{dr^3}+\frac{3\beta(\alpha^2-\beta^2+3\gamma^2)}{r^2}\frac{d^2f}{dr^2}\\
\frac{Z}{D}&=\frac{Z_1}{D}+\frac{Z_2}{D}=\frac{\gamma(\alpha^2+2\beta^2)}{r}\frac{d^3f}{dr^3}-\frac{3\gamma(\alpha^2+3\beta^2-\gamma^2)}{r^2}\frac{d^2f}{dr^2}
\end{aligned}\right\}, \ldots . (17\text{A})$$

where α, β, γ are the direction-cosines of r; X_1, Y_1, Z_1 denote the moments of the primary and X_2, Y_2, Z_2 those of the secondary electric wave.

The Magnetic Wave.—By formulae (7) and (16) the magnetic moments are

$$\left.\begin{aligned}
a&=\frac{2v_0}{v^2}\frac{d^3\phi}{dtdydz}=\frac{2v_0}{v^2}\frac{yz}{r}\frac{d^2}{dtdr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\\
b&=-\frac{v_0}{v^2}\frac{d^3\phi}{dtdxdz}=-\frac{v_0}{v^2}\frac{xz}{r}\frac{d^2}{dtdr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)\\
c&=-\frac{v_0}{v^2}\frac{d^3\phi}{dtdxdy}=-\frac{v_0}{v^2}\frac{xy}{r}\frac{d^2}{dtdr}\left(\frac{1}{r}\frac{d\phi}{dr}\right)
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

By the relations on p. 37 between the derivatives of ϕ and f, these expressions for a, b, c can be written

$$\left.\begin{aligned}
a&=\frac{2v_0}{v^2}\frac{\beta\gamma}{r}\frac{d}{dt}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3}{r^2}f\right)\\
b&=-\frac{v_0}{v^2}\frac{\alpha\gamma}{r}\frac{d}{dt}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3}{r^2}f\right)\\
c&=-\frac{v_0}{v^2}\frac{\alpha\beta}{r}\frac{d}{dt}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3}{r^2}f\right)
\end{aligned}\right\}\ldots\ldots\ldots\ldots(18\text{A})$$

We can thus conceive the given magnetic wave as composed of three single waves, whose amplitudes decrease inversely as the first, second, and third powers respectively of the distance. In the region in question, that, namely, in which formulae (17A) approximately hold, the secondary magnetic wave would, in general, be represented approximately by the terms in $\frac{1}{r^2}$ of these formulae (18A).

Regions in which the Primary Wave disappears.—From a glance at formulae (17A), it is evident that there are certain regions, in which the secondary (electric) wave alone appears* and cannot, therefore, be overlooked when compared with the primary (electric) wave, even at greater distances from the source of disturbance. These regions are characterised by the disappearance of the primary wave,* that is, they are determined by the vanishing of the coefficients of the terms of the first order of magnitude in $\frac{1}{r}$; the given particular regions, lines or surfaces, of formulae (17A) are four in number and are determined by the following sets of analytic relations:

Region 1. $\alpha(\beta^2-\gamma^2)=\beta(\alpha^2+2\gamma^2)=\gamma(\alpha^2+2\beta^2)=0.$ ………………(19)

Region 2. $\alpha(\beta^2-\gamma^2)=0,\ \beta(\alpha^2+2\gamma^2)\gtrless 0,\ \gamma(\alpha^2+2\beta^2)\gtrless 0.$ ………(20)

Region 3. $\alpha(\beta^2-\gamma^2)\gtrless 0,\ \beta(\alpha^2+2\gamma^2)=0,\ \gamma(\alpha^2+2\beta^2)\gtrless 0.$ ………(21)

Region 4. $\alpha(\beta^2-\gamma^2)\gtrless 0,\ \beta(\alpha^2+2\gamma^2)\gtrless 0,\ \gamma(\alpha^2+2\beta^2)=0.$ ………(22)

It is easy to show that the vanishing of any two of the given coefficients is identical to that of all three or to the analytic relations (19). Moreover, since formulae (17A) are symmetrical in y and z, it will be necessary to examine only the first three regions, formulae (22) which determine the fourth region being analogous to those (21) for the third.

Region 1: The Three Coordinate Axes.—The analytic relation (19) can evidently be replaced by the following:

$$\alpha=0,\ \beta=0\ \text{ and hence }\ \gamma=1;$$

$$\alpha=0,\ \alpha^2+2\gamma^2=0,\ \text{ hence }\ \gamma=0\ \text{ and }\ \beta=1;$$

and $\beta=\gamma,\ \beta=0,$ hence $\gamma=0$ and $\alpha=1,$

or the three coordinate-axes.

---

* Or, more strictly speaking, one or more of the electric component-moments of the primary wave disappear.

It thus follows that the electric moment X_1, Y_1, Z_1 vanishes along all three axes x, y, z. The electric moment X_2, Y_2, Z_2 assumes the following particular form along the same:

$$X_2 = Y_2 = Z_2 = 0 \text{ along the } x\text{-axis,}$$

$$X_2 = Z_2 = 0,\ Y_2 = -\frac{3D}{r^2}\frac{d^2f}{dr^2} \text{ along the } y\text{-axis,}$$

and
$$X_2 = Y_2 = 0,\ Z_2 = \frac{3D}{r^2}\frac{d^2f}{dr^2} \text{ along the } z\text{-axis;}$$

that is, the secondary wave disappears entirely along the x-axis, but is propagated along the y and z axes as a longitudinal wave.

Region 2: The yz and $\beta^2 = \gamma^2$ Planes.—This region evidently comprises the two regions $\alpha = 0$ or the yz-plane and $\beta^2 = \gamma^2$ or two planes passing through the origin at right angles to each other and bisecting the angles between the xy- and xz-planes.

For $\alpha = 0$, hence $\beta^2 + \gamma^2 = 1$, formulae (17A) reduce to

$$X = 0,$$

$$Y = -\frac{2\beta\gamma^2}{r}\frac{d^3f}{dr^3} - \frac{3\beta(\beta^2 - 3\gamma^2)}{r^2}\frac{d^2f}{dr^2},$$

$$Z = \frac{2\beta^2\gamma}{r}\frac{d^3f}{dr^3} - \frac{3\gamma(3\beta^2 - \gamma^2)}{r^2}\frac{d^2f}{dr^2},$$

where, for brevity, we have put $D = 1$. These values give

$$\sqrt{X_1^2 + Y_1^2 + Z_1^2} = \frac{2\beta\gamma}{r}\frac{d^3f}{dr^3},$$

and
$$\sqrt{X_2^2 + Y_2^2 + Z_2^2} = \frac{3}{r^2}\frac{d^2f}{dr^2}.$$

It thus follows that both the primary and secondary oscillations of the yz-plane take place in that plane and that the amplitude and intensity of the latter are functions only of the distance from the origin.

Formulae (18A) reduce here to

$$a = \frac{2v_0}{v^2}\frac{\beta\gamma}{r}\frac{d}{dt}\left(\frac{d^2f}{dr^2} - \frac{3}{r}\frac{df}{dr} + \frac{3}{r^2}f\right),$$

$$b = c = 0;$$

that is, the magnetic oscillations of the yz-plane take place parallel to the x-axis.

The analytic relations $\quad \alpha = 0,$

and
$$\beta(\alpha^2 + 2\gamma^2) = \gamma(\alpha^2 + 2\beta^2) \gtrless 0,$$

or, since $\alpha = 0$ and hence $\beta^2 + \gamma^2 = 1$,

$$\alpha = 0,\quad \beta^2 = \gamma^2 = \frac{1}{2}$$

correspond to a particular case of the given one. The particular region is evidently two straight lines passing through the origin at right angles to each other and the x-axis and bisecting the angles between the y and z axes.

The resultant electric and magnetic moments assume the following values along these lines:

$$\sqrt{X_1^2+Y_1^2+Z_1^2}=\frac{1}{r}\frac{d^3f}{dr^3},$$

$$\sqrt{X_2^2+Y_2^2+Z_2^2}=\frac{3}{r^2}\frac{d^2f}{dr^2},$$

and
$$\sqrt{a^2+b^2+c^2}=a=\frac{v_0}{v^2r}\frac{d}{dt}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3}{r^2}f\right).$$

For $\beta^2=\gamma^2$, hence $\alpha^2+2\beta^2=\alpha^2+2\gamma^2=1$, formulae (17A) reduce to

$$X=0,$$

$$Y=-\frac{\beta}{r}\frac{d^3f}{dr^3}+\frac{3\beta}{r^2}\frac{d^2f}{dr^2},$$

$$Z=\frac{\gamma}{r}\frac{d^3f}{dr^3}-\frac{3\gamma}{r^2}\frac{d^2f}{dr^2}, \quad D=1.$$

Formulae (18A) give here the following expression for the resultant magnetic moment:

$$\sqrt{a^2+b^2+c^2}=\frac{\sqrt{2}\,v_0}{v^2}\frac{\gamma}{r}\frac{d}{dt}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3}{r^2}f\right).$$

Both the primary and secondary oscillations of the planes $\beta^2=\gamma^2$ thus take place at right angles to the x-axis, whereas their resultant moments, and also that of the magnetic oscillations accompanying them, are entirely independent of the direction-cosine α.

Region 3. See Examples 2 and 3 at end of chapter.

Proof of General Laws.—To confirm the validity of formulae (17A) and (18A), let us next prove some of the well-known laws of electricity and magnetism for the oscillations represented by the same.

The Electric and Magnetic Oscillations take place at ⊥ to each other.—The analytic condition that electric and magnetic oscillations take place at right angles to each other is, by formula (11),

$$Xa+Yb+Zc=0.$$

That the primary (electric) and the magnetic oscillations of the given problem take place at right angles to each other, the relation must then hold

$$X_1a+Y_1b+Z_1c=0.$$

To ascertain whether this condition be fulfilled, replace here the given moments by their values (cf. formulae (17A) and (18A)), and we find

$$X_1a+Y_1b+Z_1c=C\alpha\beta\gamma[2(\beta^2-\gamma^2)+(\alpha^2+2\gamma^2)-(\alpha^2+2\beta^2)]=0,$$

where
$$C=\frac{Dv_0}{v^2r^2}\frac{d^3f}{dr^3}\frac{d}{dt}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3}{r^2}f\right).$$

Similarly, we find

$$X_2 a + Y_2 b + Z_2 c$$
$$= 4C \frac{\alpha\beta\gamma}{r} \left[-3(\beta^2 - \gamma^2) - (\alpha^2 - \beta^2 + 3\gamma^2) + (\alpha^2 + 3\beta^2 - \gamma^2) \right] = 0,$$

or the secondary (electric) and the magnetic oscillations take place at right angles to each other.

It thus follows that the electric oscillations represented by formulae (17A) and the magnetic ones accompanying the same take place at right angles to each other. Moreover, it is evident from the form of the above expressions that not only the magnetic oscillations represented by formulae (18A) but also all three component oscillations, of which the same may be conceived as composed, take place at right angles both to the primary and to the secondary (electric) oscillations. Similarly, it is easy to show that the electric oscillations of the third and fourth orders of magnitude in $\frac{1}{r}$ of formulae (17) also take place at right angles to the different magnetic ones (18A) that accompany them. Hence the general law: the (total) electric and the (total) magnetic oscillations take place at right angles to each other.

The Magnetic Oscillations take place at ⊥ to their Direction of Propagation.—A glance at formulae (18A) shows that the condition that the magnetic oscillations take place at right angles to their direction of propagation, namely, $ax + by + cz = 0$, is fulfilled.

The Primary Oscillations take place at ⊥ to their Direction of Propagation.—It is easy to show that the primary (electric) oscillations also take place at right angles to their direction of propagation. Replace X_1, Y_1, Z_1 by their values from formulae (17A) in the given condition, and we have

$$X_1 x + Y_1 y + Z_1 z$$
$$= \left[\alpha^2(\beta^2 - \gamma^2) - \beta^2(\alpha^2 + 2\gamma^2) + \gamma^2(\alpha^2 + 2\beta^2) \right] \frac{d^3 f}{dr^3} = 0. \quad (D = 1).$$

The Secondary Oscillations do not take place at ⊥ to their Direction of Propagation.—Lastly, replace X_2, Y_2, Z_2 by their values from formulae (17A) in the form* of the condition that the secondary (electric) oscillations take place at right angles to their direction of propagation, and we find

$$X_2 x + Y_2 y + Z_2 z$$
$$= \left[-6\alpha^2(\beta^2 - \gamma^2) + 3\beta^2(\alpha^2 - \beta^2 + 3\gamma^2) - 3\gamma^2(\alpha^2 + 3\beta^2 - \gamma^2) \right] \frac{1}{r} \frac{d^2 f}{dr^2} \quad (D = 1),$$
$$= 3\left[(\alpha^2 + \gamma^2)\gamma^2 - (\alpha^2 + \beta^2)\beta^2 \right] \frac{1}{r} \frac{d^2 f}{dr^2},$$

---

* The word "form" is used here as in the theory of invariants.

or, since $\alpha^2+\beta^2+\gamma^2=1$,

$$=3(\gamma^2-\beta^2)\,\frac{1}{r}\,\frac{d^2f}{dr^2}\gtrless 0\,;$$

that is, the condition, that the secondary electric oscillations take place at right angles to their direction of propagation, is, in general, not fulfilled.

Determination of the Angle of Oscillation.—Let us next determine the angle the given secondary oscillations make with their direction of propagation. We denote the given resultant moment by f_2 and its direction-cosines by λ_2, μ_2, ν_2; the angle in question, which we shall denote by (f_2, r), is then given by the familiar formula

$$\cos(f_2, r)=\lambda_2\alpha+\mu_2\beta+\nu_2\gamma, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(23)}$$

or, since λ_2, μ_2, ν_2 may be replaced by the quotients $\frac{X_2}{f_2}$, $\frac{Y_2}{f_2}$ and $\frac{Z_2}{f_2}$ respectively (cf. p. 31), where $f_2^2=X_2^2+Y_2^2+Z_2^2$,

$$\cos(f_2, r)=\frac{X_2\alpha+Y_2\beta+Z_2\gamma}{\sqrt{X_2^2+Y_2^2+Z_2^2}}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(23A)}$$

Replace here X_2, Y_2, Z_2 by their values from formulae (17A), and we have

$$\begin{aligned}\cos(f_2, r)&=\frac{-6\alpha^2(\beta^2-\gamma^2)+3\beta^2(\alpha^2-\beta^2+3\gamma^2)-3\gamma^2(\alpha^2+3\beta^2-\gamma^2)}{3\sqrt{4\alpha^2(\beta^2-\gamma^2)^2+\beta^2(\alpha^2-\beta^2+3\gamma^2)^2+\gamma^2(\alpha^2+3\beta^2-\gamma^2)^2}}\\ &=\frac{-\alpha^2\beta^2+\alpha^2\gamma^2-\beta^4+\gamma^4}{\sqrt{(\beta^2+\gamma^2)[\alpha^4+2\alpha^2(\beta^2+\gamma^2)+(\beta^2+\gamma^2)^2]}}\\ &=\frac{-(\beta^2-\gamma^2)(\alpha^2+\beta^2+\gamma^2)}{(\alpha^2+\beta^2+\gamma^2)\sqrt{\beta^2+\gamma^2}}=-\frac{\beta^2-\gamma^2}{\sqrt{\beta^2+\gamma^2}}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(24)}\end{aligned}$$

Since this expression for $\cos(f_2, r)$ assumes one and the same value along any given vector, that is, for any given ray or pencil of waves, it will be sufficient to determine its behaviour over any sphere with centre at the source of disturbance; for this purpose we replace the rectangular coordinates x, y, z by the polars r, θ, ϕ, already employed on p. 21, where θ denotes the angle, which the plane passing through the x-axis and the vector r or direction of propagation of the given wave makes with the xy-plane at the point x, $y=z=0$, and ϕ* the angle between the vector r and the x-axis (cf. figure 4, p. 22).

The following analytic relations hold between these coordinates:

$$\left.\begin{array}{lll} x=r\cos\phi, & & \alpha=\cos\phi \\ y=r\sin\phi\cos\theta, & \text{hence} & \beta=\sin\phi\cos\theta \\ z=r\sin\phi\sin\theta, & & \gamma=\sin\phi\sin\theta \end{array}\right\}. \quad\ldots\ldots\ldots\ldots\text{(25)}$$

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*On page 21 we denoted this angle by α.

The general formula (23A) for the angle (f_2, r) can, therefore, also be written

$$\cos(f_2, r) = \frac{X_2 \cos\phi + Y_2 \sin\phi \cos\theta + Z_2 \sin\phi \sin\theta}{\sqrt{X_2^2 + Y_2^2 + Z_2^2}}; \quad \ldots\ldots\ldots(26)$$

and formula (24) for the given particular wave in the simpler form

$$\cos(f_2, r) = -\sin\phi(\cos^2\theta - \sin^2\theta) = -\sin\phi\cos 2\theta. \ldots\ldots\ldots(26\text{A})$$

Regions in which the Secondary Oscillations are Longitudinal.—The given oscillations are longitudinal, when $\cos(f_2, r) = \pm 1$, that is, they are longitudinal in the regions determined by the equation

$$\pm 1 = \mp \sin\phi\cos 2\theta, \quad \text{or} \quad \sin\phi\cos 2\theta = \pm 1.$$

The region *sin φ cos 2θ = 1*: this region evidently comprises the regions

$$\sin\phi = 1, \quad \cos 2\theta = 1,$$

hence $\phi = \frac{\pi}{2}, \quad \theta = 0$ and π, or the y-axis,

and $\sin\phi = -1, \quad \cos 2\theta = -1,$

hence $\phi = \frac{3\pi}{2}, \quad \theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, or the $-z$-axis.

The region *sin φ cos 2θ = −1*: this region comprises the regions

$$\sin\phi = 1, \quad \cos 2\theta = -1,$$

hence $\phi = \frac{\pi}{2}, \quad \theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, or the z-axis,

and $\sin\phi = -1, \quad \cos 2\theta = 1,$

hence $\phi = \frac{3\pi}{2}, \quad \theta = 0$ and π, or the y-axis.

Regions in which the Secondary Oscillations are transverse.—The given oscillations are transverse, when $\cos(f_2, r) = 0$, that is, they are transverse in the regions determined by the equation

$$\sin\phi\cos 2\theta = 0,$$

or $\sin\phi = 0,$

hence $\phi = 0$ and π, or the x-axis.

and $\cos 2\theta = 0,$

hence $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$,

or two planes passing through the x-axis at right angles to each other, and bisecting the four quadrants (y, z), $(z, -y)$, $(-y, -z)$ and $(-z, y)$.

We have now seen on p. 39 that the given secondary oscillations vanish along the x-axis, and on p. 41 that they take place at right angles to that axis throughout the planes $\beta^2 = \gamma^2$ or the planes $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$, throughout which they have just been found to be transverse; their direction of oscillation is thus thereby determined uniquely, as indicated in figure 5 below. The resultant moment of these transverse oscillations is

$$\sqrt{Y_2^2+Z_2^2}=\frac{3\sqrt{\beta^2+\gamma^2}}{r^2}\frac{d^2f}{dr^2}$$

(cf. formulae on p. 41), which by formulae (25) can be written

$$\sqrt{Y_2^2+Z_2^2}=\frac{3\sin\phi}{r^2}\frac{d^2f}{dr^2};$$

that is, their amplitude increases according to *sin* ϕ, as we recede from the x-axis, where it vanishes, along any circle, intercepted on the planes $\beta^2=\gamma^2$ by any sphere with centre at source of disturbance, towards the yz-plane, where it reaches a maximum, as indicated in figure 5.

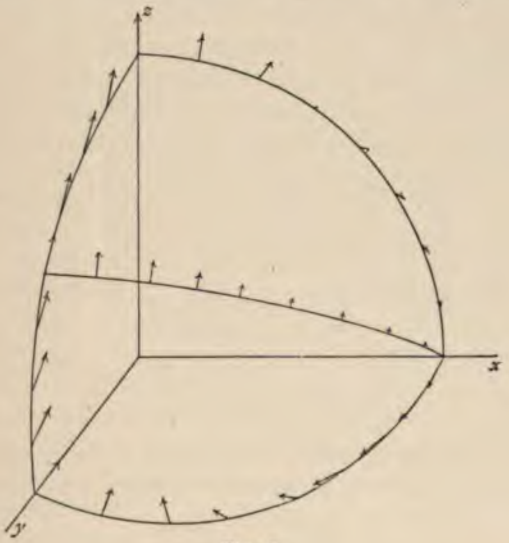

FIG. 5.

The Secondary Oscillations of the yz-Plane; Rotation of their Direction of Oscillation through 90°.—We have seen on p. 40 that the oscillations of the yz-plane take place in that plane, and that their amplitude and hence intensity remain constant for any given radius-vector. Along the lines (vectors) of intersection of this plane and those of transverse oscillation ($\beta^2=\gamma^2$), the given secondary oscillations will thus take place in the yz-plane and at right angles to their direction of propagation; that is, their direction of oscillation is determined uniquely along those lines of intersection or vectors. Moreover, it follows from the expressions for X_2, Y_2, Z_2 along the y- and z-axes (cf. p. 39), as we recede from the y-axis along the circle, intercepted on the

yz-plane by any sphere with centre at origin, towards the z-axis, that the given secondary oscillations are turned through an angle of 90° in the yz-plane or 180° with regard to their radius-vector, their amplitude and intensity remaining constant (cf. the above figure). Formula (26A) reduces here to

$$\cos(f_2, r) = -\cos 2\theta,$$

hence $$(f_2, r) = 180° - 2\theta;$$

that is, the angle (f_2, r) varies as $180° - 2\theta$.

The Secondary Oscillations of the xz-Plane; Rotation of their Direction of Oscillation through 180°. As we recede from the x-axis along the circle, intercepted on the xz-plane by any sphere with centre at origin, towards the z-axis, the given secondary oscillations are turned through an angle of 180° in the xz-plane or 90° with regard to their radius-vector, whereas their amplitude increases as $\sin\phi$, from 0 at the x-axis to a maximum at the z-axis, as indicated in the above figure. This follows from the particular form assumed by formulae (17A) and (26A) in the given plane. Formulae (17A) reduce here to

$$X_2 = \frac{6\alpha\gamma^2}{r^2}\frac{d^2f}{dr^2}, \quad Y_2 = 0, \quad Z_2 = -\frac{3\gamma(\alpha^2-\gamma^2)}{r^2}\frac{d^2f}{dr^2}, \quad (D=1),$$

which by formulae (25) can be written

$$X_2 = \frac{3\sin\phi\sin 2\phi}{r^2}\frac{d^2f}{dr^2}, \quad Y_2 = 0, \quad Z_2 = -\frac{3\sin\phi\cos 2\phi}{r^2}\frac{d^2f}{dr^2};$$

hence the resultant moment is

$$\sqrt{X_2^2 + Z_2^2} = \frac{3\gamma}{r^2}\frac{d^2f}{dr^2} = \frac{3\sin\phi}{r^2}\frac{d^2f}{dr^2},$$

or the given oscillations take place in the xz-plane, their amplitude varying as $\sin\phi$. Formula (26A) assumes here the particular form

$$\cos(f_2, r) = \sin\phi,$$

or the given oscillations are turned through an angle of 90° with regard to their radius-vector, as we pass from the z- to the x-axis; that is, they either retain one and the same direction in space or they are turned through an angle of 180° in the xz-plane. To determine the direction of oscillation in question, we seek the values of the moments X_2 and Z_2 at some intermediate point of the given arc, for example, at the point $x = z$ or $\phi = 45°$. The above formulae become here

$$X_2 = \frac{3}{\sqrt{2}\,r^2}\frac{d^2f}{dr^2}, \quad Y_2 = Z_2 = 0,$$

from which it is evident that the oscillations in question are turned through an angle of 180° in the xz-plane, as we pass from the z- to the x-axis.

The Secondary Oscillations of the xy-Plane.—Since the given secondary oscillations are symmetrical with regard to the xy- and xz-planes, except in sign (cf. formulae (17A)), the oscillations throughout the former plane will behave similarly to those of the latter, which we have just examined; namely, as we recede from the x- towards the y-axis along the circle, intercepted on the xy-plane by any sphere with centre at origin, the given oscillations are turned through an angle of 180° in the xy-plane or 90° with regard to their radius-vector, whereas their amplitude increases as $\sin\phi$, from 0 at the x-axis to a maximum at the y-axis; these oscillations are also indicated in the above figure.

The Primary Wave.—The primary wave of the given problem differs only immaterially from that of the preceding problem. We identify it, together with the magnetic wave accompanying the same, as the linearly polarized electromagnetic or light-wave, whose oscillations, both electric and magnetic, take place at right angles not only to each other but to their direction of propagation; it is, in fact, only another type—another distribution of the oscillations with regard to direction of oscillation and to amplitude over any given sphere—of the electromagnetic waves, with which we are already familiar.

The Secondary Wave.—The secondary wave of the given problem differs from that of problem 1 chiefly in that its direction of oscillation does not remain one and the same at all points—parallel the x-axis—but varies from point to point; this demonstrates that other secondary waves than those (the secondary Hertzian), whose oscillations retain one and the same direction of oscillation throughout the given region, are consistent with our differential equations. The given particular law of variation of the direction of oscillation is, of course, only one of the many possible laws (cf. also p. 63). A peculiarity of the given secondary (electric) wave, to which we may call attention, is that it is propagated along the y- and z-axes as a longitudinal wave, unaccompanied by either a primary (electric) or magnetic wave (cf. Ex. 12 at end of chapter).

Problem 3. Let U, V, W be given by the wave-functions,

$$\left.\begin{aligned} U&=\frac{d\phi_3}{dy}-\frac{d\phi_2}{dz} \\ V&=\frac{d\phi_1}{dz}-\frac{d\phi_3}{dx} \\ W&=\frac{d\phi_2}{dx}-\frac{d\phi_1}{dy} \end{aligned}\right\}, \qquad\ldots\ldots\ldots\ldots(27)$$

where ϕ_1, ϕ_2, ϕ_3 denote purely spherical wave-functions. These values evidently satisfy the required condition (4).

Since the ϕ's are functions of r and t only, we can write

$$U=\frac{y}{r}\frac{d\phi_3}{dr}-\frac{z}{r}\frac{d\phi_2}{dr},$$

with analogous expressions for V and W; or, on replacing the ϕ's by the f's (cf. formulae (31, I)) and $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ by the direction-cosines α, β, γ,

$$U=\frac{1}{r}\left(\beta\frac{df_3}{dr}-\gamma\frac{df_2}{dr}\right)-\frac{1}{r^2}(\beta f_3-\gamma f_2), \qquad (28)$$

with analogous expressions for V and W.

Let us, next, assume that the oscillations represented by the functions f_1, f_2, f_3 differ only in amplitude from one another—the more general case, where these functions shall represent oscillations of different phase, is treated in the following chapter. Moreover, let these functions be the simple *sinus* functions, namely,

$$\left.\begin{aligned} f_1&=a_1\sin\frac{2\pi}{\lambda}(vt-r)\\ f_2&=a_2\sin\frac{2\pi}{\lambda}(vt-r)\\ f_3&=a_3\sin\frac{2\pi}{\lambda}(vt-r)\end{aligned}\right\}, \qquad (29)$$

where the amplitudes a_1, a_2, a_3 shall be constant—not functions of x, y, z.

If we replace f_1, f_2, f_3 by the functions (29), formulae (28) assume the form

$$\left.\begin{aligned} U&=\frac{n}{r}(a_2\gamma-a_3\beta)\cos\omega+\frac{1}{r^2}(a_2\gamma-a_3\beta)\sin\omega\\ V&=\frac{n}{r}(a_3\alpha-a_1\gamma)\cos\omega+\frac{1}{r^2}(a_3\alpha-a_1\gamma)\sin\omega\\ W&=\frac{n}{r}(a_1\beta-a_2\alpha)\cos\omega+\frac{1}{r^2}(a_1\beta-a_2\alpha)\sin\omega\end{aligned}\right\}, \qquad (30)$$

where $$n=\frac{2\pi}{\lambda} \text{ and } \omega=n(vt-r). \qquad (31)$$

The Wave-Length λ.—The wave-length λ of the waves of light, with which we are familiar, is small in comparison to the radius-vector r, except at or near the source of the disturbance itself; this region, the examination of which offers serious difficulties (see Chapter V.), is so small—it is of the dimensions of the wave-length—that it is of little importance, if, for no other reason, for that that all empirical data concerning it are necessarily wanting. Hence, at finite distance from the given source the second terms of the above expressions for U, V, W

would, in general, be small in comparison to the first and could thus be rejected (cf. also p. 32); on the other hand, *U*, *V*, *W* would be represented by the former at or near the source of the disturbance, provided the given expressions hold there. In general we could thus put

$$\left.\begin{aligned} U&=\frac{n}{r}(a_2\gamma-a_3\beta)\cos\omega\\ V&=\frac{n}{r}(a_3\alpha-a_1\gamma)\cos\omega\\ W&=\frac{n}{r}(a_1\beta-a_2\alpha)\cos\omega \end{aligned}\right\}\text{. ....(32)}$$

We, next, replace *U*, *V*, *W* by these functions in formulae (5), and we find

$$\left.\begin{aligned} \frac{X}{D}&=\frac{n^2}{r}[a_1(\beta^2+\gamma^2)-\alpha(a_2\beta+a_3\gamma)]\sin\omega+\frac{n}{r^2}\alpha(2a_1\alpha+a_2\beta+a_3\gamma)\cos\omega\\ \frac{Y}{D}&=\frac{n^2}{r}[a_2(\alpha^2+\gamma^2)-\beta(a_1\alpha+a_3\gamma)]\sin\omega+\frac{n}{r^2}\beta(2a_2\beta+a_1\alpha+a_3\gamma)\cos\omega\\ \frac{Z}{D}&=\frac{n^2}{r}[a_3(\alpha^2+\beta^2)-\gamma(a_1\alpha+a_2\beta)]\sin\omega+\frac{n}{r^2}\gamma(2a_3\gamma+a_1\alpha+a_2\beta)\cos\omega \end{aligned}\right\}\text{.(33)}$$

For similar reasons to those just mentioned, the second terms of these expressions would, in general, be small in comparison to the first, and could thus be rejected. In those regions, however, where the coefficients of the first terms vanish (see below), the electric component-moments would be represented by the terms of the next higher order in $\frac{1}{r}$ (cf. also p. 39, and Ex. 12 at end of chapter); but these would not be the ones given here, since we have already rejected terms of that order in the above development.

Regions in which Primary Wave disappears.—The vanishing of the given coefficients signifies that certain analytic relations hold between the amplitudes a_1, a_2, a_3. and the direction-cosines α, β, γ. Since the amplitudes are entirely arbitrary, but given, and the direction-cosines variable, such relations determine given regions (cf. also p. 39). The regions in question are evidently determined by the four sets of analytic relations

$$\left.\begin{aligned} a_1(\beta^2+\gamma^2)-\alpha(a_2\beta+a_3\gamma)&=0\\ a_2(\alpha^2+\gamma^2)-\beta(a_1\alpha+a_3\gamma)&=0\\ a_3(\alpha^2+\beta^2)-\gamma(a_1\alpha+a_2\beta)&=0 \end{aligned}\right\}\text{..................(34)}$$

$$\left.\begin{gathered} a_1(\beta^2+\gamma^2)-\alpha(a_2\beta+a_3\gamma)=0\\ 0\gtrless[a_2(\alpha^2+\gamma^2)-\beta(a_1\alpha+a_3\gamma)]\gtreqqless[a_3(\alpha^2+\beta^2)-\gamma(a_1\alpha+a_2\beta)]\gtrless 0 \end{gathered}\right\}\text{,..(35)}$$

and two analogous sets, where first the second and then the third relation of the first set alone holds (cf. also formulae (19)-(22)). We observe, as on p. 39, that if any two relations of the first set, formulae (34), hold, the third necessarily follows.

We have just observed that formulae (33) do not hold in all regions, namely, in those determined by the analytic relations (34), (35), etc. To derive the formulae for those regions, we must evidently employ formulae (30) instead of the approximate ones (32) in the above development; we thus replace U, V, W by their explicit values (30) in formulae (5), and we find

$$\left.\begin{aligned}
\frac{X}{D}&=\frac{n^2}{r}[a_1(\beta^2+\gamma^2)-\alpha(a_2\beta+a_3\gamma)]\sin\omega\\
&\qquad+\frac{n}{r^2}[2a_1-3a_1(\beta^2+\gamma^2)+3\alpha(a_2\beta+a_3\gamma)]\cos\omega\\
\frac{Y}{D}&=\frac{n^2}{r}[a_2(\alpha^2+\gamma^2)-\beta(a_1\alpha+a_3\gamma)]\sin\omega\\
&\qquad+\frac{n}{r^2}[2a_2-3a_2(\alpha^2+\gamma^2)+3\beta(a_1\alpha+a_3\gamma)]\cos\omega\\
\frac{Z}{D}&=\frac{n^2}{r}[a_3(\alpha^2+\beta^2)-\gamma(a_1\alpha+a_2\beta)]\sin\omega\\
&\qquad+\frac{n}{r^2}[2a_3-3a_3(\alpha^2+\beta^2)+3\gamma(a_1\alpha+a_2\beta)]\cos\omega
\end{aligned}\right\},\ldots\ldots(36)$$

where we have rejected the terms of the third order of magnitude in $\frac{1}{r}$; the terms retained, being of the first and second orders, thus represent the primary and secondary oscillations respectively.

Region 1: the Vectors $\alpha:\beta:\gamma=a_1:a_2:a_3$.—Formulae (34) can evidently be replaced by

$$a_2\gamma=a_3\beta,\quad a_3\alpha=a_1\gamma,\quad a_1\beta=a_2\alpha;$$

these equations represent a straight line passing through the source of the disturbance; its direction is given by the proportion

$$\alpha:\beta:\gamma=a_1:a_2:a_3;$$

hence $$\alpha=ma_1,\quad \beta=ma_2,\quad \gamma=ma_3,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

where $$m=\frac{1}{\sqrt{a_1^2+a_2^2+a_3^2}}.$$

Formulae (36) assume the following simple form along these vectors:

$$\frac{X}{D}=\frac{2n}{r^2}a_1\cos\omega,\quad \frac{Y}{D}=\frac{2n}{r^2}a_2\cos\omega,\quad \frac{Z}{D}=\frac{2n}{r^2}a_3\cos\omega;$$

that is, only the secondary oscillation appears here. This and similar lines would, therefore, be suited best for an experimental research of the given secondary wave.

Region 2, a Surface.—The region determined by the analytic relation

$$a_1(\beta^2+\gamma^2)-\alpha(a_2\beta+a_3\gamma)=0,$$

is the surface

$$a_1(y^2+z^2)-(a_2y+a_3z)x=0\,;\quad\text{.....................}(38)$$

it passes through the origin and the line determined by formulae (37). The x-component of the electric moment at any point of this surface not on the line (37) still remains

$$\frac{X}{D}=\frac{2n}{r^2}a_1\cos\omega\,;\quad\text{............................}(39)$$

the other two components are given by the general formulae (36), where, however, any two of the variables α, β, γ are to be replaced as functions of the third, after the former have been determined as such from the analytic relations

$$a_1(\beta^2+\gamma^2)=\alpha(a_2\beta+a_3\gamma),$$

and

$$\alpha^2+\beta^2+\gamma^2=1.$$

Similarly, analogous formulae hold for the two regions or surfaces

$$a_2(x^2+z^2)-(a_1x+a_2z)y=0,$$

and

$$a_3(x^2+y^2)-(a_1x+a_2y)z=0.$$

The Magnetic Waves.—By formulae (7) and (32) we find the following approximate values for the component magnetic moments:

$$\left.\begin{array}{c}a=\dfrac{n^2v_0}{rv}(a_2\gamma-a_3\beta)\sin\omega,\quad b=\dfrac{n^2v_0}{rv}(a_3\alpha-a_1\gamma)\sin\omega\\ c=\dfrac{n^2v_0}{rv}(a_1\beta-a_2\alpha)\sin\omega\end{array}\right\}\text{......}(40)$$

These formulae are also approximate for the regions determined by the four sets of analytic relations

$$a_2\gamma-a_3\beta=a_3\alpha-a_1\gamma=a_1\beta-a_2\alpha=0,\quad\text{......................}(41)$$

$$a_2\gamma-a_3\beta=0,\ \ a_3\alpha-a_1\gamma\gtrless 0,\ \ a_1\beta-a_2\alpha\gtrless 0,\quad\text{...............}(42)$$

and the two analogous sets, since the terms of the second order of magnitude in $\frac{1}{r}$, which appear in the explicit formulae, contain each the same factor $(a_2\gamma-a_3\beta)$, $(a_3\alpha-a_1\gamma)$, or $(a_1\beta-a_2\alpha)$, as the respective term of the first order. The secondary magnetic wave would, therefore, always be accompanied by its primary magnetic wave.

The first set of analytic relations (41) is identical to the analogous one (cf. formulae (34)) for the electric moments; these relations determine the line $\alpha:\beta:\gamma=a_1:a_2:a_3$ (cf. p. 50).

The magnetic moments vanish along this line, whereas the electric ones were those of a secondary oscillation only. It thus follows that the *secondary oscillations* are *not*, necessarily, always *accompanied* by *magnetic* ones (cf. also p. 47).

The Wave-Length λ.—On p. 48 we have observed that the wave-length λ of all light-waves, with which we are familiar, is small in comparison to the distance r from source, and we, therefore, rejected the second terms of formulae (30) in our derivation of the formulae (33) for the electric moments; subsequently, upon the examination of those particular regions, in which the coefficients of the first terms of the latter formulae vanished, we found it necessary to retain the terms we had just rejected. The rejection of the given terms was evidently justified, as long as we were dealing with light-waves proper; but just as soon as our investigations are to be applied to electromagnetic waves of long wave-length or electric waves proper (Hertzian) (cf. also next page), those terms must be retained.

Explicit Formulae for Moments.—Instead of seeking, as above, particular formulae for the different kinds of electromagnetic waves, let us take the general (explicit) formulae for the electromagnetic waves in question and interpret the same according to the different values assigned the given quantities and constants. We replace U, V, W by their explicit values (30) in formulae (5) and (7), and we find

$$\left.\begin{aligned}
\frac{X}{D} &= \frac{n^2}{r}\left[a_1(\beta^2+\gamma^2)-\alpha(a_2\beta+a_3\gamma)\right]\sin\omega \\
&\quad+\frac{n}{r^2}\left[2a_1-3a_1(\beta^2+\gamma^2)+3\alpha(a_2\beta+a_3\gamma)\right]\cos\omega \\
&\quad+\frac{1}{r^3}\left[2a_1-3a_1(\beta^2+\gamma^2)+3\alpha(a_2\beta+a_3\gamma)\right]\sin\omega \\
\frac{Y}{D} &= \frac{n^2}{r}\left[a_2(\alpha^2+\gamma^2)-\beta(a_1\alpha+a_3\gamma)\right]\sin\omega \\
&\quad+\frac{n}{r^2}\left[2a_2-3a_2(\alpha^2+\gamma^2)+3\beta(a_1\alpha+a_3\gamma)\right]\cos\omega \\
&\quad+\frac{1}{r^3}\left[2a_2-3a_2(\alpha^2+\gamma^2)+3\beta(a_1\alpha+a_3\gamma)\right]\sin\omega \\
\frac{Z}{D} &= \frac{n^2}{r}\left[a_3(\alpha^2+\beta^2)-\gamma(a_1\alpha+a_2\beta)\right]\sin\omega \\
&\quad+\frac{n}{r^2}\left[2a_3-3a_3(\alpha^2+\beta^2)+3\gamma(a_1\alpha+a_2\beta)\right]\cos\omega \\
&\quad+\frac{1}{r^3}\left[2a_3-3a_3(\alpha^2+\beta^2)+3\gamma(a_1\alpha+a_2\beta)\right]\sin\omega
\end{aligned}\right\}, \quad \ldots\ldots(43)$$

and

$$\left.\begin{aligned}
a &= \frac{nv_0}{v}(a_2\gamma-a_3\beta)\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right) \\
b &= \frac{nv_0}{v}(a_3\alpha-a_1\gamma)\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right) \\
c &= \frac{nv_0}{v}(a_1\beta-a_2\alpha)\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right)
\end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots(44)$$

Distinction between Light and Electric Waves; the Quantity n.—The expressions for the electric moments X, Y, Z are composed each of three terms, arranged according to the different powers of $\frac{1}{r}$ or n. The order of magnitude of any such term evidently depends not only upon these powers of $\frac{1}{r}$ and n, but also upon the actual distance r from the given source and the value of the quantity n. There are now two classes of electromagnetic waves, characterized by n's of quite different orders of magnitude and between which we shall, therefore, have to discriminate; these are the light-waves proper and the electric (Hertzian) waves. The former have wave-lengths of the dimensions 10^{-3} mm.; by formula (31) n would, therefore, be very large for all light-waves. On the other hand, the wave-length λ of the electric waves proper is of the dimensions of the metre or of the other quantities (r) that appear in formulae (43) for the given electric moments; their n would, therefore, be of the same order of magnitude as their λ or the other quantities. It thus follows: For light-waves, the second and third terms of the expressions for the electric moments X, Y, Z of formulae (43) will be very (infinitely) small in comparison to the first terms of the same, and may, therefore, be rejected not alone at great or finite distances from the source—due to the functions $\frac{1}{r}$—but also in its immediate neighbourhood—due to the value of n; in other words, the amplitude of the secondary wave that accompanies any (primary) light-wave will be so very (infinitely) small in comparison to that of the light-wave proper, that we cannot expect to detect the same except in the source itself. Consequently, we may conceive *light-waves* proper as *unaccompanied* by *secondary electric disturbances.* For electric waves, the second and third terms of the given expressions will vanish, when compared with the first terms of the same, only at greater distances from the source; the primary electric wave will thus be accompanied by a secondary electric wave to a considerable distance from its source, the intensity of the latter evidently being of the same order of magnitude as that of the former in the immediate neighbourhood of the source, but decreasing somewhat more rapidly than that of the primary wave, as we recede from the same. On the other hand, the secondary wave will evidently be represented by the second terms of the given expressions, except in the immediate neighbourhood of the source.

For one and the same amplitudes a_1, a_2, a_3 the amplitude of the light-wave would be very (infinitely) large in comparison to that of the electric wave. To obtain amplitudes of the dimensions of those of the light-wave or of the same dimensions as the amplitudes of the electric wave for the light-wave, we replace the quantities a_1, a_2, a_3, the amplitudes of the functions f_1, f_2, f_3 for electric waves, by the quantities $\frac{a_1}{m^2}$, $\frac{a_2}{m^2}$, $\frac{a_3}{m^2}$, respectively for the light-waves, where m^2 shall denote a quantity (number) of the same order of magnitude as n; the secondary electric light-wave will then vanish, as above, whereas the wave-length λ will remain unaltered.

Electric and Magnetic Oscillations at $\perp$ to each other.—To confirm the general law that "the electric and magnetic oscillations take place at right angles to each other" in the given case, we must employ the explicit and not the approximate expressions for the moments. Let us denote the quantities of the first and second orders of magnitude in $\frac{1}{r}$ by suffixing the indices 1 and 2 to the same; we can then write

$$X = X_1 + X_2, \quad Y = Y_1 + Y_2, \quad Z = Z_1 + Z_2,$$

and

$$a = a_1 + a_2,^* \quad b = b_1 + b_2, \quad c = c_1 + c_2,$$

where the quantities or terms of the third order of magnitude in $\frac{1}{r}$, X_3, Y_3, Z_3, have been rejected (cf. p. 52); for the proof of the given general law, where the terms of the third order of magnitude in $\frac{1}{r}$ are retained, see Ex. 10 at end of chapter.

The analytic condition that the electric and magnetic oscillations take place at right angles to each other, namely,

$$Xa + Yb + Zc = 0$$

(cf. formula (11)) becomes here

$$(X_1 + X_2)(a_1 + a_2) + (Y_1 + Y_2)(b_1 + b_2) + (Z_1 + Z_2)(c_1 + c_2) = 0.$$

Since terms of only the same order of magnitude in $\frac{1}{r}$ can evidently be compared with one another, this condition can be replaced by the three,

$$X_1a_1 + Y_1b_1 + Z_1c_1 = 0, \text{ terms of the second order,}$$

$$(X_1a_2 + X_2a_1) + (Y_1b_2 + Y_2b_1) + (Z_1c_2 + Z_2c_1) = 0, \quad \text{,,} \quad \text{third} \quad \text{,,}$$

and

$$X_2a_2 + Y_2b_2 + Z_2c_2 = 0, \quad \text{,,} \quad \text{fourth} \quad \text{,,}$$

To ascertain whether these conditions be fulfilled, we replace the given moments by their values and evaluate the forms† in question. Let us examine here the second condition; we have

$$X_1a_2 + X_2a_1 = C[a_1 - 2a_1(\beta^2 + \gamma^2) + 2a(a_2\beta + a_3\gamma)](a_2\gamma - a_3\beta),$$

---

[*] The moments a_1 and a_2 are not to be confounded with the amplitudes a_1, a_2, (a_3) of the wave-functions f_1, f_2, f_3.

[†] See footnote on p. 42.

and, similarly,

$$Y_1 b_2 + Y_2 b_1 = C[a_2 - 2a_2(\alpha^2+\gamma^2) + 2\beta(a_1\alpha + a_3\gamma)](a_3\alpha - a_1\gamma),$$

and $$Z_1 c_2 + Z_2 c_1 = C[a_3 - 2a_3(\alpha^2+\beta^2) + 2\gamma(a_1\alpha + a_2\beta)](a_1\beta - a_2\alpha),$$

where $$C = \frac{2n^2 D v_0}{r^3 v} \sin\omega \cos\omega.$$

The coefficient of a_1 of the form* of the given condition is thus

$$\begin{aligned} C\{&[1 - 2(\beta^2+\gamma^2)](a_2\gamma - a_3\beta) \\ &- \gamma[a_2 - 2a_2(\alpha^2+\gamma^2) + 2\beta(a_1\alpha + a_3\gamma)] + 2a_3\alpha^2\beta \\ &+ \beta[a_3 - 2a_3(\alpha^2+\beta^2) + 2\gamma(a_1\alpha + a_2\beta)] - 2a_2\alpha^2\gamma\}, \end{aligned}$$

which evidently vanishes. The form itself thus reduces to

$$\begin{aligned} C\{&2\alpha(a_2\beta + a_3\gamma)(a_2\gamma - a_3\beta) + [a_2 - 2a_2(\alpha^2+\gamma^2) + 2a_3\beta\gamma]a_3\alpha \\ &- [a_3 - 2a_3(\alpha^2+\beta^2) + 2a_2\beta\gamma]a_2\alpha\}, \end{aligned}$$

the different terms of which evidently cancel one another, and the given condition is thus fulfilled. The proof of the validity of the other two conditions offers no difficulties.

Magnetic and Primary Electric Oscillations at ⊥ to Direction of Propagation.—From a glance at formulae (44) for the magnetic moments, it is evident that the magnetic oscillations take place at right angles to their direction of propagation.

It is, likewise, easy to show that the primary oscillations X_1, Y_1, Z_1 also take place at right angles to their direction of propagation; this is not, however, true of the secondary oscillations X_2, Y_2, Z_2, as the ensuing development will show.

Determination of the Angle (f_2, r).—Let us, next, determine the angle of oscillation (f_2, r), which the given secondary oscillations make with their direction of propagation; we denote their resultant moment by f_2 and the direction-cosines of that moment by λ_2, μ_2, ν_2, as on p. 43, and we have

$$\cos(f_2, r) = \lambda_2\alpha + \mu_2\beta + \nu_2\gamma$$

(cf. formula (23)). Replace here λ_2, μ_2, ν_2 by the respective moments (cf. p. 43) from formulae (43), and we have

$$\begin{aligned} \cos(f_2, r) = \frac{nD}{r^2 f_2}\{&[2a_1 - 3a_1(\beta^2+\gamma^2) + 3\alpha(a_2\beta + a_3\gamma)]\alpha \\ &+ [2a_2 - 3a_2(\alpha^2+\gamma^2) + 3\beta(a_1\alpha + a_3\gamma)]\beta \\ &+ [2a_3 - 3a_3(\alpha^2+\beta^2) + 3\gamma(a_1\alpha + a_2\beta)]\gamma\}\cos\omega \\ = \frac{2nD}{r^2 f_2}&(a_1\alpha + a_2\beta + a_3\gamma)\cos\omega, \end{aligned}$$

---

* See foot-note, p. 42.

where f_2 is to be replaced by

$$f_2 = \sqrt{X_2^2 + Y_2^2 + Z_2^2}$$

$$= \frac{nD}{r^2}\sqrt{a_1^2(1+3\alpha^2) + a_2^2(1+3\beta^2) + a_3^2(1+3\gamma^2) + 6(a_1a_2\alpha\beta + a_1a_3\alpha\gamma + a_2a_3\beta\gamma)}\cos\omega$$

$$= \frac{nD}{r^2}\sqrt{a_1^2 + a_2^2 + a_3^2 + 3(a_1\alpha + a_2\beta + a_3\gamma)^2}\cos\omega.$$

We thus have

$$\cos(f_2, r) = \frac{2(a_1\alpha + a_2\beta + a_3\gamma)}{\sqrt{a_1^2 + a_2^2 + a_3^2 + 3(a_1\alpha + a_2\beta + a_3\gamma)^2}} \quad \ldots\ldots\ldots\ldots(45)$$

This general expression for $\cos(f_2, r)$ is evidently too complicated to admit of a simple analysis. In order to acquire some knowledge of the behaviour of these secondary (electric) waves—and among other properties one of the most important is the variation of their angle of oscillation throughout the given region—we shall undertake to examine the expression (45) for some particular case, for example, that, where

$$a_1 = a_2 = a_3 = a.$$

The expression for $\cos(f_2, r)$ then reduces to

$$\cos(f_2, r) = \frac{2(\alpha + \beta + \gamma)}{\sqrt{6(1 + \alpha\beta + \alpha\gamma + \beta\gamma)}} \quad \ldots\ldots\ldots\ldots(46)$$

Since this expression, as also the general one, assumes one and the same value along any given vector, it will suffice to examine its behaviour over the surface of any given sphere with centre at origin. The evaluation of the same at different points on the surface of any such sphere will evidently be facilitated by the introduction of the polar coordinates r, θ, ϕ employed in the preceding problem (cf. p. 43). By formulae (25) the given expression (46) can then be written

$$\cos(f_2, r) = \frac{2[\cos\phi + \sin\phi(\sin\theta + \cos\theta)]}{\sqrt{6[1 + \sin\phi\cos\phi(\sin\theta + \cos\theta) + \sin^2\phi\sin\theta\cos\theta]}} \quad \ldots(47)$$

$$= \frac{2[\cos\phi + \sin\phi(\sin\theta + \cos\theta)]}{\sqrt{3[2 + \sin 2\phi(\sin\theta + \cos\theta) + \sin^2\phi\sin 2\theta]}} \quad \ldots\ldots\ldots(47\text{A})$$

To determine the angle (f_2, r) at any point, we plot the curves $\theta = 0°$, 15°, 30° ... 180° for different values of ϕ between 0° and 360°, choosing the angle (f_2, r)—its degrees—as ordinate and the angle ϕ—its degrees—as abscissa. To plot these curves, we shall find it sufficient to determine the angle (f_2, r) for every 15° of ϕ, except in the case of the curve $\theta = 45°$ between $\phi = 45°$ and 60° (see below).

The Hemisphere $\theta=0°$ to $90°$.—It is evident that the expression for $\cos(f_2, r)$ will remain unchanged when we replace θ by $90°-\theta$; hence we need plot only the curves $\theta=0°$, $15°$, $30°$, and $45°$, the other curves, $\theta=60°$, $75°$, and $90°$, being identical to the curves $\theta=30°$, $15°$, and $0°$ respectively.

The formula (47) for $\cos(f_2, r)$ reduces for the given hemisphere to the following:

For (1) $\theta=0°$ or $90°$,

$$\cos(f_2, r)=\frac{2(\cos\phi+\sin\phi)}{\sqrt{6(1+\sin\phi\cos\phi)}}; \qquad (48)$$

(2) $\theta=15°$ or $75°$,

$$\cos(f_2, r)=\frac{2(\cos\phi+1{\cdot}2247\sin\phi)}{\sqrt{6(1+1{\cdot}2247\sin\phi\cos\phi+0{\cdot}25\sin^2\phi)}}; \qquad (49)$$

(3) $\theta=30°$ or $60°$,

$$\cos(f_2, r)=\frac{2(\cos\phi+1{\cdot}366\sin\phi)}{\sqrt{6(1+1{\cdot}366\sin\phi\cos\phi+0{\cdot}433\sin^2\phi)}}; \qquad (50)$$

and (4) $\theta=45°$,

$$\cos(f_2, r)=\frac{2(\cos\phi+\sqrt{2}\sin\phi)}{\sqrt{6(1+\sqrt{2}\sin\phi\cos\phi+0{\cdot}5\sin^2\phi)}}. \qquad (51)$$

Upon evaluating these expressions for $\phi=0°$, $15° \ldots 180°$, we find the values given in foot-note* for the angle (f_2, r), which evidently suffice for the plotting of the curves in question, with the exception of the curve $\theta=45°$ between $\phi=45°$ and $60°$. It is now easy to show that this curve touches the ϕ-axis between these two values; in which case formula (51) would assume the particular form

$$1=\frac{2(\cos\phi+\sqrt{2}\sin\phi)}{\sqrt{6(1+\sqrt{2}\sin\phi\cos\phi+0{\cdot}5\sin^2\phi)}}. \qquad (52)$$

* The curves	$\theta=0°$ or $90°$,	$\theta=15°$ or $75°$,	$\theta=30°$ or $60°$ and	$\theta=45°$.
$\phi=0°$	35° 17′	35° 17′	35° 17′	35° 17′
15°	26° 34′	24° 25′	23° 2′	22° 34′
30°	21° 10′	17°	13° 57′	12° 57′
45°	19° 28′	13°	7° 40′	4° 55′
60°	21° 10′	13° 35′	7° 3′	2° 40′
75°	26° 34′	18° 25′	12° 45′	10° 30′
90°	35° 17′	26° 26′	21° 18′	19° 26′
105°	48° 11′	38° 20′	32° 50′	31° 3′
120°	66° 47′	55° 38′	49° 21′	47° 21′
135°	90°	79° 33′	73° 11′	72° 14′
150°	113° 13′	106° 30′	102°	100° 26′
165°	131° 49′	128° 57′	127° 5′	126° 26′
180°	144° 43′	144° 43′	144° 43′	144° 43′

and 54° 43′ $(f_2, r)=0°$.

This equation must then serve for the determination of ϕ or that point, at which the given curve touches the ϕ-axis.

Equation (52) gives

$$2(\cos^2\phi + 2\sqrt{2}\sin\phi\cos\phi + 2\sin^2\phi) = 3(1 + \sqrt{2}\sin\phi\cos\phi + \tfrac{1}{2}\sin^2\phi),$$

or $$\sin^2\phi - 2 = -2\sqrt{2}\sin\phi\cos\phi = -2\sqrt{2}\sin\phi\sqrt{1-\sin^2\phi};$$

which squared gives $$9\sin^4\phi - 12\sin^2\phi + 4 = 0,$$

hence $$\sin\phi = \sqrt{\frac{2}{3}}, \text{ or } \phi = 54^\circ\, 43', \quad\text{.......................(53)}$$

which is also included among the values of ϕ in foot-note on p. 57.

We observe that, if the resulting equation for ϕ had no real root, the above assumption, that the given curve touch the ϕ-axis, would have to be abandoned (cf. Ex. 7 at end of chapter). Upon including this particular value of ϕ among those above, we can plot the given curves, as in fig. 6 on next page.

For $\phi > 180^\circ = 180^\circ + \phi'$, the general expression (47) for $\cos(f_2, r)$ remains unchanged, except in sign, since

$$\sin(180^\circ + \phi') = -\sin\phi',$$

and $$\cos(180^\circ + \phi') = -\cos\phi'.$$

It thus follows that

$$\cos(f_2, r)_{\phi'} = -\cos(f_2, r)_\phi = \cos\{180^\circ - (f_2, r)_\phi\}$$

or $$(f_2, r)_{\phi'} = 180^\circ - (f_2, r)_\phi, \quad\text{.........................(54)}$$

where the indices ϕ and ϕ' denote that the angle (f_2, r) is to be taken in the regions $\phi = 0^\circ$ to 180° and $\phi' = 180^\circ$ to 360° $(0^\circ \leqq \theta \leqq 90^\circ)$ respectively.

The values of the angle (f_2, r) in the region $\phi = 180^\circ$ to 360° $(0^\circ \leqq \theta \leqq 90^\circ)$ thus follow directly, by formula (54), from the values for that angle in the region $\phi = 0^\circ$ to 180° $(0^\circ \leqq \theta \leqq 90^\circ)$ (cf. foot-note, p. 57). We can evidently obtain the curves represented by these values, upon revolving the plane $\phi = 0^\circ$ to 180° and $(f_2, r) = 0^\circ$ to 180°, together with its curves $\theta = 0^\circ$, 15°, 30°, and 45°, through 180° about the line $(f_2, r) = 90^\circ$ in that plane as axis, and then displacing the same (plane and curves) the distance 180° along the ϕ-axis.

The Hemisphere $\theta = 90^\circ$ to 180°.—Our general formula (47) reduces here to the following:

For (1) $\theta = 105^\circ = 90^\circ + \theta'$ and $\theta = 165^\circ = 90^\circ + \theta'$,

$$\cos(f_2, r) = \frac{2(\cos\phi \pm 0{\cdot}7071\sin\phi)}{\sqrt{6(1 \pm 0{\cdot}7071\sin\phi\cos\phi - 0{\cdot}25\sin^2\phi)}}, \quad\text{........(55)}$$

where the plus-sign is to be taken for $\theta' = 15^\circ$ and the minus-sign for $\theta' = 75^\circ$;

(2) $\theta = 120^\circ = 90^\circ + \theta'$ and $\theta = 150^\circ = 90^\circ + \theta'$,

$$\cos(f_2, r) = \frac{2(\cos\phi \pm 0{\cdot}366 \sin\phi)}{\sqrt{6(1 \pm 0{\cdot}366 \sin\phi\cos\phi - 0{\cdot}433 \sin^2\phi)}}, \quad \ldots\ldots\ldots(56)$$

the plus-sign to be taken for $\theta' = 30^\circ$ and the minus-sign for $\theta' = 60^\circ$;

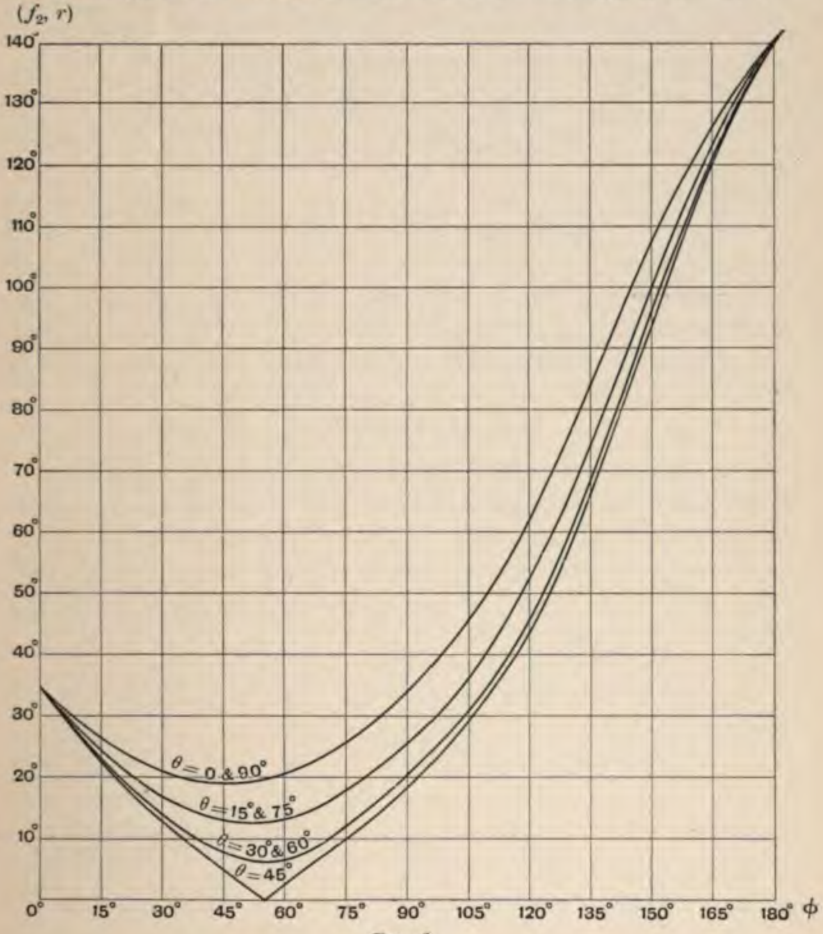

FIG. 6.

(3) $\theta = 135^\circ = 90^\circ + \theta'$,

$$\cos(f_2, r) = \frac{2\cos\phi}{\sqrt{6(1 - 0{\cdot}5\sin^2\phi)}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(57)$$

and (4) $\theta = 180^\circ = 90^\circ + \theta'$,

$$\cos(f_2, r) = \frac{2(\cos\phi - \sin\phi)}{\sqrt{6(1 - \sin\phi\cos\phi)}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58)$$

Upon evaluating these expressions for the different values of ϕ between 0° and 180°, we observe that

$$\cos(f_2, r)_{90°+\phi,\, 90°-\theta'} = -\cos(f_2, r)_{90°-\phi,\, \theta'},$$

where the given indices denote, as above, that the angles in question are to be taken in the regions $90° + \phi$, $90° - \theta'$ and $90° - \phi$, θ' respectively, the angles ϕ and θ' being assigned the values

$$\phi = 15°,\ 30°,\ 45°,\ 60°,\ \text{and}\ 75°$$

and $$\theta' = 15°\ \text{and}\ 30°.$$

It thus follows that

$$(f_2, r)_{90°+\phi,\, 90°-\theta'} = 180° - (f_2, r)_{90°-\phi,\, \theta'} \quad \text{...............} (59)$$

(cf. formula (54)).

By this and formulae (55)–(58) we find the values given in foot-note* for the angle (f_2, r), which give the six curves of fig. 7 on next page.

As above, the expressions for $\cos(f_2, r)$ remain unchanged, except in sign, when we put $\phi = 180° + \phi'$; it thus follows that

$$(f_2, r)_{\phi'} = 180° - (f_2, r)_{\phi}.$$

The values of the angle (f_2, r) in the region $\phi = 180°$ to 360° $(90° < \theta < 180°)$ thus follow directly from those in the foot-note* below, whereas we can obtain the curves represented by the same, as above, upon revolving the curves $\theta' = 15°, 30° \ldots 90°$ through 180° about the line $(f_2, r) = 90°$ as axis and then displacing the same the distance 90° along the ϕ-axis.

It is evident from the above systems of curves that:

1. The given (secondary) oscillations are longitudinal along the two vectors $\theta = 45°$, $\phi = 54° 43'$ and $\theta = 45°$, $\phi = 234° 43'$.

2. The longitudinal oscillations (1) take place in opposite directions with regard to their vectors or in the same direction in space.

*The curves	$\theta' = 15°$,	$\theta' = 30°$,	$\theta' = 45°$,	$\theta' = 60°$,	$\theta' = 75°$ and	$\theta' = 90°$.
$\phi =$ 0°	35° 17′	35° 17′	35° 17′	35° 17′	35° 17′	35° 17′
15°	29° 26′	32° 15′	36° 39′	40° 41′	44° 37′	48° 12′
30°	26° 47′	33° 18′	40° 54′	49° 22′	58° 14′	66° 36′
45°	27° 15′	36° 39′	48° 12′	61° 49′	76° 27′	90°
60°	30° 50′	43° 4′	58° 54′	78°	97° 24′	113° 24′
75°	37° 40′	52° 56′	73° 11′	96° 15′	116° 48′	131° 48′
90°	48° 12′	66° 37′	90°	113° 23′	131° 48′	144° 43′
105°	63° 12′	83° 45′	106° 49′	127° 4′	142° 20′	153° 24′
120°	82° 36′	102°	121° 6′	136° 56′	149° 10′	158° 42′
135°	103° 33′	118° 11′	131° 48′	143° 21′	152° 45′	160° 42′
150°	121° 46′	130° 38′	139° 6′	146° 42′	153° 13′	158° 42′
165°	135° 23′	139° 19′	143° 21′	147° 10′	150° 34′	153° 24′
180°	144° 43′	144° 43′	144° 43′	144° 43′	144° 43′	144° 43′

The Longitudinal Secondary Electric Wave.—We have seen on pp. 50-51 that the secondary (electric) wave was unaccompanied by either a primary (electric) or a magnetic wave along the two vectors $\alpha:\beta:\gamma = a_1:a_2:a_3$; these were the only vectors, along which the primary (electric) or magnetic wave did not appear. From the given formulae for the secondary (electric) wave along these vectors it is evident that these secondary oscillations take place *along* those vectors, that is, that they are longitudinal along the same. It is now easy to show that these longitudinal oscillations and those of (1) and (2) above are the same, the latter corresponding only to the particular case of the former, where $a_1 = a_2 = a_3 = a$. It thus follows that the longitudinal oscillations of (1) and (2) are unaccompanied by either a

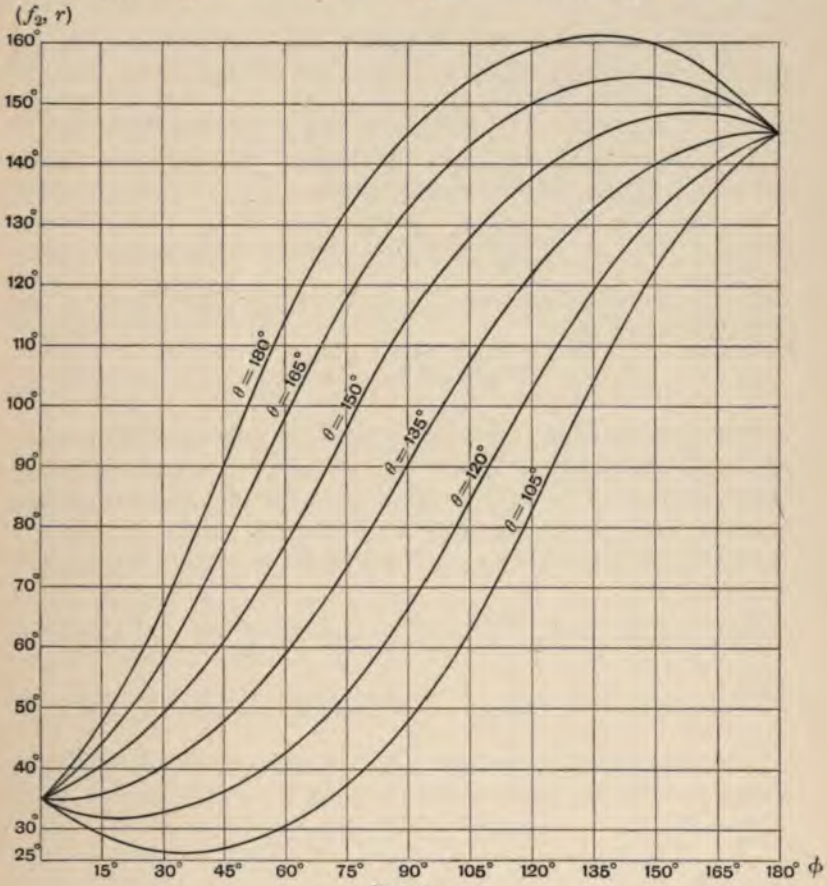

FIG. 7.

primary (electric) or a magnetic wave; hence the general law: In those regions, where the primary (electric) and magnetic waves fail to appear, the secondary (electric) wave is longitudinal and conversely the longitudinal secondary (electric) oscillations are thereby characterized that they are unaccompanied by either a primary (electric) or a magnetic wave.

The Transverse Secondary Electric Wave.—By formulae (46) the secondary (electric) oscillations are transverse throughout the plane

$$\alpha+\beta+\gamma=0\,(a_1=a_2=a_3), \qquad (60)$$

which passes through the origin. Throughout this plane the electric moments evidently assume the simpler form

$$\frac{X_1}{D}=\frac{Y_1}{D}=\frac{Z_1}{D}=\frac{n^2}{r}\sin\omega,$$

or the moments of the primary (electric) wave are independent of the direction-cosines, and

$$\frac{X_2}{D}=\frac{Y_2}{D}=\frac{Z_2}{D}=-\frac{n}{r^2}\cos\omega,$$

or the moments of the secondary (electric) wave are also independent of the direction-cosines. It is evident that the primary and secondary waves, represented by the more general formulae (43), possess this same property throughout the plane $a_1\alpha+a_2\beta+a_3\gamma=0$.

By formulae (44)—$a_1=a_2=a_3$—the resultant moment of the magnetic wave is

$$\sqrt{a^2+b^2+c^2}=\frac{nv_0}{v}\sqrt{(\gamma-\beta)^2+(\alpha-\gamma)^2+(\beta-\alpha)^2}\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right)$$

$$=\frac{nv_0}{v}\sqrt{2-2(\alpha\beta+\alpha\gamma+\beta\gamma)}\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right).$$

The secondary (electric) oscillations are transverse throughout the plane $\alpha+\beta+\gamma=0$, that is, here the relation holds $\alpha+\beta+\gamma=0$, which squared gives

$$\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)=0,$$

or, since

$$\alpha^2+\beta^2+\gamma^2=1,$$

$$2(\alpha\beta+\alpha\gamma+\beta\gamma)=-1. \qquad (61)$$

The analytic relation (60) between the direction-cosines can thus be replaced by this relation (61), and hence the above expression for the resultant magnetic moment throughout the plane $\alpha+\beta+\gamma=0$ written

$$\sqrt{a^2+b^2+c^2}=\frac{\sqrt{3}nv_0}{v}\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right),$$

or the resultant moment of the magnetic wave is here independent of the direction-cosines. It is also easy to show that the resultant moment of the magnetic wave represented by formulae (44)—$a_1 \gtreqless a_2 \gtreqless a_3$—is independent of the direction-cosines throughout the plane $a_1\alpha + a_2\beta + a_3\gamma = 0$ (cf. Ex. 16 at end of chapter).

The above results can evidently be summed up as follows:

The transverse secondary (electric) wave is thereby characterized that (1) its amplitude is independent of the direction-cosines, depending only upon r, the distance from the source, and (2) it is accompanied by a primary (electric) and a magnetic wave, whose amplitudes, likewise, depend alone upon the distance from the source; and conversely, in those regions, the plane $a_1\alpha + a_2\beta + a_3\gamma = 0$, where the amplitudes of the primary (electric) and the magnetic waves are functions only of the distance from the source, the secondary (electric) wave is transverse.

The Primary Electric Waves.—The primary wave of this problem differs only immaterially from those of problems 1 and 2; it is the (primary) electromagnetic or light-wave, with which we are already familiar; it reveals only another law of distribution of the amplitudes with regard to magnitude and direction of oscillation (over any given sphere).

The Secondary Electric Waves; the Roentgen (X) Rays.—The secondary wave of the given problem is also similar to those of the preceding problems, in that it belongs to one and the same class of wave-motion, namely that which is thereby characterized that the oscillations do not, in general, take place at right angles to their direction of propagation. Moreover, the secondary waves of all three problems display certain properties that are common to all. One of the most striking such properties is that there are certain regions, throughout which the secondary (electric) wave is unaccompanied by either a primary (electric) or a magnetic wave, and that in those regions the secondary wave is longitudinal; in problem 1 the given region was the x-axis (cf. Ex. 12), in problem 2 the y- and z-axes (cf. p. 39) and in problem 3 the vectors $\alpha : \beta : \gamma = a_1 : a_2 : a_3$ (cf. p. 61). In this respect the longitudinal secondary electric waves would resemble the Roentgen (X) rays, which have not yet been found to be influenced by magnetic disturbances. Another similarity between these waves and the Roentgen rays is the empirical confirmation* that the latter advance with the velocity of light, which is evidently the velocity of propagation of the former (cf. formulae (43)). As to the law of intensity of the Roentgen rays, the few empirical data we have would tend to show that it is not according to the inverse square of the distance from source, but that their intensity diminishes much more rapidly, perhaps according to the fourth power of the distance, the law of variation of our secondary electric waves.

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* "Sur l'égalité de la vitesse de propagation des rayons X et de la vitesse de la lumière dans l'air." Note de M. R. Blondlot. *Comptes Rendus*, Tome CXXXV., No. 18, Nov., 1902.

Summary.—Lastly, let us compare the results found on pp. 61-62 pertaining to the longitudinal and transverse secondary (electric) waves and their respective primary (electric) and magnetic waves of problem 3 with the results of Exs. 2, 4 and 12-16 at end of chapter pertaining to the respective waves of problems 1 and 2; we find the following general results:

1. In those regions, where the primary (electric) and the magnetic waves do not appear, the secondary wave is either longitudinal, as in problems 1, 2 and 3, or it does not appear at all, as in problem 2 along the x-axis.

2. The longitudinal secondary (electric) wave is unaccompanied by either a primary (electric) or a magnetic wave (problems 1, 2 and 3).

3. In those regions, where the secondary (electric) wave is transverse, its amplitude is independent of the direction-cosines, that is, one and the same for any $r = const.$; and, conversely, in those regions, where the amplitude of the secondary (electric) wave is independent of the direction-cosines or a function only of r, the same is transverse (problems 1 and 3).

4. The transverse wave is accompanied by a primary (electric) and a magnetic wave, whose amplitudes are independent of the direction-cosines, that is, remain the same for any $r = const.$; and, conversely, in those regions, where the amplitudes of the primary (electric) and magnetic waves are independent of the direction-cosines or functions only of r, the secondary (electric) wave is transverse (problems 1 and 3). 3 and 4 do not hold for the waves of problem 2; the explanation of this is evidently to be sought in the particular form chosen for the auxiliary functions U, V, W, which are unsymmetrical with regard to the coordinate-axes. On the other hand, we call special attention to their validity for the waves of problem 3, where the auxiliary functions U, V, W have been chosen symmetrical with regard to the x, y, z axes, but as derivatives of *three arbitrary* wave-functions ϕ_1, ϕ_2, ϕ_3, since waves of most various types can evidently be represented by the derivatives of three such arbitrary functions ($a_1 \gtreqless a_2 \gtreqless a_3$).

EXAMPLES.

1. The Hertzian oscillations are represented by Hertz by the following derivatives of the function Π, which is assumed to be a purely spherical wave-function:

$$P=-\frac{d^2\Pi}{dx\,dz},\quad Q=-\frac{d^2\Pi}{dy\,dz},\quad R=\frac{d^2\Pi}{dx^2}+\frac{d^2\Pi}{dy^2},$$

and

$$\bar{\alpha}=\frac{1}{v}\frac{d^2\Pi}{dt\,dy},\quad \bar{\beta}=-\frac{1}{v}\frac{d^2\Pi}{dt\,dx},\quad \bar{\gamma}=0,$$

where P, Q, R (X, Y, Z)* and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ (L, M, N)* denote the electric and magnetic forces respectively.†

Show, on assuming that the function Π has the form

$$\Pi=\frac{El\sin m(r-vt)}{r},$$

where El is a constant and $m=\frac{\pi}{\lambda}$, that P, Q, R and $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are given by the expressions

$$P=\frac{\alpha\gamma}{r}Elm^2\sin m(r-vt)+\frac{3\alpha\gamma}{r^2}Elm\cos m(r-vt)$$
$$-\frac{3\alpha\gamma}{r^3}El\sin m(r-vt),$$

$$Q=\frac{\beta\gamma}{r}Elm^2\sin m(r-vt)+\frac{3\beta\gamma}{r^2}Elm\cos m(r-vt)$$
$$-\frac{3\beta\gamma}{r^3}El\sin m(r-vt),$$

$$R=-\frac{\alpha^2+\beta^2}{r}Elm^2\sin m(r-vt)+\frac{2-3(\alpha^2+\beta^2)}{r^2}Elm\cos m(r-vt)$$
$$-\frac{2-3(\alpha^2+\beta^2)}{r^3}El\sin m(r-vt),$$

and

$$\bar{\alpha}=\frac{\beta}{r}Elm^2\sin m(r-vt)+\frac{\beta}{r^2}Elm\cos m(r-vt),$$
$$\bar{\beta}=-\frac{\alpha}{r}Elm^2\sin m(r-vt)-\frac{\alpha}{r^2}Elm\cos m(r-vt),$$
$$\bar{\gamma}=0$$

(cf. formulae (39), p. 83, of my *Theory of Electricity and Magnetism*), where α, β, γ are the direction-cosines of r.

These (Hertzian) oscillations are evidently those already examined in problem 1 of the text, being referred only to a different system of coordinates (cf. formulae (10) and (13)).

2. Show that region 3 of problem 2, determined by the analytic conditions

$$\alpha(\beta^2-\gamma^2)\gtreqless 0,\quad \beta(\alpha^2+2\gamma^2)=0,\quad \gamma(\alpha^2+2\beta^2)\gtreqless 0,$$

comprises the two regions $\beta=0$ or the xz-plane and $\alpha^2+2\gamma^2=0$, hence $\beta^2=1$ or the y-axis; and that throughout the former the resultant electric moments are

$$\sqrt{X_1^2+Y_1^2+Z_1^2}=\sqrt{X_1^2+Z_1^2}=\frac{\alpha\gamma}{r}\frac{d^3f}{dr^3},$$
$$\sqrt{X_2^2+Y_2^2+Z_2^2}=\sqrt{X_2^2+Z_2^2}=\frac{3\gamma}{r}\frac{d^2f}{dr^2},$$

and along the latter

$$\sqrt{X_1^2+Y_1^2+Z_1^2}=0,\quad \sqrt{X_2^2+Y_2^2+Z_2^2}=Y_2=-\frac{3}{r^2}\frac{d^2f}{dr^2}.\quad (D=1.)$$

---

* The Hertzian notation.

† Cf. Hertz, *Untersuchungen ueber die Ausbreitung der elektrischen Kraft*, p. 150; and Curry, *Theory of Electricity and Magnetism*, formulae (28) and (29), p. 77.

Throughout the xz-plane both the primary and secondary oscillations thus take place in that plane, whereby the intensity of the former is a function of the direction-cosines α and γ and that of the latter a function of γ alone, whereas along the y-axis the primary wave disappears entirely and the secondary one is propagated as a longitudinal wave.

The moments of the magnetic wave that accompanies the given electric wave are

$$a=c=0, \quad b=-\frac{v_0}{v^2}\frac{\alpha\gamma}{r}\frac{d}{dt}\left(\frac{d^2f}{dr^2}-\frac{3}{r}\frac{df}{dr}+\frac{3}{r^2}f\right)$$

throughout the xz-plane, and

$$a=b=c=0$$

along the y-axis. The magnetic oscillations of the xz-plane thus take place at right angles to that plane, whereas no magnetic disturbance whatever appears along the y-axis.

The only disturbance that appears along the y-axis is, therefore, a secondary (electric) wave, which is propagated along that axis as a longitudinal wave. The appearance of a secondary (electric) wave, unaccompanied by either a primary (electric) or magnetic wave, along one vector, at least, is thus consistent with our differential equations.

3. The analytic conditions

$$\beta(\alpha^2+2\gamma^2)=0, \quad \alpha(\beta^2-\gamma^2)=\gamma(\alpha^2+2\beta^2)\gtrless 0$$

determine the region $\alpha=-\gamma=\sqrt{\frac{1}{2}}$, $\beta=0$ or a straight line passing through the origin, lying in the xz-plane and bisecting the quadrant x, $-z$; the resultant moments along this vector are

$$\sqrt{X_1^2+Y_1^2}=-\frac{1}{2r}\frac{d^3f}{dr^3} \text{ and } \sqrt{X_2^2+Z_2^2}=-\frac{3}{\sqrt{2}\,r^2}\frac{d^2f}{dr^2}. \quad (D=1.)$$

4. Examine the electric and magnetic waves in region 4 of problem 2.

Show that a secondary (electric) wave, unaccompanied by either a primary (electric) or a magnetic wave, is propagated in longitudinal oscillations along the z-axis.

5. Show, when $a_1=a_2=a_3$, that no region is determined by the following particular form of formulae (35):

$$\beta^2+\gamma^2=\alpha(\beta+\gamma),$$
$$\alpha^2+\gamma^2-\beta(\alpha+\gamma)=\alpha^2+\beta^2-\gamma(\alpha+\beta)\gtrless 0.$$

6. To find in problem 3 the electric and magnetic moments in the region determined by the analytic conditions

$$a_2\gamma-a_3\beta=0, \quad a_3\alpha-a_1\gamma\gtrless 0, \quad a_1\beta-a_2\alpha\gtrless 0$$

(cf. formulae (42)), replace β and γ by

$$\beta=\frac{a_2}{\sqrt{a_2^2+a_3^2}}\sqrt{1-\alpha^2}, \quad \gamma=\frac{a_3}{\sqrt{a_2^2+a_3^2}}\sqrt{1-\alpha^2}$$

in formulae (36) and (40) respectively.

Also show that no region is defined by the analytic conditions

$$a_2\gamma-a_3\beta=0, \quad a_3\alpha-a_1\gamma=a_1\beta-a_2\alpha\gtrless 0.$$

7. The curves $\theta=0°$, 15°, and 30° of Fig. 6 (p. 59) do not touch the ϕ-axis.

For $\theta=0$ our general formula (47) would reduce to the following at any point on the ϕ-axis:

$$2(\cos\phi+\sin\phi)=\sqrt{6(1+\sin\phi\cos\phi)}\,; \quad \text{..........................}(a)$$

which would give

$$\sin 2\phi=2.$$

Since the *sine* of an angle cannot be greater than unity, it follows that there is no value ϕ that satisfies the given equation (a) (assumed), and hence that the given curve does not touch the ϕ-axis. We find similar equations, that cannot be satisfied, for $\theta = 15^\circ$ and $\theta = 30^\circ$.

8. In problem 1 show that the angle of oscillation of the secondary (electric) oscillations is given by

$$\cos(f_2, r) = \frac{2\alpha}{\sqrt{1+3\alpha^2}}; \quad \dots\dots\dots\dots\dots\dots\dots (a)$$

moreover, that the angle of oscillation of the (electric) oscillations represented by the terms of the third order of magnitude in $\frac{1}{r}$ is given by the same expression (a) with sign reversed.

9. Show that the electric oscillations of the third and fourth orders of magnitude in $\frac{1}{r}$ of problem 2 make one and the same angle of oscillation with their direction of propagation, namely

$$\cos(f_{3,4}, r) = \frac{3(\beta^2-\gamma^2)}{\sqrt{4(\beta^2+\gamma^2)+5(\beta^2-\gamma^2)^2}} = \frac{3\sin\phi\cos 2\theta}{\sqrt{4+5\sin^2\phi\cos^2 2\theta}}.$$

To find this expression, write the coefficients of the given component moments in the form

$$15\alpha\sqrt{(1+\alpha^2)(\beta^2+\gamma^2)-4\beta^2\gamma^2}, \quad 3\beta(5\alpha^2+10\gamma^2-3), \quad 3\gamma(5\alpha^2+10\beta^2-3).$$

Show that the oscillations in question, those represented by the terms of the third and fourth orders of magnitude in $\frac{1}{r}$, are transverse or longitudinal in the same regions, in which the secondary oscillations that are represented by the terms of the second order of magnitude in $\frac{1}{r}$ are transverse or longitudinal respectively.

10. In problem 3 show that the electric oscillations that are represented by the terms of the third order of magnitude in $\frac{1}{r}$ make the same angle of oscillation with their direction of propagation as the secondary electric oscillations that are represented by the terms of the second order of magnitude.

It thus follows that the secondary electric oscillations proper or those represented by the terms of all higher orders of magnitude in $\frac{1}{r}$ than the first are transverse or longitudinal in the same regions, in which the secondary electric oscillations that are represented alone by the terms of the second order of magnitude are transverse or longitudinal respectively. This law is quite general (cf. Exs. 8 and 9).

11. In problem 3 the electric and magnetic oscillations take place at right angles to each other—this has been proved on pp. 54-55 for only the approximate values of the moments. To confirm this law for the exact values of X, Y, Z and a, b, c, we evidently need prove the validity of only the two additional equations

$$X_3a_1 + Y_3b_1 + Z_3c_1 = 0$$

and

$$X_3a_2 + Y_3b_2 + Z_3c_2 = 0$$

(cf. p. 54). Since now the moments X_3, Y_3, Z_3 and X_2, Y_2, Z_2 have one and the same coefficients in a_1, a_2, a_3 and α, β, γ (cf. formulae (43)), and the moments a_2, b_2, c_2 and a_1, b_1, c_1 also (cf. formulae (44)), the validity of these two equations follows directly from those confirmed on pp. 54-55.

Moreover, since both $X_2a_1+Y_2b_1+Z_2c_1=0$

and $X_1a_2+Y_1b_2+Z_1c_2=0$,

it follows: The electric and magnetic oscillations of not only the same but different orders of magnitude in $\frac{1}{r}$ take place at right angles to each other. It would, therefore, be impossible to separate or pair off the electric and magnetic waves of the same order of magnitude by means of the property that they take place at right angles to each other.

12. In problem 1 the only region, in which primary (electric) and magnetic waves do not appear and the secondary (electric) wave becomes longitudinal, is the x-axis.

By the formulae on p. 35, the moments of the given primary wave are

$$X_1=\frac{\beta^2+\gamma^2}{r}\frac{d^2f}{dr^2},\quad Y_1=-\frac{\alpha\beta}{r}\frac{d^2f}{dr^2},\quad Z_1=-\frac{\alpha\gamma}{r}\frac{d^2f}{dr^2}\quad (D=1),\ldots\ldots\ldots\ldots(a)$$

and those of the secondary

$$X_2=\frac{2-3(\beta^2+\gamma^2)}{r^2}\frac{df}{dr},\quad Y_2=\frac{3\alpha\beta}{r^2}\frac{df}{dr},\quad Z_2=\frac{3\alpha\gamma}{r^2}\frac{df}{dr},\quad (D=1),\ldots\ldots\ldots\ldots(b)$$

where we have rejected the terms of the third order of magnitude in $\frac{1}{r}$.

The resultant moment of the primary wave is thus

$$\sqrt{X_1^2+Y_1^2+Z_1^2}=\sqrt{(\beta^2+\gamma^2)^2+\alpha^2\beta^2+\alpha^2\gamma^2}\,\frac{1}{r}\frac{d^2f}{dr^2}$$

$$=\sqrt{\beta^2+\gamma^2}\,\frac{1}{r}\frac{d^2f}{dr^2},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c)$$

which can vanish only when $\beta^2+\gamma^2=0$, hence $\alpha^2=1$, or the x-axis.

Replace λ_2, μ_2, ν_2 by their values from formulae (b) (cf. p. 43) in formulae (23) for $\cos(f_2, r)$, and we have

$$\cos(f_2, r)=\frac{2\alpha-3\alpha(\beta^2+\gamma^2)+3\alpha\beta^2+3\alpha\gamma^2}{\sqrt{4-12(\beta^2+\gamma^2)+9(\beta^2+\gamma^2)^2+9\alpha^2\beta^2+9\alpha^2\gamma^2}}$$

$$=\frac{2\alpha}{\sqrt{4-3(\beta^2+\gamma^2)}}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(d)$$

That these oscillations be longitudinal, we must evidently have

$$\pm1=\frac{2\alpha}{\sqrt{4-3(\beta^2+\gamma^2)}},$$

hence $4-3(\beta^2+\gamma^2)=4\alpha^2$, or $\alpha^2=1$, or the x-axis.

The resultant moment of the magnetic wave is

$$\sqrt{a^2+b^2+c^2}=\frac{v_0}{v^2}\frac{\sqrt{\beta^2+\gamma^2}}{r}\left(\frac{d^2f}{dt\,dr}-\frac{1}{r}\frac{df}{dt}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(e)$$

(cf. formulae for a, b, c on p. 33); that this moment vanish, we must have

$\beta^2+\gamma^2=0$, or the x-axis. Q.E.D.

13. In problem 1 the secondary (electric) wave is transverse throughout the yz-plane only, throughout which both its resultant moment X_2, Y_2, Z_2 and those of the primary (electric) and the magnetic waves are independent of the direction-cosines, that is, are constant for $r=const.$

By formula (d), Ex. 12, the given secondary oscillations are evidently transverse only, when $\alpha=0$, that is, throughout the yz-plane.

By formula (*b*), Ex. 12, the resultant moment X_2, Y_2, Z_2 is

$$\sqrt{X_2^2+Y_2^2+Z_2^2}=\sqrt{4-3(\beta^2+\gamma^2)}\,\frac{1}{r^2}\frac{df}{dr},$$

or throughout the *yz*-plane, where the secondary wave is transverse,

$$\sqrt{X_2^2+Y_2^2+Z_2^2}=\frac{1}{r}\frac{df}{dr}.$$

By formulae (*c*) and (*e*), Ex. 12, the resultant moments X_1, Y_1, Z_1 and a, b, c are evidently independent of the direction-cosines throughout the *yz*-plane. Q.E.D.

14. In problem 2 show that the secondary (electric) wave is longitudinal along the *y*- and *z*-axes only, whereas primary (electric) and magnetic waves fail to appear along all three coordinate-axes.

15. In problem 2 the secondary (electric) wave is transverse throughout the planes $\beta^2=\gamma^2$ (cf. p. 44); show that throughout these planes the resultant moment X_2, Y_2, Z_2 and those of the primary (electric) and magnetic waves are functions of the direction-cosines, that is, vary for $r=const.$; moreover, that the only regions, where the resultant moments X_1, Y_1, Z_1 and a, b, c are independent of the direction-cosines, are the *y*- and *z*-axes, along which the same vanish entirely.

16. The amplitude of the magnetic wave of problem 3 is independent of the direction-cosines throughout the plane

$$a_1\alpha+a_2\beta+a_3\gamma=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

By formulae (44) the resultant moment of the given wave is

$$\sqrt{a^2+b^2+c^2}=\frac{nv_0}{v}\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right)\sqrt{(a_2\gamma-a_3\beta)^2+(a_3\alpha-a_1\gamma)^2+(a_1\beta-a_2\alpha)^2}$$

$$=\frac{nv_0}{v}\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right)$$

$$\times\sqrt{a_1^2(\beta^2+\gamma^2)+a_2^2(\alpha^2+\gamma^2)+a_3^2(\alpha^2+\beta^2)-2(a_1a_2\alpha\beta+a_1a_3\alpha\gamma+a_2a_3\beta\gamma)}. \quad \ldots\ldots (b)$$

The condition (*a*) representing the plane, throughout which the given secondary electric wave is transverse, can now be replaced by the condition

$$(a_1\alpha+a_2\beta+a_3\gamma)^2=0,$$

or

$$a_1^2\alpha^2+a_2^2\beta^2+a_3^2\gamma^2+2(a_1a_2\alpha\beta+a_1a_3\alpha\gamma+a_2a_3\beta\gamma)=0\,;$$

by which the expression (*b*) for the resultant moment can be written

$$\sqrt{a^2+b^2+c^2}=\frac{nv_0}{v}\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right)$$

$$\times\sqrt{a_1^2(\beta^2+\gamma^2)+a_2^2(\alpha^2+\gamma^2)+a_3^2(\alpha^2+\beta^2)+a_1^2\alpha^2+a_2^2\beta^2+a_3^2\gamma^2},$$

or, since

$$\alpha^2+\beta^2+\gamma^2=1,$$

$$\sqrt{a^2+b^2+c^2}=\frac{nv_0}{v}\left(\frac{n}{r}\sin\omega-\frac{1}{r^2}\cos\omega\right)\sqrt{a_1^2+a_2^2+a_3^2},$$

which is independent of the direction-cosines.

17. Examine, in detail, problem 3 for the particular case, where

$$a_3=0 \quad (a_1 \gtreqless a_2).$$

18. Show, on replacing the functions f_1, f_2, f_3 of formulae (29) by

$$f_1 = a_1 \sin\frac{2\pi}{\lambda}(vt - r),$$

$$f_2 = a_2 \sin\frac{2\pi}{\lambda}(vt - r),$$

$$f_3 = a_3 \cos\frac{2\pi}{\lambda}(vt - r),$$

that the primary electric wave is represented by the moments

$$\frac{X_1}{D} = n^2[a_1(\beta^2+\gamma^2) - a_2\alpha\beta]\frac{\sin\omega}{r} - n^2 a_3\alpha\gamma\frac{\cos\omega}{r},$$

$$\frac{Y_1}{D} = n^2[a_2(\alpha^2+\gamma^2) - a_1\alpha\beta]\frac{\sin\omega}{r} - n^2 a_3\beta\gamma\frac{\cos\omega}{r},$$

$$\frac{Z_1}{D} = -n^2\gamma(a_1\alpha + a_2\beta)\frac{\sin\omega}{r} + n^2 a_3(\alpha^2+\beta^2)\frac{\cos\omega}{r},$$

the secondary electric wave by the moments

$$\frac{X_2}{D} = -3na_3\alpha\gamma\frac{\sin\omega}{r^2} + n[2a_1 - 3a_1(\beta^2+\gamma^2) + 3a_2\alpha\beta]\frac{\cos\omega}{r^2},$$

$$\frac{Y_2}{D} = -3na_3\beta\gamma\frac{\sin\omega}{r^2} + n[2a_2 - 3a_2(\alpha^2+\gamma^2) + 3a_1\alpha\beta]\frac{\cos\omega}{r^2},$$

$$\frac{Z_2}{D} = na_3[3(\alpha^2+\beta^2) - 2]\frac{\sin\omega}{r} + 3n\gamma(a_1\alpha + a_2\beta)\frac{\cos\omega}{r^2},$$

and the magnetic wave by the moments

$$a = -\frac{v_0}{v}\left[\frac{n^2}{r}(a_3\beta\cos\omega - a_2\gamma\sin\omega) + \frac{n}{r^2}(a_2\gamma\cos\omega + a_3\beta\sin\omega)\right],$$

$$b = \frac{v_0}{v}\left[\frac{n^2}{r}(a_3\alpha\cos\omega - a_1\gamma\sin\omega) + \frac{n}{r^2}(a_1\gamma\cos\omega + a_3\alpha\sin\omega)\right]$$

$$c = \frac{v_0}{v}\left[\frac{n^2}{r}(a_1\beta - a_2\alpha)\sin\omega - \frac{n}{r^2}(a_1\beta - a_2\alpha)\cos\omega\right].$$

19. Show that for the electromagnetic waves of Ex. 18 the analytic relation holds

$$(X_1 + X_2)a + (Y_1 + Y_2)b + (Z_1 + Z_2)c = 0,$$

or the resultant electric and magnetic moments are always at right angles to each other.

20. Show that the primary (electric) oscillations of Ex. 18 take place in planes that are at right angles to the direction of propagation, and that the angle between the vector f_2 of any element of the secondary (electric) oscillations of the same and the direction of propagation is for the particular case, where $a_1 = a_2 = a_3$, determined by the formula

$$\cos^2(f_2, r) = \frac{4[-\gamma\sin\omega + (\alpha+\beta)\cos\omega]^2}{(1+3\gamma^2)\sin^2\omega + 18\alpha\gamma\sin\omega\cos\omega + [2+6\alpha\beta+3(\alpha^2+\beta^2)]\cos^2\omega},$$

or in polars

$$\cos^2(f_2, r) = \frac{8[-\sin\phi\sin\theta\sin\omega + (\cos\phi + \sin\phi\cos\theta)\cos\omega]^2}{\begin{array}{l}2(1+3\sin^2\phi\sin^2\theta)\sin^2\omega + 9\sin 2\phi\sin\theta\sin 2\omega \\ \qquad + 2[2+3(\cos^2\phi + \sin 2\phi\cos\theta + \sin^2\phi\cos^2\theta)]\cos^2\omega\end{array}}.$$

21. Show that there is no vector, along which the magnetic moments of Ex. 18 vanish, provided $a_3 \gtrless 0$; and that for $a_3 = 0$ they vanish along the vectors

$$\alpha = \pm \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \quad \beta = \pm \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, \quad \gamma = 0;$$

along which the electric moments assume the form

$$X_1 = Y_1 = Z_1 = 0,$$

and

$$X_2 = 2na_1 \frac{\cos \omega}{r^2} + 2a_1 \frac{\sin \omega}{r^3},$$

$$Y_2 = 2na_2 \frac{\cos \omega}{r^2} + 2a_2 \frac{\sin \omega}{r^3},$$

$$Z_2 = 0. \qquad (D = 1.)$$

22. Examine, in detail, the electromagnetic waves of Ex. 18 for the particular case, where $a_2 = 0$.

CHAPTER III.

LINEARLY, CIRCULARLY, AND ELLIPTICALLY POLARIZED OSCILLATIONS; GENERAL PROBLEM OF ELLIPTICALLY POLARIZED ELECTROMAGNETIC OSCILLATIONS.

Different Kinds of Light.—In the foregoing chapters we have examined certain periodic oscillations of the ether without attempting to identify them directly with what we call "light"; still, we recognize, if light is to be regarded as an electromagnetic phenomenon, it has already been identified with that periodic state of the ether, where two kinds of transverse oscillations, known as the electric and magnetic, which are closely allied to each other (cf. Chapter II.), are taking place (at right angles to each other). Whether the light-wave is to be regarded as a particular kind* of *electric* or *magnetic* disturbance is a matter of little consequence. Likewise, no attempt was made in Chapters I. and II. to discriminate between the different kinds of light. The first distinction to be made is that between ordinary and so-called "polarized" light.

Polarized Light.—A ray of light is termed "polarized" when its behaviour is not one and the same round its direction of propagation, circularly polarized light excepted†; the (extraordinary) ray that emerges from a plate of tourmaline and passes through a second such plate is known to vary in intensity, as we rotate the latter (plate) round the ray (its direction of propagation); the ray emerging from the first plate is thus termed "polarized." Or, to express ourselves analytically, we call a ray "polarized" when its wave-front elements describe similar and similarly situated paths (during given finite intervals); if the paths described are parallel straight lines, the ray is termed "linearly" polarized; if the given paths are similar and similarly situated ellipses or circles, the ray is then said to be "elliptically" or "circularly" polarized.

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* Oscillations of very short wave-lengths, those of light waves.

† Although circularly polarized light exhibits the same properties round its direction of propagation, it differs materially from ordinary light, as Chapter VIII. on the behaviour of light in crystals will show.

One of the simplest kinds of linearly polarized oscillations or waves can be represented by equations of the form

$$y = a \sin \frac{2\pi}{\lambda}(vt - x), \quad \text{..........................} (1)$$

which represents an infinite succession of similar changes or vibrations in a given (the xy-) plane. Equations of this form may be used to represent linearly polarized light.

Ordinary Light.—When the behaviour of a ray is one and the same round its direction of propagation, or, more strictly, when the particles or elements of its wave-front describe quite arbitrary paths or similar and similarly situated paths for only infinitely short intervals, the ray (light) is termed "ordinary." We can thus imagine any particle of an *ordinary* light ray as oscillating for an infinitely short time in any given path, for example, in a straight line, in the next interval in another path, a flat ellipse, then in a circle, and so on, and assume the number of such changes in polarization during the (finite) interval required for light to impart an impression on the retina of the eye to be so large that the mean of the displacements in any and every direction (at right angles to the direction of propagation) during that interval becomes approximately one and the same. This conception of ordinary light not only explains the empirical fact that a ray of ordinary light shows one and the same behaviour round its direction of propagation, but it also agrees with the observations made by Michelson,* that a change of polarization is possible after the elapse of 540,000 vibrations, which would correspond to thousands of changes in polarization during the interval required for an impression of light on the retina of the eye. Moreover, the given conception will enable us to explain certain empirical laws on the interference of polarized and ordinary light (cf. Chapter IV.).

Homogeneous waves are those of one and the same wave-length (colour) or period of oscillation and *heterogeneous* those of different wave-lengths (colours) or periods of oscillation; when the different wave-lengths are equally represented in the given waves, we have waves of so-called "*white*" light.

Plane of Polarization.—The methods for obtaining polarized from ordinary light are familiar to us all; of these that by reflection is of special interest on account of the terminology used. We know, namely from experiment, that there is one angle of incidence, the so-called "angle of polarization" or the "polarizing angle," for which ordinary light upon falling on certain bodies, as a glass mirror, is reflected as linearly polarized light. If we now let this linearly polarized light fall at the polarizing angle upon a second mirror, the intensity of the reflected light will be found to depend upon the angle the plane of incidence chosen makes with the first plane of incidence; namely the smaller this angle the greater the intensity, and the nearer this angle approaches a right angle the smaller the intensity. That particular plane of incidence, in which the light is most copiously reflected, is now known as the "plane of polarization"; this plane is evidently the plane of incidence or reflection of the polarizing surface or first mirror. Since now the oscillations reflected from the polarizing surface or the first mirror evidently take place in some particular plane, as an examination of them by the polariscope will show, it is natural to assume some characteristic plane as plane of oscillation; this would naturally be either the plane of polarization or that at right angles to it. In the elastic theory of light it is a pure matter of taste, which of these planes be chosen as plane of oscillation; Fresnel assumes that the light oscillations take place at right angles to the plane of polarization, and Neumann in the plane of polarization. In this respect the electromagnetic theory of light differs materially from the elastic; the former demands *two* just such characteristic planes (at right angles to each other), the one for the electric and the other for the magnetic oscillations; which one of these, the plane of polarization or that at right angles to it, be the plane of (electric) oscillation, is also apparently a matter of choice; this is not, however, the case, as the chapter on the behaviour of light in crystals will show; we shall find, namely, that the electric oscillations take place at right angles to and the magnetic ones in the plane of polarization.

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* A. A. Michelson: *American Journal of Science*, vol. xxxiv. p. 427, 1887.

Elliptically Polarized Oscillations.—We know from experiment that it is possible to obtain other kinds of polarized light than the linearly polarized, also that the most general form of polarization is the elliptic. This suggests the supposition, that an elliptically polarized oscillation be identical to two linearly polarized oscillations of the same period of oscillation, but of different amplitudes and phases, that are taking place at right angles to each other; this is only another or somewhat more general form of the principle of the resolution and composition of forces or displacements. Let us examine the resultant of two such oscillations, for example, the two rectangular (linearly polarized) periodic oscillations

$$\left.\begin{aligned} x &= a_1 \sin n\,[vt - (z + \delta_1)] \\ y &= a_2 \sin n\,[vt - (z + \delta_2)] \quad \left(n = \frac{2\pi}{\lambda}\right) \end{aligned}\right\}, \text{ ....(2)}$$

their planes of oscillation being the xz- and yz-planes, and the direction of propagation the z-axis, where $\delta_1 - \delta_2$ denotes their difference in phase. To find the path described by any element under the simultaneous action of these two displacements (oscillations), we must eliminate the time t from the two equations (2). For this purpose we write the same explicitly

$$x = a_1 \sin n(vt - z) \cos n\delta_1 - a_1 \cos n(vt - z) \sin n\delta_1$$

$$y = a_2 \sin n(vt - z) \cos n\delta_2 - a_2 \cos n(vt - z) \sin n\delta_2\,;$$

which give

$$\frac{x}{a_1} \cos n\delta_2 - \frac{y}{a_2} \cos n\delta_1 = -\cos n(vt - z) \sin n(\delta_1 - \delta_2)$$

and

$$\frac{x}{a_1} \sin n\delta_2 - \frac{y}{a_2} \sin n\delta_1 = -\sin n(vt - z) \sin n(\delta_1 - \delta_2)\,;$$

and these, squared and added,

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2\frac{xy}{a_1 a_2} \cos n(\delta_1 - \delta_2) = \sin^2 n(\delta_1 - \delta_2). \text{ ....(3)}$$

This is the equation of an elliptic cylinder (cf. Ex. 22), whose infinitely long axis is the z-axis. The path of oscillation of any particle of the wave represented by formulae (2) is evidently the ellipse intersected by this cylinder on the plane $z = a$, where a denotes the distance of that particle from the origin. It thus follows that two linearly polarized oscillations of the form (2) compound to an elliptically polarized oscillation.

Mode of Propagation of Elliptic Oscillations.—To form a conception of an elliptically polarized wave, we choose its direction of propagation as axis of an elliptic cylinder, and imagine a wire wound loosely round that cylinder; the spiral described by the wire would represent an elliptically polarized wave at any given time, and the uniform displacement of that spiral along the surface of the cylinder in the direction of its axis, the manner in which that wave were propagated. For a circularly polarized wave the elliptic cylinder would have to be replaced by a circular one.

Circularly Polarized Oscillations.—Let us examine the analytic equation (3). That the oscillation (polarization) (3) be circular, the following conditions must evidently be satisfied:

$$a_1 = a_2,$$

or the amplitudes of the given oscillations (2) are the same, and

$$\cos n(\delta_1 - \delta_2) = 0, \quad \text{hence} \quad n(\delta_1 - \delta_2) = \frac{\pi}{2} \text{ or } \frac{3\pi}{2},$$

or, since

$$n = \frac{2\pi}{\lambda}, \quad \delta_1 - \delta_2 = \frac{\lambda}{4} \text{ or } \frac{3\lambda}{4},$$

or the oscillations differ in phase by quarter of a wave-length.

We must, however, discriminate here between the two cases

$$\delta_1 - \delta_2 = \frac{\lambda}{4} \quad \text{and} \quad \delta_1 - \delta_2 = \frac{3\lambda}{4};$$

in the former the circularly polarized oscillations are evidently represented by equations of the form

$$x = a \sin n\,[vt - (z + \delta_1)],$$

$$y = a \sin n\left[vt - \left(z + \delta_1 - \frac{\lambda}{4}\right)\right] = a \cos n\,[vt - (z + \delta_1)],$$

and in the latter by

$$x = a \sin n\,[vt - (z + \delta_1)],$$

$$y = a \sin n\left[vt - \left(z + \delta_1 - \frac{3\lambda}{4}\right)\right] = -a \cos n\,[vt - (z + \delta_1)].$$

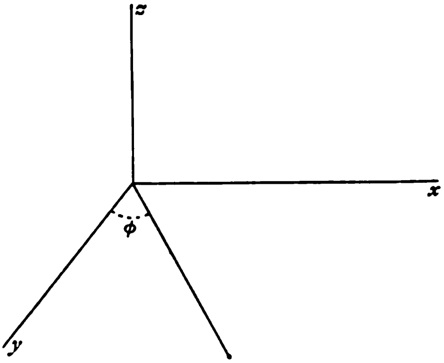

FIG. 8.

Right and Left-handed Circular (Elliptic) Oscillations.—The difference between the two above circular oscillations or waves becomes apparent upon the determination of their so-called "azimuths"; the azimuth is the angle ϕ (cf. the above figure), which the vector from the position of rest of any given element (particle) to any point of the path described by the same makes with any such fixed vector, as the y-axis. Let us denote the azimuths of the two oscillations in question by ϕ_1 and ϕ_2 respectively, measuring the same from the y-axis, as indicated in figure 8; we have then

$$\phi_1 = \arctan\frac{x}{y} = n\,[vt - (z + \delta_1)], \quad \phi_2 = -n\,[vt - (z + \delta_1)].$$

As t increases, ϕ_1 increases and ϕ_2 decreases. For an observer at any point on the positive z-axis, along which the given waves are advancing, ϕ_1 is thus rotating from right to *left* and ϕ_2 from left to *right*; the former is, therefore, known as a "left-handed" and the latter as a "right-handed" circular oscillation. The same distinction is, of course, to be made between the elliptic oscillations.

Linearly Polarized Oscillations.—That the resultant of two rectangular linear oscillations remain linear, $\sin n(\delta_1 - \delta_2)$ of formula (3) must evidently vanish; that is,

$$n(\delta_1 - \delta_2) = 0 \text{ or } \pi,$$

hence

$$\delta_1 - \delta_2 = 0 \text{ or } \frac{\lambda}{2}.$$

In which case formula (3) reduces to

$$\frac{x^2}{a_1{}^2} + \frac{y^2}{a_2{}^2} \mp \frac{2xy}{a_1 a_2} = 0,$$

hence

$$\frac{x}{a_1} - \frac{y}{a_2} = 0$$

or

$$\frac{x}{a_1} + \frac{y}{a_2} = 0$$

respectively; which are the equations of straight lines.

The component rectangular oscillations sought are, therefore,

$$x = a_1 \sin n[vt - (z + \delta_2)],$$
$$y = a_2 \sin n[vt - (z + \delta_2)], \text{ for } \delta_1 - \delta_2 = 0,$$

and

$$x = -a_1 \sin n[vt - (z + \delta_2)],$$
$$y = a_2 \sin n[vt - (z + \delta_2)], \text{ for } \delta_1 - \delta_2 = \frac{\lambda}{2};$$

hence

$$\tan \phi = \pm \frac{a_1}{a_2},$$

or the azimuth is constant, that is, the resultant oscillation is linear in both cases,

$$\delta_1 - \delta_2 = 0 \text{ and } \frac{\lambda}{2}.$$

The resultant amplitude is in both cases

$$\sqrt{x^2 + y^2} = \sqrt{a_1{}^2 + a_2{}^2} \sin n[vt - (z + \delta_2)].$$

The Elliptic Polarization the most general.—We have seen on pp. 74-75 that two rectangular oscillations of the form (2) compound to an elliptic oscillation. Let us next show that the path described by any particle under the simultaneous action of three rectangular (linear) periodic oscillations of different amplitudes and phases, but of the same period of oscillation, is an ellipse, that is, that the most general form of polarization, obtained from the composition of rectangular (linear) oscillations, is the elliptic. Three such rectangular oscillations are

$$\left.\begin{aligned} x &= a_1 \sin n(vt - \delta_1) \\ y &= a_2 \sin n(vt - \delta_2) \\ z &= a_3 \sin n(vt - \delta_3) \end{aligned}\right\} . \qquad (4)$$

To determine the path described by any particle under the simultaneous action of these three oscillations, we must eliminate the time t from the same. We first write the given expressions (4) explicitly, namely,

$$\left.\begin{aligned} x &= a_1 \sin nvt \cos n\delta_1 - a_1 \cos nvt \sin n\delta_1 \\ y &= a_2 \sin nvt \cos n\delta_2 - a_2 \cos nvt \sin n\delta_2 \\ z &= a_3 \sin nvt \cos n\delta_3 - a_3 \cos nvt \sin n\delta_3 \end{aligned}\right\} , \qquad (4\text{A})$$

multiply the first by $\frac{\sin n(\delta_2 - \delta_3)}{a_1}$, the second by $\frac{\sin n(\delta_3 - \delta_1)}{a_2}$ and the third by $\frac{\sin n(\delta_1 - \delta_2)}{a_3}$, add, and we have

$$\left.\begin{aligned} &\frac{\sin n(\delta_2 - \delta_3)}{a_1} x + \frac{\sin n(\delta_3 - \delta_1)}{a_2} y + \frac{\sin n(\delta_1 - \delta_2)}{a_3} z \\ &= (\sin nvt \cos n\delta_1 - \cos nvt \sin n\delta_1)(\sin n\delta_2 \cos n\delta_3 - \cos n\delta_2 \sin n\delta_3) \\ &+ (\sin nvt \cos n\delta_2 - \cos nvt \sin n\delta_2)(\sin n\delta_3 \cos n\delta_1 - \cos n\delta_3 \sin n\delta_1) \\ &+ (\sin nvt \cos n\delta_3 - \cos nvt \sin n\delta_3)(\sin n\delta_1 \cos n\delta_2 - \cos n\delta_1 \sin n\delta_2) \\ &= 0 \end{aligned}\right\} ; \ldots (5)$$

that is, since a *linear* equation holds between the three variables x, y, z, the path of oscillation of the given particle will lie in a plane, the one determined by that (linear) equation.

To determine the path described in the plane of oscillation (5), we seek relations between the different pairs of the three variables, x, y, z, which will give the projections of the path of oscillation on the coordinate-planes.

The first two equations (4A) give

$$\frac{\sin n\delta_2}{a_1} x - \frac{\sin n\delta_1}{a_2} y = -\sin nvt \sin n(\delta_1 - \delta_2)$$

and

$$\frac{\cos n\delta_2}{a_1} x - \frac{\cos n\delta_1}{a_2} y = -\cos nvt \sin n(\delta_1 - \delta_2) ;$$

and these, squared and added,

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2 \cos n(\delta_1 - \delta_2) \frac{xy}{a_1 a_2} = \sin^2 n(\delta_1 - \delta_2),$$

rith similar equations in x, z, and y, z, which is the equation of an ellipse (cf. Ex. 22 at end of chapter). The projections of the path of oscillation on the coordinate planes are, therefore, ellipses, that is, the path of oscillation itself is an ellipse (in the plane of oscillation (5)). The rectangular oscillations (4) thus represent an elliptic oscillation, and hence conversely the elliptic oscillation (polarization) is the most general form of oscillation (polarization), as maintained above.

The Electromagnetic Waves of Chapter II.—The oscillations just examined represent fundamental types of polarized wave-motion; they are, in the strictest sense, polarized oscillations. Electromagnetic waves, like light waves, may be either polarized or not; those examined in the preceding chapter are not, strictly speaking, polarized, except at infinite distance from their source. At greater distances from its source any such electromagnetic wave or ray may now be regarded as polarized, since the paths described by the different elements of its wave-front remain approximately similar and similarly situated (during finite intervals). Although the disturbances treated in the preceding chapter are not, in the strictest sense, polarized, it is, nevertheless, of interest to examine the paths described by the elements of given rays of the same; we shall find that they are linear.

The primary (electric) oscillations of Problem 1, Chapter II., are represented by the moments

$$X_1 = \frac{\beta^2+\gamma^2}{r}\frac{d^2f}{dr^2}, \quad Y_1 = -\frac{\alpha\beta}{r}\frac{d^2f}{dr^2}, \quad Z_1 = -\frac{\alpha\gamma}{r}\frac{d^2f}{dr^2},$$

where $D=1$. To find the path described by any element (at any given point), we eliminate the time t or $\frac{d^2f}{dr^2}$ from these formulae, and we have

$$X_1 : Y_1 : Z_1 = \beta^2+\gamma^2 : -\alpha\beta : -\alpha\gamma,$$

which is the equation of a straight line; for different values of α, β, γ, the direction of this line evidently changes.

The secondary (electric) oscillations of Problem 1, Chapter II., are represented by the moments

$$X_2 = \frac{2-3(\beta^2+\gamma^2)}{r^2}\frac{df}{dr}, \quad Y_2 = \frac{3\alpha\beta}{r^2}\frac{df}{dr}, \quad Z_2 = \frac{3\alpha\gamma}{r^2}\frac{df}{dr}, \quad (D=1),$$

which give $X_2 : Y_2 : Z_2 = [2-3(\beta^2+\gamma^2)] : 3\alpha\beta : 3\alpha\gamma,$

that is, these oscillations take place along the lines determined by this proportion.

Similarly, we find that the primary and secondary (electric) oscillations of Problems 2 and 3, Chapter II., also take place along lines determined by similar proportions.

We observe that the oscillations examined in the preceding chapter all take place along straight lines; moreover, that the directions of these lines of oscillation are functions alone of the direction-cosines α, β, γ. In any region situated at a distance r from the source of the disturbance, that is large in comparison to the dimensions of that region, the directions of oscillation of the given waves would, therefore, be approximately parallel; that is, at any point at considerable distance from the source any ray or pencil of rays could be regarded as approximately linearly polarized. We observe, moreover, that the direction-cosines appear in the above expressions for the determination of the directions of oscillation, not in the first, but in the second and third powers; this will evidently correspond to a more complete polarization in distant regions.

More General Problem; Elliptically Polarized Electromagnetic Oscillations. A most general case of an electromagnetic disturbance in elliptic paths can be obtained, if we somewhat generalize Problem 3 of the preceding chapter; let the auxiliary functions U, V, W be the same functions of the purely spherical wave-functions ϕ_1, ϕ_2, ϕ_3 as in Problem 3, but let the functions f_1, f_2, f_3, which differed there from one another only in amplitude, differ here also in phase; namely let

$$\left.\begin{aligned} f_1 &= a_1 \sin \omega_1 = a_1 \sin n\left[vt - (r + \delta_1)\right] \\ f_2 &= a_2 \sin \omega_2 = a_2 \sin n\left[vt - (r + \delta_2)\right] \\ f_3 &= a_3 \sin \omega_3 = a_3 \sin n\left[vt - (r + \delta_3)\right] \end{aligned}\right\} . \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

We replace f_1, f_2, f_3 by these functions in formulae (28, II) for U, V, W, and we have

$$U = \frac{n}{r}(a_2\gamma \cos \omega_2 - a_3\beta \cos \omega_3) + \frac{1}{r^2}(a_2\gamma \sin \omega_2 - a_3\beta \sin \omega_3),$$

$$V = \frac{n}{r}(a_3\alpha \cos \omega_3 - a_1\gamma \cos \omega_1) + \frac{1}{r^2}(a_3\alpha \sin \omega_3 - a_1\gamma \sin \omega_1),$$

$$W = \frac{n}{r}(a_1\beta \cos \omega_1 - a_2\alpha \cos \omega_2) + \frac{1}{r^2}(a_1\beta \sin \omega_1 - a_2\alpha \sin \omega_2).$$

We then replace U, V, W by these values in formulae (5, II), and we find

$$\left.\begin{aligned} \frac{X}{D} = \frac{n^2}{r} & \left[a_1(\beta^2 + \gamma^2) \sin \omega_1 - a_2\alpha\beta \sin \omega_2 - a_3\alpha\gamma \sin \omega_3\right] \\ & + \frac{n}{r^2}\{\left[2a_1 - 3a_1(\beta^2 + \gamma^2)\right] \cos \omega_1 + 3a_2\alpha\beta \cos \omega_2 + 3a_3\alpha\gamma \cos \omega_3\} \\ & + \frac{1}{r^3}\{\left[2a_1 - 3a_1(\beta^2 + \gamma^2)\right] \sin \omega_1 + 3a_2\alpha\beta \sin \omega_2 + 3a_3\alpha\gamma \sin \omega_3\} \end{aligned}\right\} \quad (7)$$

and similar expressions for Y and Z. The different terms of these expressions for the moments represent simple waves; these waves evidently interfere with one another (cf. Chapter IV.), and may give rise to phenomena of interference. But here we are considering only the resultants of these simple waves, the compound waves themselves, and not the phenomena due to the interference of the former.

Electric and Magnetic Moments at $\perp$ to each other.—It is easy to show that the resultant electric moment X, Y, Z of formulae (7) and the resultant moment a, b, c of the magnetic disturbance accompanying the given electric one always stand at right angles to each other (cf. Ex. 13). This is the form which the law for linearly polarized oscillations (cf. p. 54) assumes for elliptically polarized ones.

The Primary Wave.—We conceive the path described by any element of any given wave-front of the disturbance represented by formulae (7) as the resultant of the paths described by that element due to the passage of the waves represented by the terms of the different orders of magnitude in $1/r$. We shall, first, examine the path described by any element due to the passage of the primary wave; but, beforehand, let us call attention to a property of the primary wave that will be of service to us in the examination of the path described by any element of the same.

The Vector X_1, Y_1, Z_1 at $\perp$ to Direction of Propagation.—The primary (electric) wave is represented by the moments

$$\left.\begin{aligned} X_1 &= \frac{n^2}{r}\left[a_1(\beta^2+\gamma^2)\sin\omega_1 - a_2\alpha\beta\sin\omega_2 - a_3\alpha\gamma\sin\omega_3\right] \\ Y_1 &= \frac{n^2}{r}\left[a_2(\alpha^2+\gamma^2)\sin\omega_2 - a_3\beta\gamma\sin\omega_3 - a_1\alpha\beta\sin\omega_1\right] \\ Z_1 &= \frac{n^2}{r}\left[a_3(\alpha^2+\beta^2)\sin\omega_3 - a_1\alpha\gamma\sin\omega_1 - a_2\beta\gamma\sin\omega_2\right], \quad (D=1) \end{aligned}\right\} \quad \ldots\ldots(8)$$

It is now evident from the form of these expressions that the resultant moment X_1, Y_1, Z_1 always stands at right angles to the direction of propagation of the wave represented by the same; that is, the primary oscillations take place in planes at right angles to their direction of propagation. For replace X_1, Y_1, Z_1 by these values in formula (23A, II) for the angle $(f_1$[*]$, r)$, and we find

$$\cos(f_1, r) = 0, \quad \text{hence} \quad (f_1, r) = \frac{\pi}{2}$$

(cf. also Ex. 17 at end of chapter).

The Path of Oscillation.—To obtain the path described by any element of the primary wave represented by formulae (8), we must eliminate the time t from those equations (8); for this purpose we write the same in the explicit form

---

[*] This vector (moment) f_1 is not to be confounded with the wave-function f_1 of formulae (6).

$$\left.\begin{aligned}\frac{X_1 r}{n^2} = a_1(\beta^2+\gamma^2)\,[\sin n\,(vt-r)\cos n\delta_1 - \cos n\,(vt-r)\sin n\delta_1] \\ - a_2\alpha\beta\,[\sin n\,(vt-r)\cos n\delta_2 - \cos n\,(vt-r)\sin n\delta_2] \\ - a_3\alpha\gamma\,[\sin n\,(vt-r)\cos n\delta_3 - \cos n\,(vt-r)\sin n\delta_3]\end{aligned}\right\}, \;\ldots\text{.(9)}$$

with similar expressions for Y_1 and Z_1, and eliminate first the $\sin n\,(vt-r)$ and then the $\cos n\,(vt-r)$ from any two of the same; the elimination of the former function from the first two equations gives

$$A_2\frac{X_1 r}{n^2} - A_1\frac{Y_1 r}{n^2} = (A_1B_2 - A_2B_1)\cos\tau, \;\ldots\ldots\ldots\ldots\ldots\text{.(10)}$$

where

$$\left.\begin{aligned}A_1 &= a_1(\beta^2+\gamma^2)\cos n\delta_1 - a_2\alpha\beta\cos n\delta_2 - a_3\alpha\gamma\cos n\delta_3 \\ B_1 &= a_1(\beta^2+\gamma^2)\sin n\delta_1 - a_2\alpha\beta\sin n\delta_2 - a_3\alpha\gamma\sin n\delta_3 \\ A_2 &= a_2(\alpha^2+\gamma^2)\cos n\delta_2 - a_3\beta\gamma\cos n\delta_3 - a_1\alpha\beta\cos n\delta_1 \\ B_2 &= a_2(\alpha^2+\gamma^2)\sin n\delta_2 - a_3\beta\gamma\sin n\delta_3 - a_1\alpha\beta\sin n\delta_1\end{aligned}\right\}, \;\ldots\ldots\text{.(11)}$$

and

$$\tau = n\,(vt-r);\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(12)}$$

and the elimination of $\cos n\,(vt-r)$ from the same two equations

$$B_2\frac{X_1 r}{n^2} - B_1\frac{Y_1 r}{n^2} = (A_1B_2 - A_2B_1)\sin\tau. \;\ldots\ldots\ldots\ldots\ldots\text{.(13)}$$

We, next, eliminate the function τ from equations (10) and (13); for this purpose we square the same, add, and we have

$$(A_2^2+B_2^2)\frac{X_1^2r^2}{n^4} - 2(A_1A_2+B_1B_2)\frac{X_1Y_1r^2}{n^4} + (A_1^2+B_1^2)\frac{Y_1^2r^2}{n^4}$$
$$= (A_1B_2 - A_2B_1)^2. \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{.(14)}$$

Upon evaluating the coefficients of this equation, we find

$$\left.\begin{aligned}A_2^2+B_2^2 &= a_1^2\alpha^2\beta^2 + a_2^2(\alpha^2+\gamma^2)^2 + a_3^2\beta^2\gamma^2 \\ &\quad - 2a_1a_2\alpha\beta(\alpha^2+\gamma^2)\cos n(\delta_1-\delta_2) + 2a_1a_3\alpha\beta^2\gamma\cos n(\delta_1-\delta_3) \\ &\quad - 2a_2a_3\beta\gamma(\alpha^2+\gamma^2)\cos n(\delta_2-\delta_3) \\ A_1^2+B_1^2 &= a_1^2(\beta^2+\gamma^2)^2 + a_2^2\alpha^2\beta^2 + a_3^2\alpha^2\gamma^2 \\ &\quad - 2a_1a_2\alpha\beta(\beta^2+\gamma^2)\cos n(\delta_1-\delta_2) - 2a_1a_3\alpha\gamma(\beta^2+\gamma^2)\cos n(\delta_1-\delta_3) \\ &\quad + 2a_2a_3\alpha^2\beta\gamma\cos n(\delta_2-\delta_3) \\ A_1A_2+B_1B_2 &= -a_1^2\alpha\beta(\beta^2+\gamma^2) - a_2^2\alpha\beta(\alpha^2+\gamma^2) + a_3^2\alpha\beta\gamma^2 \\ &\quad + a_1a_2(2\alpha^2\beta^2+\gamma^2)\cos n(\delta_1-\delta_2) + a_1a_3\beta\gamma(2\alpha^2-1)\cos n(\delta_1-\delta_3) \\ &\quad + a_2a_3\alpha\gamma(2\beta^2-1)\cos n(\delta_2-\delta_3) \\ \text{and}\quad (A_1B_2-A_2B_1)^2 &= [-a_1a_2\gamma^2\sin n(\delta_1-\delta_2) + a_1a_3\beta\gamma\sin n(\delta_1-\delta_3) \\ &\quad - a_2a_3\alpha\gamma\sin n(\delta_2-\delta_3)]^2\end{aligned}\right\}\text{.(15)}$$

Equation (14) is that of a cylindrical surface parallel to the z-axis. It will thus suffice to determine the curve intersected by the same on the xy-plane.

The Conic (14) an Ellipse.—As equation (14) is of the second degree, the corresponding curve will be a conic. To determine the particular conic in question, we make use of the well-known properties peculiar to the same. The general equation of a conic can be written

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0. \qquad \text{(16)}$$

Upon comparing equation (14) with this one, we observe that the coefficients A, B, ... of the former assume here the particular form

$$\left.\begin{array}{ll} A=(A_2{}^2+B_2{}^2)\dfrac{r^2}{n^4}, & B=-(A_1A_2+B_1B_2)\dfrac{r^2}{n^4} \\ C=(A_1{}^2+B_1{}^2)\dfrac{r^2}{n^4}, \quad D=E=0, & F=-(A_1B_2-A_2B_1)^2 \end{array}\right\} \qquad \text{(17)}$$

We, first, evaluate the determinates

$$\alpha=\begin{vmatrix} A & B \\ B & C \end{vmatrix}^{*} \text{ and } \Delta=\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

of the given conic (14): we replace here A, B, ... by their values (17), and we have

$$\alpha=[(A_2{}^2+B_2{}^2)^2-(A_1A_2+B_1B_2)^2]\frac{r^4}{n^8}=(A_1B_2-A_2B_1)^2\frac{r^4}{n^8}$$

and $\quad \Delta=(AC-B^2)F=\alpha F=-(A_1B_2-A_2B_1)^4\dfrac{r^4}{n^8};$

hence $\qquad \alpha>0 \text{ and } \Delta<0, \qquad \text{(18)}$

except in the particular case where

$$A_1B_2=A_2B_1. \qquad \text{(19)}$$

The conditions (18) do not suffice for the determination of the conic in question; we must also know the value or sign of the quotient $\dfrac{\Delta}{A}$; we have

$$\frac{\Delta}{A}=-\frac{(A_1B_2-A_2B_1)^4}{A_2{}^2+B_2{}^2}\frac{r^2}{n^4}, \text{ hence } \frac{\Delta}{A}<0, \qquad \text{(20)}$$

except where $\qquad A_1B_2=A_2B_1.$

Equation (14) is now determined uniquely as that of an ellipse by the conditions (18) and (20).

The Particular Case $A_1B_2=A_2B_1$; here the determinates α and Δ vanish, and the conic in question is determined by the values of the coefficients A and C and the determinates

$$\beta=\begin{vmatrix} A & D \\ D & F \end{vmatrix} \text{ and } \gamma=\begin{vmatrix} C & E \\ E & F \end{vmatrix}.^{\dagger}$$

---

[*] This determinate α is not to be confounded with the direction-cosine α.

[†] These determinates are not to be confounded with the direction-cosines β and γ.

We replace here A, B, ... by their values (17) and (19), and we have

$$A=(A_2^2+B_2^2)\frac{r^2}{n^4}>0, \quad C=(A_1^2+B_1^2)\frac{r^2}{n^4}>0,$$

and $$\beta=\gamma=0;$$

by which conditions equation (14), where $A_1B_2=A_2B_1$, is determined as that of a double straight line. To confirm this, replace A_1 by its value $\frac{A_2B_1}{B_2}$ from formula (19) in equation (14), and we have

$$(A_2^2+B_2^2)\frac{X_1^2r^2}{n^4}-\frac{2B_1}{B_2}(A_2^2+B_2^2)\frac{X_1Y_1r^2}{n^4}+\frac{B_1^2}{B_2^2}(A_2^2+B_2^2)\frac{Y_1^2r^2}{n^4}=0,$$

hence $$\left(B_2\frac{X_1r}{n^2}-B_1\frac{Y_1r}{n^2}\right)^2=0. \quad \text{.........................(21)}$$

It thus follows that the cylinder represented by equation (14) intersects the xy-plane in an ellipse, except, where $A_1B_2=A_2B_1$, when the given ellipse contracts to a double straight line.

To interpret the condition (19), we recall the last of formulae (15), by which we can write the same in the form

$$a_1a_2\gamma^2\sin n(\delta_1-\delta_2)-a_1a_3\beta\gamma\sin n(\delta_1-\delta_3)+a_2a_3\alpha\gamma\sin n(\delta_2-\delta_3)=0. \quad (22)$$

This can be replaced by the *two* conditions

$$\gamma=0$$

or

$$a_1a_2\gamma\sin n(\delta_1-\delta_2)-a_1a_3\beta\sin n(\delta_1-\delta_3)+a_2a_3\alpha\sin n(\delta_2-\delta_3)=0. \quad (22\text{A})$$

The xy-plane is defined by the former, and a plane, passing through the origin and making angles with the coordinate-axes, that are determined as functions of the quantities a_1, a_2, a_3 and δ_1, δ_2, δ_3, which are given, by the latter condition (cf. Ex. 12).

Path of Oscillation determined by Intersection of Elliptic Cylinders; Primary Wave Elliptically Polarized.—Equation (14) determines the path of any element, set in oscillation by the passage of the given primary (electric) wave, with regard to the x and y axes; that is, the path sought lies on the elliptic cylinder defined by this equation. To determine the path described *on* this cylinder (14) by the given element, we must evidently seek a second equation, in X_1, Z_1 or Y_1, Z_1, which represents a surface, upon which the given path also lies. This equation is derived in a similar manner to the one above (14) and is evidently also similar to it in form. The intersection of the two cylinders represented by these equations gives then the path (in space), *along* which the given element is oscillating.

The equation in X_1, Z_1 similar to (14) is evidently

$$(A_3^2+B_3^2)\frac{X_1^2r^2}{n^4}-2(A_1A_3+B_1B_3)\frac{X_1Z_1r^2}{n^4}+(A_1^2+B_1^2)\frac{Z_1^2r^2}{n^4}$$
$$=(A_1B_3-A_3B_1)^2, \quad \ldots\ldots(23)$$

where
$$\left.\begin{aligned}A_3&=a_3(\alpha^2+\beta^2)\cos n\delta_3-a_1\alpha\gamma\cos n\delta_1-a_2\beta\gamma\cos n\delta_2\\ B_3&=a_3(\alpha^2+\beta^2)\sin n\delta_3-a_1\alpha\gamma\sin n\delta_1-a_2\beta\gamma\sin n\delta_1\end{aligned}\right\}. \quad \ldots\ldots(24)$$

Surfaces of the second degree intersect, in general, in a curve of the fourth degree. We have now seen on p. 81 that the given oscillations take place in planes that are at right angles to the direction of propagation. The elliptic cylinders represented by equations (14) and (23) must thus intersect in a curve that lies in a plane. A more thorough examination of the form of these elliptic cylinders, the relative position of their principal axes to each other and the lengths of the same (cf. Exs. 20, 21, 23, and 24), shows that they intersect in a curve that lies in two given planes or better in two curves, the one lying in the one and the other in the other plane. Since now an elliptic cylinder and a plane, for example the plane, in which one of these curves lies, intersect in an ellipse (provided, of course, they intersect), the given cylinders will evidently also intersect in (two) ellipses. Of these two ellipses that one determines the path of oscillation of the given element, which lies in the plane that is at right angles to the direction of propagation; it can also be determined as follows: the equation in Y_1, Z_1 similar to equations (14) and (23) represents an elliptic cylinder parallel to the x-axis, which intersects either of the other two elliptic cylinders (14) or (23), for example the former, in two ellipses, each lying in a plane; of these two ellipses one and only one is identical to one of the above two ellipses, the intersections of the elliptic cylinders (14) and (23), and that ellipse is evidently the one sought or that of oscillation of the given particle or element. The given primary electric wave is thus elliptically polarized.

The Secondary Wave; Determination of the Angle (f_2, r).—Let us, next, determine the path described by any ether-element upon the passage of the secondary electric wave, represented by the moments X_2, Y_2, Z_2 of formulae (7). For this purpose we, first, determine the angle (f_2, r), which the vector f_2* from the position of rest of that element to its position at any time t makes with the direction of propagation of the wave, to which that element belongs. By formula (23A, II) the angle (f_2, r) is given by the formula

$$\cos(f_2, r)=\frac{X_2\alpha+Y_2\beta+Z_2\gamma}{\sqrt{X_2^2+Y_2^2+Z_2^2}},$$

---

* This vector f_2 is not to be confounded with the wave-function f_2 of formulae (6).

where X_2, Y_2, Z_2 are to be replaced by their values

$$\left.\begin{aligned}
X_2&=\frac{n}{r^2}\{[2a_1-3a_1(\beta^2+\gamma^2)]\cos\omega_1+3a_2\alpha\beta\cos\omega_2+3a_3\alpha\gamma\cos\omega_3\}\\
Y_2&=\frac{n}{r^2}\{[2a_2-3a_2(\alpha^2+\gamma^2)]\cos\omega_2+3a_3\beta\gamma\cos\omega_3+3a_1\alpha\beta\cos\omega_1\}\\
Z_2&=\frac{n}{r^2}\{[2a_3-3a_3(\alpha^2+\beta^2)]\cos\omega_3+3a_1\alpha\gamma\cos\omega_1+3a_2\beta\gamma\cos\omega_2\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(D=1)
\end{aligned}\right\}\ldots.(25)$$

(cf. formulae (7)). We thus find

$$\cos(f_2, r)=\frac{2(a_1\alpha\cos\omega_1+a_2\beta\cos\omega_2+a_3\gamma\cos\omega_3)}{\sqrt{\begin{aligned}&a_1^2(1+3\alpha^2)\cos^2\omega_1+a_2^2(1+3\beta^2)\cos^2\omega_2\\&+a_3^2(1+3\gamma^2)\cos^2\omega_3+6(a_1a_2\alpha\beta\cos\omega_1\cos\omega_2\\&+a_1a_3\alpha\gamma\cos\omega_1\cos\omega_3+a_2a_3\beta\gamma\cos\omega_2\cos\omega_3)\end{aligned}}}\ \ldots.(26)$$

$$\text{or }\cos(f_2, r)=\frac{2(a_1\alpha\cos\omega_1+a_2\beta\cos\omega_2+a_3\gamma\cos\omega_3)}{\sqrt{\begin{aligned}&a_1^2\cos^2\omega_1+a_2^2\cos^2\omega_2+a_3^2\cos^2\omega_3\\&+3(a_1\alpha\cos\omega_1+a_2\beta\cos\omega_2+a_3\gamma\cos\omega_3)^2\end{aligned}}}.\ (26\text{A})$$

The Vector X_2, Y_2, Z_2 rotates in a Plane.—At any given point (α, β, γ) $\cos(f_2, r)$ is evidently a function of the time t only. Is now this expression for $\cos(f_2, r)$ such a function of t that as t varies the vector f_2 rotates in one and the same plane, like the vector f_1 of any element of the primary wave represented by formulae (8)? If this be the case, there must then evidently be a line n passing through the position of rest of the given element, for which $\cos(f_2, n)=0$ for all values of t. On the other hand, if this condition can be satisfied, such a line n must exist and, conversely, the direction of the same thereby be determined.

If the line n exist, then $\cos(f_2, n)$ must vanish for all values of t. We write $\cos(f_2, n)$ in the familiar form

$$\cos(f_2, n)=\cos(f_2, x)\cos(n, x)+\cos(f_2, y)\cos(n, y)+\cos(f_2, z)\cos(n, z),$$

replace the cosines (f_2, x), (f_2, y), and (f_2, z) by $\frac{X_2}{f_2}$, $\frac{Y_2}{f_2}$, and $\frac{Z_2}{f_2}$ rsspectively, and we have

$$\cos(f_2, n)=\frac{X_2\cos(n, x)+Y_2\cos(n, y)+Z_2\cos(n, z)}{f_2}.$$

That this expression vanish, we must have

$$X_2\cos(n, x)+Y_2\cos(n, y)+Z_2\cos(n, z)=0.$$

Replace here X_2, Y_2, Z_2 by their values (25), and we find, upon expanding the $\cos\omega$'s as functions of the angle $n(vt-r)$ and the $n\delta$'s (cf. formulae (6)),

$$\begin{aligned}&[A_1'\cos n(vt-r)+B_1'\sin n(vt-r)]\cos(n,x)\\+&[A_2'\cos n(vt-r)+B_2'\sin n(vt-r)]\cos(n,y)\\+&[A_3'\cos n(vt-r)+B_3'\sin n(vt-r)]\cos(n,z)=0,\end{aligned}$$

where

$$\left.\begin{aligned}A_1'&=[2a_1-3a_1(\beta^2+\gamma^2)]\cos n\delta_1+3a_2\alpha\beta\cos n\delta_2+3a_3\alpha\gamma\cos n\delta_3\\B_1'&=[2a_1-3a_1(\beta^2+\gamma^2)]\sin n\delta_1+3a_2\alpha\beta\sin n\delta_2+3a_3\alpha\gamma\sin n\delta_3\\A_2'&=[2a_2-3a_2(\alpha^2+\gamma^2)]\cos n\delta_2+3a_3\beta\gamma\cos n\delta_3+3a_1\alpha\beta\cos n\delta_1\\B_2'&=[2a_2-3a_2(\alpha^2+\gamma^2)]\sin n\delta_2+3a_3\beta\gamma\sin n\delta_3+3a_1\alpha\beta\sin n\delta_1\\A_3'&=[2a_3-3a_3(\alpha^2+\beta^2)]\cos n\delta_3+3a_1\alpha\gamma\cos n\delta_1+3a_2\beta\gamma\cos n\delta_2\\B_3'&=[2a_3-3a_3(\alpha^2+\beta^2)]\sin n\delta_3+3a_1\alpha\gamma\sin n\delta_1+3a_2\beta\gamma\sin n\delta_2\end{aligned}\right\},\ldots(27)$$

or

$$\begin{aligned}&[A_1'\cos(n,x)+A_2'\cos(n,y)+A_3'\cos(n,z)]\cos n(vt-r)\\+&[B_1'\cos(n,x)+B_2'\cos(n,y)+B_3'\cos(n,z)]\sin n(vt-r)=0.\end{aligned}$$

That this equation hold for all values of t, the coefficients of $\cos n(vt-r)$ and $\sin n(vt-r)$ must evidently vanish; that is, we must have

$$\left.\begin{aligned}&A_1'\cos(n,x)+A_2'\cos(n,y)+A_3'\cos(n,z)=0\\\text{and}\quad&B_1'\cos(n,x)+B_2'\cos(n,y)+B_3'\cos(n,z)=0\end{aligned}\right\}\ldots\ldots\ldots\ldots(28)$$

These two equations can evidently always be satisfied, provided the cosines (n,x), (n,y), and (n,z) be so chosen that they are determined by the same and the analytic relation

$$\cos^2(n,x)+\cos^2(n,y)+\cos^2(n,z)=1.\ \ldots\ldots\ldots\ldots\ldots\ldots(29)$$

On the other hand, these three equations suffice for the unique determination of the direction of the line n.

Determination of Normal to Plane of Oscillation.—Upon eliminating $\cos(n,x)$ and $\cos(n,y)$ from equations (28) and (29), we find the following expression for $\cos^2(n,z)$:

$$\begin{aligned}\cos^2(n,z)&=\frac{A_1'^2(A_1'B_2'-A_2'B_1')^2}{\begin{aligned}(A_1'^2+A_2'^2)(A_1'B_3'-A_3'B_1')^2-2A_2'A_3'(A_1'B_2'-A_2'B_1')\\\times(A_1'B_3'-A_3'B_1')+(A_1'^2+A_3'^2)(A_1'B_2'-A_2'B_1')^2\end{aligned}}\\&=\frac{(A_1'B_2'-A_2'B_1')^2}{(A_1'B_2'-A_2'B_1')^2+(A_1'B_3'-A_3'B_1')^2+(A_2'B_3'-A_3'B_2')^2}\end{aligned}$$

and similarly

$$\begin{aligned}\cos^2(n,x)&=\frac{(A_2'B_3'-A_3'B_2')^2}{(A_1'B_2'-A_2'B_1')^2+(A_1'B_3'-A_3'B_1')^2+(A_2'B_3'-A_3'B_2')^2}\\\cos^2(n,y)&=\frac{(A_3'B_1'-A_1'B_3')^2}{(A_1'B_2'-A_2'B_1')^2+(A_1'B_3'-A_3'B_1')^2+(A_2'B_3'-A_3'B_2')^2}\end{aligned}\quad\Bigg\}.(30)$$

To express these direction-cosines in terms of the given quantities a_1, a_2, a_3 and δ_1, δ_2, δ_3 and the direction-cosines α, β, γ of the vector r, we must, first, evaluate the three expressions

$$A_1'B_2' - A_2'B_1', \quad A_2'B_3' - A_3'B_2' \text{ and } A_3'B_1' - A_1'B_3';$$

by formulae (27) we find

$$\left.\begin{aligned}
&A_1'B_2' - A_2'B_1' \\
&\quad = a_1a_2(2 - 3\gamma^2)\sin n\delta_{12} + 3a_1a_3\beta\gamma \sin n\delta_{13} - 3a_2a_3\alpha\gamma \sin n\delta_{23} \\
&A_2'B_3' - A_3'B_2' \\
&\quad = a_2a_3(2 - 3\alpha^2)\sin n\delta_{23} - 3a_1a_2\alpha\gamma \sin n\delta_{12} + 3a_1a_3\alpha\beta \sin n\delta_{13} \\
&A_3'B_1' - A_1'B_3' \\
&\quad = -a_1a_3(2 - 3\beta^2)\sin n\delta_{13} - 3a_2a_3\alpha\beta \sin n\delta_{23} - 3a_1a_2\beta\gamma \sin n\delta_{12} \\
&\text{where} \qquad \delta_{12} = \delta_1 - \delta_2,\ \delta_{13} = \delta_1 - \delta_3 \text{ and } \delta_{23} = \delta_2 - \delta_3
\end{aligned}\right\} \ldots(31).$$

Replace the given expressions by these in formulae (30), and we find

$$\left.\begin{aligned}
&\cos^2(n, x) \\
&\quad = \frac{[a_2a_3(2 - 3\alpha^2)\sin n\delta_{23} - 3a_1a_2\alpha\gamma \sin n\delta_{12} + 3a_1a_3\alpha\beta \sin n\delta_{13}]^2}{\begin{array}{l} a_1^2a_2^2(4 - 3\gamma^2)\sin^2 n\delta_{12} + a_1^2a_3^2(4 - 3\beta^2)\sin^2 n\delta_{13} \\ \qquad + a_2^2a_3^2(4 - 3\alpha^2)\sin^2 n\delta_{23} + 6a_1a_2a_3(a_1\beta\gamma \sin n\delta_{12} \sin n\delta_{13} \\ \qquad - a_2\alpha\gamma \sin n\delta_{12} \sin n\delta_{23} + a_3\alpha\beta \sin n\delta_{13} \sin n\delta_{23}) \end{array}} \\
&\quad = \frac{[a_2a_3(2 - 3\alpha^2)\sin n\delta_{23} - 3a_1a_2\alpha\gamma \sin n\delta_{12} + 3a_1a_3\alpha\beta \sin n\delta_{13}]^2}{\begin{array}{l} 4(a_1^2a_2^2 + a_1^2a_3^2 + a_2^2a_3^2) \\ \qquad - 3(a_1a_2\gamma \sin n\delta_{12} - a_1a_3\beta \sin n\delta_{13} + a_2a_3\alpha \sin n\delta_{23})^2 \end{array}} \\
&\cos^2(n, y) \\
&\quad = \frac{[-a_1a_3(2 - 3\beta^2)\sin n\delta_{13} - 3a_2a_3\alpha\beta \sin n\delta_{23} - 3a_1a_2\beta\gamma \sin n\delta_{12}]^2}{\begin{array}{l} 4(a_1^2a_2^2 + a_1^2a_3^2 + a_2^2a_3^2) \\ \qquad - 3(a_1a_2\gamma \sin n\delta_{12} - a_1a_3\beta \sin n\delta_{13} + a_2a_3\alpha \sin n\delta_{23})^2 \end{array}} \\
&\cos^2(n, z) \\
&\quad = \frac{[a_1a_2(2 - 3\gamma^2)\sin n\delta_{12} + 3a_1a_3\beta\gamma \sin n\delta_{13} - 3a_2a_3\alpha\gamma \sin n\delta_{23}]^2}{\begin{array}{l} 4(a_1^2a_2^2 + a_1^2a_3^2 + a_2^2a_3^2) \\ \qquad - 3(a_1a_2\gamma \sin n\delta_{12} - a_1a_3\beta \sin n\delta_{13} + a_2a_3\alpha \sin n\delta_{23})^2 \end{array}}
\end{aligned}\right\} . \ldots(32)$$

It thus follows that there is a fixed line n, passing through the position of rest of the given oscillating element, and with which the vector f_2 always makes a right angle; the direction-cosines of that line are given by formulae (32). The given secondary (electric) oscillations, like the primary ones, which they are accompanying, thus take place in planes (cf. also Ex. 16 at end of chapter); these planes of oscillation do not, however, in general, stand at right angles to the direction of propagation of the waves, as was the case with the primary oscillations, but they make angles with the same, which vary from point to point and for different values of the quantities a_1, a_2, a_3 and δ_1, δ_2, δ_3. For another proof of the above, namely that the given oscillations take place in planes, see Ex. 16 at end of chapter, where the equation of those planes is determined.

Regions in which the Secondary Oscillations take place at ⊥ to Direction of Propagation.—We have just seen that the secondary electric oscillations take place in planes that do not, in general, stand at right angles to the direction of propagation. Are there now vectors α, β, γ, along which the secondary wave is propagated in planes, that stand at right angles to its direction of propagation? and, if there be such vectors, let us determine the same. That the given oscillations take place in planes that stand at right angles to the direction of propagation, $\cos(f_2, r)$ must vanish for all values of t; that is, the following relation must evidently hold between the direction-cosines sought and the given quantities a_1, a_2, a_3 and δ_1, δ_2, δ_3 for all values of t:

$$a_1\alpha\cos\omega_1 + a_2\beta\cos\omega_2 + a_3\gamma\cos\omega_3 = 0$$

(cf. formula 26)), or explicitly

$$\begin{aligned} &a_1\alpha[\cos n(vt-r)\cos n\delta_1 + \sin n(vt-r)\sin n\delta_1] \\ &+ a_2\beta[\cos n(vt-r)\cos n\delta_2 + \sin n(vt-r)\sin n\delta_2] \\ &+ a_3\gamma[\cos n(vt-r)\cos n\delta_3 + \sin n(vt-r)\sin n\delta_3] = 0. \end{aligned}$$

That this equation hold for all values of t, the coefficients of $\cos n(vt-r)$ and $\sin n(vt-r)$ must evidently vanish; that is, the two equations

$$a_1\alpha\cos n\delta_1 + a_2\beta\cos n\delta_2 + a_3\gamma\cos n\delta_3 = 0$$

and

$$a_1\alpha\sin n\delta_1 + a_2\beta\sin n\delta_2 + a_3\gamma\sin n\delta_3 = 0$$

must be satisfied, and also the analytic condition between the direction-cosines

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

We have here three equations for the determination of the three quantities (α, β, γ) sought; the former can, therefore, be satisfied, provided the latter be determined thereby. The first two equations give

$$a_2\beta\sin n\delta_{12} = -a_3\gamma\sin n\delta_{13},$$

the last two

$$(a_2\beta\sin n\delta_2 + a_3\gamma\sin n\delta_3)^2 + a_1^2(\beta^2+\gamma^2)\sin^2 n\delta_1 = a_1^2\sin^2 n\delta_1,$$

and the elimination of β from these the following value for γ:

$$\left.\begin{aligned} &\gamma = \pm a_1a_2\sin n\delta_1\sin n\delta_{12}F^{-1} \\ \text{and hence}\quad &\beta = \pm a_1a_3\sin n\delta_1\sin n\delta_{13}F^{-1} \\ \text{and}\quad &\alpha = \pm a_2a_3\sin n\delta_1\sin n\delta_{23}F^{-1} \\ \text{where}\quad &F^2 = a_2^2(a_1^2\sin^2 n\delta_1 + a_3^2\sin^2 n\delta_3)\sin^2 n\delta_{12} \\ &\qquad + a_3^2(a_1^2\sin^2 n\delta_1 + a_2^2\sin^2 n\delta_2)\sin^2 n\delta_{13} \\ &\qquad - 2a_2^2a_3^2\sin n\delta_2\sin n\delta_3\sin n\delta_{12}\sin n\delta_{13} \end{aligned}\right\} \cdots\cdots\cdots(33)$$

Path of Oscillation.—To obtain the path described by any element of the given secondary (electric) wave in its plane of oscillation, which, as we have seen above, is thereby determined that its normal (n) is given by formulae (32), we must eliminate the time t from formulae (25), by which that wave is represented. For this purpose we write these formulae explicitly, expanding the cos ω's as functions of the angle $n(vt-r)$ and the $n\delta$'s, as on p. 87, and we have

$$\left.\begin{aligned} \frac{X_2r^2}{n} &= A_1'\cos\tau + B_1'\sin\tau \\ \frac{Y_2r^2}{n} &= A_2'\cos\tau + B_2'\sin\tau \\ \frac{Z_2r^2}{n} &= A_3'\cos\tau + B_3'\sin\tau \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

where A_1', B_1', ... are given by formulae (27) and

$$\tau = n(vt-r).$$

The first two equations (34) give

$$B_2'\frac{X_2r^2}{n} - B_1'\frac{Y_2r^2}{n} = (A_1'B_2' - A_2'B_1')\cos\tau$$

and

$$A_2'\frac{X_2r^2}{n} - A_1'\frac{Y_2r^2}{n} = (A_2'B_1' - A_1'B_2')\sin\tau\,;$$

and these, squared and added, the following quadratic equation in X_2, Y_2:

$$(A_2'^2 + B_2'^2)\frac{X_2^2r^4}{n^2} - (2A_1'A_2' + B_1'B_2')\frac{X_2Y_2r^4}{n^2} + (A_1'^2 + B_1'^2)\frac{Y_2^2r^4}{n^2}$$
$$= (A_1'B_2' - A_2'B_1')^2. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

Upon evaluating the coefficients of this equation, we find

$$\left.\begin{aligned} A_2'^2 + B_2'^2 &= a_2^2[2-3(\alpha^2+\gamma^2)]^2 + 9a_3^2\beta^2\gamma^2 + 9a_1^2\alpha^2\beta^2 \\ &\quad + 6a_2a_3[2-3(\alpha^2+\gamma^2)]\cos n\delta_{23} + 6a_1a_2\alpha\beta \\ &\quad \times [2-3(\alpha^2+\gamma^2)]\cos n\delta_{12} + 18a_1a_3\alpha\beta^2\gamma\cos n\delta_{13} \\ A_1'A_2' + B_1'B_2' &= 3a_1^2\alpha\beta[2-3(\beta^2+\gamma^2)] + 3a_2^2\alpha\beta[2-3(\alpha^2+\gamma^2)] \\ &\quad + 9a_3^2\alpha\beta\gamma^2 - a_1a_2(2-3\gamma^2-18\alpha^2\beta^2)\cos n\delta_{12} \\ &\quad - 3a_1a_3\beta\gamma(1-6\alpha^2)\cos n\delta_{13} \\ &\quad - 3a_2a_3\alpha\gamma(1-6\beta^2)\cos n\delta_{23} \\ \text{and } A_1'^2 + B_1'^2 &= a_1^2[2-3(\beta^2+\gamma^2)]^2 + 9a_2^2\alpha^2\beta^2 + 9a_3^2\alpha^2\gamma^2 \\ &\quad + 6a_1a_2\alpha\beta[2-3(\beta^2+\gamma^2)]\cos n\delta_{12} \\ &\quad + 6a_1a_3\alpha\gamma[2-3(\beta^2+\gamma^2)]\cos n\delta_{13} \\ &\quad + 18a_2a_3\alpha^2\beta\gamma\cos n\delta_{23}\,; \\ A_1'B_2' - A_2'B_1' &\text{ is given by formulae (31).} \end{aligned}\right\} \ldots (36)$$

Equation (35) evidently represents a cylindrical surface parallel to the z-axis. It is the same equation as that (14) examined on pp. 83, 84, differing only in the values of its coefficients. The cylindrical surface represented by this equation is thus elliptical, except where

$$A_1'B_2' - A_2'B_1' = 0$$

(cf. p. 83) or upon the surface

$$a_1a_2(2-3\gamma^2)\sin n\delta_{12} + 3a_1a_3\beta\gamma\sin n\delta_{13} - 3a_2a_3\alpha\gamma\sin n\delta_{23} = 0 \quad \text{.....(37)}$$

(cf. formulae (31)); in which particular case the elliptical cylinder contracts to a double plane; that is, the ellipse, intersected on the xy-plane, contracts to a double straight line upon the surface (37).

Path of Oscillation determined by Intersection of Elliptic Cylinders; Secondary Wave Elliptically Polarized.—Equation (35) determines the path of the given oscillating element with regard to the x- and y-axes only. As on pp. 84, 85, we must also seek the equations in X_2, Z_2 or Y_2, Z_2 representing surfaces, upon which the given element also lies. The intersection of these surfaces will then determine the path described. The equations in X_2, Z_2, and Y_2, Z_2 are obtained in a similar manner to the one above (35) in X_2, Y_2 and are evidently also similar to the same in form, representing elliptic cylinders parallel to the y- and x-axes respectively.

Since now equation (35) and the two analogous ones in X_2, Z_2 and Y_2, Z_2 are the same equations as those (14), (23), etc., already examined, differing only in the values of their coefficients, the results deduced on p. 85 for the latter will also hold here: namely, since the secondary oscillations X_2, Y_2, Z_2 take place in planes, as we have seen above, the elliptic cylinders (35), etc., will intersect in curves that lie in planes, that is, in ellipses, and the oscillations themselves will thus take place in elliptic paths. The primary and secondary waves, represented by the moments X_1, Y_1, Z_1 of formulae (8) and X_2, Y_2, Z_2 of formulae (25) respectively, and belonging to any given pencil, will thus be elliptically polarized; the only material difference between the paths of these two waves is that the planes of oscillation of the former are always at right angles to the direction of propagation, whereas those of the latter make variable angles with the same.

Confirmation that the Elliptic Cylinders intersect in Plane Closed Curves.—The conclusions drawn on p. 85 and applied above to the secondary oscillations also, namely that the elliptic cylinders (14) and (23) and (35) and the analogous one in X_2, Z_2 intersect in plane ellipses, were founded on the fact that by the formula on p. 81 for $\cos(f_1, r)$ and formula (26) for $\cos(f_2, r)$ the vectors f_1 and f_2 of any oscillating element always made right angles either with the direction of propagation of the wave or with some fixed line n in space (cf. also Ex. 16 at end of chapter). That now the two elliptic cylinders (14) and (23) or (35) and the analogous one in X_2, Z_2 or Y_2, Z_2 intersect in two *plane closed* curves—only *closed* curves would come into consideration as paths of oscillation—the cylinders themselves must evidently be of such dimensions that their breadths with regard to that coordinate axis, which stands at right angles to the plane passing through the two infinitely long axes of the given cylinders, be the

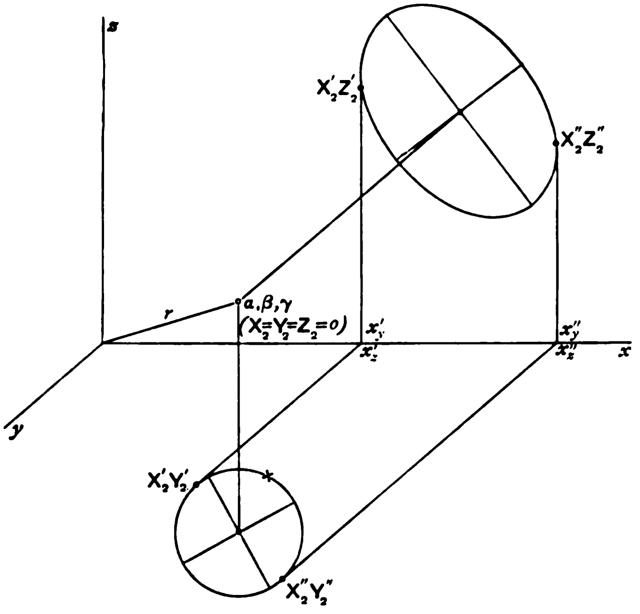

FIG. 9.

same; for example, the breadth of the cylinder (35) with regard to the x-axis, which breadth we denote, as indicated in the annexed figure, by the distance $x_z'' - x_z'$, must be the same as that of the analogous cylinder in X_2Z_2 with regard to the same axis (x), denoted by the distance $x_y'' - x_y'$, as in figure. Let us now confirm this proposition for the two cylinders X_2Y_2 and X_2Z_2, whose intersection determines the path of the oscillating element of the given secondary wave at any point α, β, γ, the origin of our coordinates X_2, Y_2, Z_2, $X_2 = Y_2 = Z_2 = 0$ being the position of rest of that element. For the proof of this proposition for the cylinders (14) and (23), whose intersection determines the path of any oscillating element of the primary wave, see Ex. 29.

To determine the points x_2' and x_2'', where the tangents to the ellipse (35) that are parallel to the y-axis intersect the x-axis (cf. the above figure), we first seek those values of Y_2 of the given equation (35), for which X_2 is a maximum and minimum. For this purpose we first express X_2 as function of Y_2: we write the given equation

$$aX_2^2+bX_2Y_2+cY_2^2+d=0, \quad\text{........................}(38)$$

putting

$$\left.\begin{array}{ll} a=(A_2'^2+B_2'^2)\dfrac{r^4}{n^2}, & b=-2(A_1'A_2'+B_1'B_2')\dfrac{r^4}{n^2} \\ c=(A_1'^2+B_1'^2)\dfrac{r^4}{n_2}, & d=-(A_1'B_2'-A_2'B_1')^2 \end{array}\right\}, \quad\text{.......}(39)$$

and we have

$$X_2=-\frac{bY_2}{2a}\pm\frac{1}{2a}\sqrt{(b^2-4ac)Y_2^2-4ad}=f(Y_2). \quad\text{............}(40)$$

The equation

$$\frac{d}{dY_2}f(Y_2)=0 \quad\text{.................................}(41)$$

determines now, as we know, those values of Y_2 of the curve (38), for which X_2 becomes a maximum and minimum.

By formula (40) this equation (41) can be written

$$\frac{d}{dY_2}f(Y_2)=-\frac{b}{2a}\pm\frac{1}{2a}\,\frac{b^2-4acY_2}{\sqrt{(b^2-4ac)Y_2^2-4ad}}=0,$$

which gives the following equation for the determination of Y_2:

$$(b-4ac)^2Y_2^2=b^2[(b^2-4ac)Y_2^2-4ad],$$

hence

$$Y_2^2=\frac{b^2d}{c(b^2-4ac)},$$

that is, the two values

$$Y_2'=b\sqrt{\frac{d}{c(b^2-4ac)}} \quad\text{and}\quad Y_2''=-b\sqrt{\frac{d}{c(b^2-4ac)}}\,; \quad\text{........}(42)$$

which of these values is that, for which X_2 becomes a maximum or minimum, is evidently immaterial.

We then determine those values of X_2, to which these values (42) of Y_2 belong, upon replacing Y_2 by those values (42) in equation (38) or better (40); we evidently have

$$\left.\begin{array}{l} X_2'=-\dfrac{b^2}{2a}\sqrt{\dfrac{d}{c(b^2-4ac)}}\pm\dfrac{1}{2a}\sqrt{\dfrac{(b^2-4ac)d}{c}} \\ X_2''=+\dfrac{b^2}{2a}\sqrt{\dfrac{d}{c(b^2-4ac)}}\pm\dfrac{1}{2a}\sqrt{\dfrac{(b^2-4ac)d}{c}} \end{array}\right\}; \quad\text{...........}(43)$$

that is, two values for X_2' and two for X_2''. Of the two values for

X_2' the one is evidently the minimum (cf. figure 9) sought, whereas the other is that other or smaller value of X_2', which belongs to the same value Y_2' of Y_2 and together with the latter determines that point of the given ellipse, which we have marked in the above figure with a cross (×). To determine which of these two values for X_2' is the value (minimum) sought, we must evidently compare the same with regard to their absolute values and choose the larger of the two.

We have

$$\left.\begin{aligned} X_2' &= -\frac{b^2}{2a}\sqrt{\frac{d}{c(b^2-4ac)}}+\frac{1}{2a}\sqrt{\frac{(b^2-4ac)d}{c}} = -2\sqrt{\frac{cd}{b^2-4ac}} \\ \text{and}\qquad & \\ X_2' &= -\frac{b^2}{2a}\sqrt{\frac{d}{c(b^2-4ac)}}-\frac{1}{2a}\sqrt{\frac{(b^2-4ac)d}{c}} \\ &= -\frac{b^2-2ac}{a}\sqrt{\frac{d}{c(b^2-4ac)}} \end{aligned}\right\} \ldots\ldots(44)$$

Let us now assume that the former of these expressions be the larger (in absolute value) of the two. The following inequality must then hold:

$$\left[-2\sqrt{\frac{cd}{b^2-4ac}}\right]^2 > \left[-\frac{b^2-2ac}{a}\sqrt{\frac{d}{c(b^2-4ac)}}\right]^2$$

—we take the squares of the given expressions, since we are comparing their absolute values—hence

$$4c > \frac{(b^2-2ac)^2}{a^2c}$$

or

$$4a^2c^2 > b^4 - 4b^2ac + 4a^2c^2$$

or

$$0 > b^2(b^2-4ac).$$

Replace here a, b, c by their values (39), and we have

$$0 > 16(A_1'A_2'+B_1'B_2')^2[(A_1'A_2'+B_1'B_2')^2-(A_1'^2+B_1'^2)(A_2'^2+B_2'^2)]\frac{r^{16}}{n^8}$$

or

$$0 > -16(A_1'A_2'+B_1'B_2')^2(A_1'B_2'-A_2'B_1')^2\frac{r^{16}}{n^8};$$

which is evidently always the case.

The above assumption, namely that the minimum value of X_2' be given by the former of the above expressions (44), namely

$$X_2' = -2\sqrt{\frac{cd}{b^2-4ac}}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(45)$$

is thus correct.

Similarly, we can show that the maximum value of X_2'' is given by the expression

$$X_2'' = \frac{b^2}{2a}\sqrt{\frac{d}{c(b^2-4ac)}} - \frac{1}{2a}\sqrt{\frac{(b^2-4ac)d}{c}} = 2\sqrt{\frac{cd}{b^2-4ac}}. \quad \text{.........(46)}$$

The breadth of the cylinder X_2Y_2 with regard to the x-axis is now evidently determined by the absolute value of the difference between the maximum and minimum values of X_2, which are given by these formulae (45) and (46); we thus have

$$|x_z''-x_z'| = 4\sqrt{\frac{cd}{b^2-4ac}}, \quad \text{..........................(47)}$$

where the vertical lines denote that the absolute value of the given expression is to be taken.

Similarly, we evidently find the following analogous expression for the breadth of the cylinder X_2Z_2 with regard to the x-axis:

$$|x_y''-x_y'| = 4\sqrt{\frac{c'd'}{b'-4a'c'}}, \quad \text{........................(48)}$$

where

$$\left.\begin{array}{ll} a' = (A_3'^2+B_3'^2)\dfrac{r^4}{n^2}, & b' = -2(A_1'A_3'+B_1'B_3')\dfrac{r^4}{n^2} \\ c' = (A_1'^2+B_1'^2)\dfrac{r^4}{n^2}, & d' = -(A_1'B_3'-A_3'B_1')^2 \end{array}\right\} \quad \text{......(49)}$$

Replace *abcd* and *a'b'c'd'* by their values (39) and (49) in formulae (47) and (48), and we find

$$\begin{aligned} |x_z''-x_z'| &= 4\sqrt{\frac{-(A_1'^2+B_1'^2)(A_1'B_2'-A_2'B_1')^2\dfrac{r^4}{n^2}}{4[(A_1'A_2'+B_1'B_2')^2-(A_1'^2+B_1'^2)(A_2'^2+B_2'^2)]\dfrac{r^8}{n^4}}} \\ &= \frac{2n}{r^2}(A_1'B_2'-A_2'B_1')\sqrt{\frac{-(A_1'^2+B_1'^2)}{2A_1'A_2'B_1'B_2'-A_1'^2B_2'^2-A_2'^2B_1'^2}} \\ &= \frac{2n}{r^2}\sqrt{A_1'^2+B_1'^2}, \quad \text{..(50)} \end{aligned}$$

and similarly

$$|x_y''-x_y'| = \frac{2n}{r^2}\sqrt{A_1'^2+B_1'^2}; \quad \text{.......................(51)}$$

that is, one and the same expression for the breadths of the given cylinders with regard to the x-axis.

The proposition stated on p. 92 is thus confirmed, and hence the conclusions drawn therefrom, which were founded on the same.

For a further examination of the ellipses (14), (23), (35), etc., the determination of the angles, which their principal axes make with the coordinate-axes, and of the lengths of those (major and minor) axes, see Exs. 20-21 and 23-25 at end of chapter.

The Magnetic Waves Elliptically Polarized in Planes at ⊥ to Direction of Propagation.—The paths of oscillation of the magnetic (primary and secondary) oscillations are likewise determined by the intersections of elliptic cylinders (cf. Exs. 10 and 11 at end of chapter). It is easy to show that these cylinders intersect in *plane* ellipses, and that the planes of oscillation of both the primary and the secondary oscillations stand at right angles to the direction of propagation. In this respect the secondary electric and the secondary magnetic oscillations differ from each other, the former taking place in planes that make variable angles, not always 90°, with the direction of propagation, whereas the planes of oscillation of the latter always stand at right angles to the same. On the other hand, the vector X_2, Y_2, Z_2 always stands at right angles to both magnetic vectors, a_1, b_1, c_1 and a_2, b_2, c_2, whereas the vector X_1, Y_1, Z_1 stands at right angles to the former magnetic vector (a_1, b_1, c_1) but not to the latter (cf. Ex. 14 at end of chapter).

EXAMPLES.

1. Show, when the rectangular oscillations (2) differ in phase by $\frac{\pi}{2}$, that the principal axes of the ellipse described by any element under the simultaneous action of those oscillations coincide with their directions of oscillation.

2. Show that $\sin n(\delta_1-\delta_2)$ and $\cos n(\delta_1-\delta_2)$ of formulae (3) can be interpreted geometrically as follows:

$$\sin n(\delta_1-\delta_2)=\frac{OC}{OA}=\frac{OD}{OB},$$

$$\cos n(\delta_1-\delta_2)=\frac{AE}{AP}=\frac{BF}{BP},$$

where E and F denote the points of contact of the tangents to the given ellipse

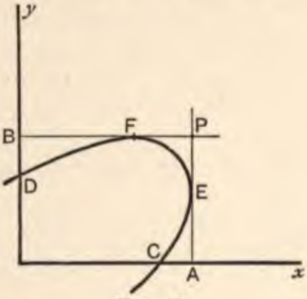

FIG. 10.

parallel to the y- and x-axes respectively, P the point of intersection of those tangents, A and B the points, in which the same intersect the x- and y-axes respectively, and C and D the points of intersection of the given ellipse and the x- and y-axes respectively, as indicated in the annexed figure.

3. Show that the directions of the principal axes of the ellipse described by the elliptic oscillation
$$x=a\sin\omega t,\qquad y=b\sin(\omega t+\delta)$$
are given by
$$\tan 2\phi=\frac{2ab}{a^2-b^2}\cos\delta,\text{*}$$
where ϕ denotes the angle these axes make with the x-, y-axes.

4. The velocity of oscillation of a circular oscillation is uniform.

Take the circular oscillation
$$x=a\sin n[vt-(z+\delta_1)],$$
$$y=a\cos n[vt-(z+\delta_1)];$$
and we have
$$\text{vel. of osc.}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$
$$=anv=2\pi\frac{a}{T},$$
which is constant with regard to t.

5. The circular oscillation is the only one, whose velocity of oscillation is uniform.

The velocity of oscillation V of the elliptic oscillation
$$x=a_1\sin n[vt-(z+\delta_1)]=a_1\sin\omega_1,$$
$$y=a_2\cos n[vt-(z+\delta_2)]=a_2\cos\omega_2$$
is
$$V=nv\sqrt{a_1{}^2\cos^2\omega_1+a_2{}^2\sin^2\omega_2}.$$

That this velocity remain uniform, we must evidently have
$$\frac{dV}{dt}=-n^2v^2\frac{a_1{}^2\cos\omega_1\sin\omega_1-a_2{}^2\sin\omega_2\cos\omega_2}{\sqrt{a_1{}^2\cos^2\omega_1+a_2{}^2\sin^2\omega_2}}=0,$$
hence
$$a_1{}^2\sin 2\omega_1-a_2{}^2\sin 2\omega_2=0,$$
or explicitly
$$[a_1{}^2\cos 2n\delta_1-a_2{}^2\cos 2n\delta_2]\sin 2n(vt-z)-[a_1{}^2\sin 2n\delta_1-a_2{}^2\sin 2n\delta_2]\cos 2n(vt-z)=0.$$

That this equation hold for *all values* of t, its coefficients must evidently vanish, that is, the relations must hold
$$a_1{}^2\cos 2n\delta_1-a_2{}^2\cos 2n\delta_2=0,$$
and
$$a_1{}^2\sin 2n\delta_1-a_2{}^2\sin 2n\delta_2=0.$$

Show that these relations between the a's and δ's can be satisfied only when $a_1=a_2$ and $n(\delta_1-\delta_2)=\frac{\pi}{2}$, that is, when the given oscillation is circular.

6. For $a_1=a_2=a_3=1$ the coefficients in formulae (14) and (23) assume the following form throughout the xy-plane:
$$A_1{}^2+B_1{}^2=\beta^2-2\alpha\beta^3\cos n\delta_{12},$$
$$A_2{}^2+B_2{}^2=\alpha^2-2\alpha^3\beta\cos n\delta_{12},$$
$$A_3{}^2+B_3{}^2=1,$$
$$A_1A_2+B_1B_2=-\alpha\beta+2\alpha^2\beta^2\cos n\delta_{12},$$
$$A_1A_3+B_1B_3=\beta^2\cos n\delta_{13}-\alpha\beta\cos n\delta(\delta_{13}-\delta_{12}),$$
$$A_1B_2-A_2B_1=0,$$
$$A_1B_3-A_3B_1=-\beta^2\sin n\delta_{13}+\alpha\beta\sin n(\delta_{13}-\delta_{12}),$$
where
$$\delta_{12}=\delta_1-\delta_2\quad\text{and}\quad\delta_{13}=\delta_1-\delta_3.$$

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* Cf. Preston's *Theory of Light* (second edition), Ex. 5, p. 56.

The given formulae thus assume here the form

$$\alpha^2(1-2\alpha\beta\cos n\delta_{12})\frac{X_1{}^2r^2}{n^4}+2\alpha\beta(1-2\alpha\beta\cos n\delta_{12})\frac{X_1Y_1r^2}{n^4}+\beta^2(1-2\alpha\beta\cos n\delta_{12})\frac{Y_1{}^2r^2}{n^4}=0$$

or
$$\left(\alpha\frac{X_1r}{n^2}+\beta\frac{Y_1r}{n^2}\right)^2=0, \qquad \text{.................................}(a)$$

a double plane passing through the origin and the z-axis—observe that the angles this plane makes with the x- and y-axes are functions of α and β only and not of the δ's—and

$$\left.\begin{aligned}&\frac{X_1{}^2r^2}{n^4}-2\beta[\beta\cos n\delta_{13}-\alpha\cos n(\delta_{13}-\delta_{12})]\frac{X_1Z_1r^2}{n^4}\\&\quad+\beta^2(1-2\alpha\beta\cos n\delta_{12})\frac{Z_1{}^2r^2}{n^4}=\beta^2[\beta\sin n\delta_{13}-\alpha\sin n(\delta_{13}-\delta_{12})]^2\end{aligned}\right\}\text{.......}(b)$$

Confirm that this equation is that of an elliptic cylinder parallel to the y-axis or an ellipse in the xz-plane (cf. p. 85). The familiar conditions

$$a>0,\quad \Delta\gtrless 0,\quad \text{and}\quad \frac{\Delta}{A}<0 \quad \text{(cf. p. 83)}$$

must then hold. We have here

$$^*a=AC-B^2=\frac{r^2}{n^4}\beta^2[1-2\alpha\beta\cos n\delta_{12}-\beta^2\cos^2 n\delta_{13}$$
$$\alpha^2\cos^2 n(\delta_{13}-\delta_{12})+2\alpha\beta\cos n\delta_{13}\cos n(\delta_{13}-\delta_{12})],$$

or, since here $\alpha^2+\beta^2=1$,

$$\begin{aligned}a&=\frac{r^2}{n^4}\beta^2\{\alpha^2\sin^2 n(\delta_{13}-\delta_{12})+\beta^2\sin^2 n\delta_{13}\\&\qquad+2\alpha\beta[\cos n\delta_{13}(\cos n\delta_{13}\cos n\delta_{12}+\sin n\delta_{13}\sin n\delta_{12})-\cos n\delta_{12}]\}\\&=\frac{r^2}{n^4}\beta^2[\alpha^2\sin^2 n(\delta_{13}-\delta_{12})+\beta^2\sin^2 n\delta_{13}-2\alpha\beta\sin n\delta_{13}\sin n(\delta_{13}-\delta_{12})]\\&=\frac{r^2}{n^4}\beta^2[\alpha\sin n(\delta_{13}-\delta_{12})-\beta\sin n\delta_{13}]^2>0\,; \qquad \text{.............................. ......}(c)\end{aligned}$$

moreover, since
$$\Delta=aF \quad \text{(cf. p. 83)}$$

and here
$$F=-\beta^2[\beta\sin n\delta_{13}-\alpha\sin n(\delta_{13}-\delta_{12})]^2,$$

by formula (c)

$$\Delta=-\frac{r^2}{n^4}\beta^4[\alpha\sin n(\delta_{13}-\delta_{12})-\beta\sin n\delta_{13}]^2[\beta\sin n\delta_{13}-\alpha\sin n(\delta_{13}-\delta_{12})]^2<0\,;$$

and lastly, since $A=1$,

$$\frac{\Delta}{A}=\Delta<0. \qquad \text{Q.E.D.}$$

Equations (a) and (b) thus intersect in an ellipse, that is, the given oscillations are elliptic throughout the xy-plane (cf. p. 85).

7. Show that the vectors (in the xy-plane), along which the oscillations of Ex. 6 become linear, are

$$\alpha^2=1,\quad \beta^2=0,\quad (\gamma=0)$$

and

$$\alpha^2=\frac{\sin^2 n\delta_{13}}{\sin^2 n(\delta_{13}-\delta_{12})+\sin^2 n\delta_{13}},\quad \beta^2=\frac{\sin^2 n(\delta_{13}-\delta_{12})}{\sin^2 n(\delta_{13}-\delta_{12})+\sin^2 n\delta_{13}},\quad (\gamma=0).$$

---

* This determinate a is not to be confounded with the direction cosine α.

8. For $a_1=a_2=a_3=a$ the coefficients (36) in equation (35) assume the following form in the xy-plane:

$$A_2'^2+B_2'^2=4-3\alpha^2+6\alpha\beta(2-3\alpha^2)\cos n\delta_{12},$$
$$A_1'A_2'+B_1'B_2'=3\alpha\beta-2(1-9\alpha^2\beta^2)\cos n\delta_{12},$$
$$A_1'^2+B_1'^2=4-3\beta^2+6\alpha\beta(2-3\beta^2)\cos n\delta_{12},$$
$$A_1'B_2'-A_2'B_1'=2\sin n\delta_{12},$$

where
$$\delta_{12}=\delta_1-\delta_2.$$

Confirm that the given equation (35) in X_2Y_2 is that of an ellipse, showing that

$$\begin{aligned}AC-B^2&=[4-3\alpha^2+6\alpha\beta(2-3\alpha^2)\cos n\delta_{12}][(4-3\beta^2)+6\alpha\beta(2-3\beta^2)\cos n\delta_{12}]\\&\qquad-[3\alpha\beta-2(1-9\alpha^2\beta^2)\cos n\delta_{12}]^2\\&=4\sin^2 n\delta_{12}>0, \text{ etc.}\end{aligned}$$

The coefficients in the equation in X_2 and Z_2 similar to (35) assume here the form

$$A_3'^2+B_3'^2=1,\quad A_1'^2+B_1'^2 \text{ same as above,}$$
$$A_1'A_3'+B_1'B_3'=-(2-3\beta^2)\cos n\delta_{13}-3\alpha\beta\cos n(\delta_{12}-\delta_{13}),$$
$$A_1'B_3'-A_3'B_1'=(2-3\beta^2)\sin n\delta_{13}-3\alpha\beta\sin n(\delta_{12}-\delta_{13}),$$

where
$$\delta_{12}=\delta_1-\delta_2 \quad\text{and}\quad \delta_{13}=\delta_1-\delta_3.$$

To confirm that the equation in X_2, Z_2 is here that of an ellipse, we first replace the A''s, B''s, C''s by these values in the determinate α, and we have

$$\begin{aligned}AC-B^2&=4-3\beta^2+6\alpha\beta(2-3\beta^2)\cos n\delta_{12}\\&\quad-[(2-3\beta^2)^2\cos^2 n\delta_{13}+9\alpha^2\beta^2\cos^2 n(\delta_{12}-\delta_{13})+6\alpha\beta(2-3\beta^2)\cos n\delta_{13}\cos n(\delta_{12}-\delta_{13})];\end{aligned}$$

which we can write

$$\begin{aligned}&=4-3\beta^2-(2-3\beta^2)^2(1-\sin^2 n\delta_{13})-9\beta^2(1-\beta^2)[1-\sin^2 n(\delta_{12}-\delta_{13})]\\&\qquad+6\alpha\beta(2-3\beta^2)[\cos n\delta_{12}-\cos n\delta_{13}(\cos n\delta_{12}\cos n\delta_{13}+\sin n\delta_{12}\sin n\delta_{13})]\\&=(2-3\beta^2)^2\sin^2 n\delta_{13}+9\beta^2(1-\beta^2)\sin^2 n(\delta_{12}-\delta_{13})\\&\qquad+6\alpha\beta(2-3\beta^2)(\cos n\delta_{12}\sin^2 n\delta_{13}-\sin n\delta_{12}\sin n\delta_{13}\cos n\delta_{13})\\&=(2-3\beta^2)^2\sin^2 n\delta_{13}+9\alpha^2\beta^2\sin^2 n(\delta_{12}-\delta_{13})\\&\qquad-6\alpha\beta(2-3\beta^2)\sin n\delta_{13}\sin n(\delta_{12}-\delta_{13})\\&=[(2-3\beta^2)\sin n\delta_{13}-3\alpha\beta\sin n(\delta_{12}-\delta_{13})]^2>0, \text{ etc.}\end{aligned}$$

We observe that the equation (35) in X_2, Y_2 contains here δ_{12} only; for $\delta_{12}=0$ it evidently reduces to that of a plane. It is thus evident that the path of oscillation of the given secondary wave ($a_1=a_2=a_3=a$) will be that of an ellipse throughout the xy-plane, when $\delta_{12}=\delta_1-\delta_2=0$.

In the xy-plane the equation in X_2, Z_2 reduces to that of a plane, when

$$(2-3\beta^2)\sin n\delta_{13}=3\alpha\beta\sin n(\delta_{12}-\delta_{13}). \quad\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Along the vectors α and β determined by this equation the given elliptic cylinder will thus contract to a plane, and hence the given secondary oscillations take place in ellipses.

Lastly, show, when the relation (a) holds and $\delta_{12}=0$, that the given secondary oscillations become linear along the four vectors,

$$\alpha_{11}=\sqrt{\frac{5-\sqrt{17}}{12}}=0{\cdot}271,\qquad \beta_{11}=\sqrt{\frac{7+\sqrt{17}}{12}}=0{\cdot}9627,\quad (\gamma=0);$$
$$\alpha_{12}=-0{\cdot}271,\qquad \beta_{12}=-0{\cdot}9627,\quad (\gamma=0);$$
$$\alpha_{21}=-\sqrt{\frac{5+\sqrt{17}}{12}}=-0{\cdot}872,\qquad \beta_{21}=\sqrt{\frac{7-\sqrt{17}}{12}}=0{\cdot}4898,\quad (\gamma=0);$$

and
$$\alpha_{22}=0{\cdot}872,\qquad \beta_{22}=-0{\cdot}4898,\quad (\gamma=0).$$

9. Show that along those vectors, for which the secondary oscillations of Ex. 8 become linear, the same do not take place at right angles to their direction of propagation.

10. Show that the moments a_1, b_1, c_1 of the primary magnetic oscillations that accompany the electric oscillations represented by formulae (7) are determined by the formulae

$$\left.\begin{aligned} a_1' &= A_1 \sin\tau - B_1 \cos\tau \\ b_1' &= A_2 \sin\tau - B_2 \cos\tau \\ c_1' &= A_3 \sin\tau - B_3 \cos\tau \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(a)$$

where $$a_1' = \frac{v}{v_0}\frac{r}{n^2}a_1, \quad b_1' = \frac{v}{v_0}\frac{r}{n^2}b_1, \quad c_1' = \frac{v}{v_0}\frac{r}{n^2}c_1,$$

$$\begin{aligned} A_1 &= a_2\gamma\cos n\delta_2 - a_3\beta\cos n\delta_3, & B_1 &= a_2\gamma\sin n\delta_2 - a_3\beta\sin n\delta_3, \\ A_2 &= a_3\alpha\cos n\delta_3 - a_1\gamma\cos n\delta_1, & B_2 &= a_3\alpha\sin n\delta_3 - a_1\gamma\sin n\delta_1, \\ A_3 &= a_1\beta\cos n\delta_1 - a_2\alpha\cos n\delta_2, & B_3 &= a_1\beta\sin n\delta_1 - a_2\alpha\sin n\delta_2, \end{aligned}$$

and $$\tau = n(vt - r);$$

moreover, that these oscillations take place in elliptic paths, which lie in planes that are at right angles to the direction of propagation and are determined by the intersection of the elliptic cylinders

$$\left.\begin{aligned} &(A_2^2+B_2^2)a_1'^2 - 2(A_1A_2+B_1B_2)a_1'b_1' + (A_1^2+B_1^2)b_1'^2 = (A_1B_2 - A_2B_1)^2 \\ \text{and}\quad &(A_3^2+B_3^2)a_1'^2 - 2(A_1A_3+B_1B_3)a_1'c_1' + (A_1^2+B_1^2)c_1'^2 = (A_1B_3 - A_3B_1)^2 \\ \text{or}\quad &(A_3^2+B_3^2)b_1'^2 - 2(A_2A_3+B_2B_3)b_1'c_1' + (A_2^2+B_2^2)c_1'^2 = (A_2B_3 - A_3B_2)^2 \end{aligned}\right\}; \ldots(b)$$

and, lastly, that these cylinders contract to double planes along the planes

$$\gamma=0, \quad \beta=0 \quad \text{and} \quad \alpha=0 \text{ respectively}, \quad \ldots\ldots\ldots\ldots(c)$$

and all three (cylinders) along the plane

$$a_1a_2\gamma\sin n(\delta_1-\delta_2) - a_1a_3\beta\sin n(\delta_1-\delta_3) + a_2a_3\alpha\sin n(\delta_2-\delta_3) = 0 \quad \ldots\ldots(d)$$

(cf. also p. 84), throughout which (*d*) the given oscillations thus become linear.

11. Show that the moments a_2, b_2, c_2 of the secondary magnetic oscillations that accompany the electric oscillations represented by formulae (7) are determined by the formulae

$$\left.\begin{aligned} a_2' &= A_1' \sin\tau - B_1' \cos\tau \\ b_2' &= A_2' \sin\tau - B_2' \cos\tau \\ c_2' &= A_3' \sin\tau - B_3' \cos\tau \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(a)$$

where $$a_2' = -\frac{v}{v_0}\frac{r^2}{n}a_2, \quad b_2' = -\frac{v}{v_0}\frac{r^2}{n}b_2, \quad c_2' = -\frac{v}{v_0}\frac{r^2}{n}c_2,$$

$$\begin{aligned} A_1' &= a_2\gamma\sin n\delta_2 - a_3\beta\sin n\delta_3, & B_1' &= a_2\gamma\cos n\delta_2 - a_3\beta\cos n\delta_3, \\ A_2' &= a_3\alpha\sin n\delta_3 - a_1\gamma\sin n\delta_1, & B_2' &= a_3\alpha\cos n\delta_3 - a_1\gamma\cos n\delta_1, \\ A_3' &= a_1\beta\sin n\delta_1 - a_2\alpha\sin n\delta_2, & B_3' &= a_1\beta\cos n\delta_1 - a_2\alpha\cos n\delta_2, \end{aligned}$$

and $$\tau = n(vt - r);$$

moreover, that these oscillations, like the primary magnetic, also take place in ellipses, which lie in planes that make *right angles with the direction of propagation* and are determined by the intersection of the elliptic cylinders

$$\left.\begin{aligned} &(A_2'^2+B_2'^2)a_2'^2 - 2(A_1'A_2'+B_1'B_2')a_2'b_2' + (A_1'^2+B_1'^2)b_2'^2 = (A_1'B_2' - A_2'B_1')^2 \\ &\text{and} \\ &(A_3'^2+B_3'^2)a_2'^2 - 2(A_1'A_3'+B_1'B_3')a_2'c_2' + (A_1'^2+B_1'^2)c_2'^2 = (A_1'B_3' - A_3'B_1')^2 \\ &\text{or} \\ &(A_3'^2+B_3'^2)b_2'^2 - 2(A_2'A_3'+B_2'B_3')b_2'c_2' + (A_2'^2+B_2'^2)c_2'^2 = (A_2'B_3' - A_3'B_2')^2 \end{aligned}\right\}; (b)$$

and, lastly, that these cylinders contract to double planes throughout the same planes (*c*) and (*d*), Ex. 10, as the elliptic cylinders (*b*), Ex. 10, for the primary magnetic oscillations.

12. Show that the plane determined by formula (22A) or formula (*d*), Ex. 10, passes through the origin and that the direction-cosines of its normal are

$$\begin{aligned}\cos(n,\ x)&=a_2a_3\sin n(\delta_2-\delta_3),\\ \cos(n,\ y)&=-a_1a_3\sin n(\delta_1-\delta_3),\\ \cos(n,\ z)&=a_1a_2\sin n(\delta_1-\delta_2).\end{aligned}$$

13. Show that the total resultant electric moment X, Y, Z of formulae (7) and the total resultant magnetic moment a, b, c of the magnetic wave that accompanies the given electric wave always stand—their respective vectors—at right angles to each other.

14. Show, for the general problem treated in the text, that the resultant electric moment X_2, Y_2, Z_2, formulae (25), always stands at right angles to the resultant (magnetic) moments a_1, b_1, c_1 and a_2, b_2, c_2 of the magnetic wave that is accompanying the given electric wave, formulae (7); moreover, that the resultant electric moment X_1, Y_1, Z_1, formulae (8), and the resultant magnetic moment a_1, b_1, c_1 also always stand at right angles to each other; and, lastly, that the resultant electric and magnetic moments X_1, Y_1, Z_1 and a_2, b_2, c_2 make right angles with each other only throughout the plane

$$a_1a_2\gamma\sin n\delta_{12}-a_1a_3\beta\sin n\delta_{13}+a_2a_3\alpha\sin n(\delta_{13}-\delta_{12})=0.$$

15. Show that the magnetic wave that accompanies the electric wave represented by formulae (7) can vanish only when $\delta_1=\delta_2=\delta_3$ and then only along the vectors $\alpha:\beta:\gamma=a_1:a_2:a_3$ (cf. also p. 51).

16. The following linear equation holds between the component-moments X_2, Y_2, Z_2 of formulae (25):

$$(A_3'B_2'-A_2'B_3')X_2+(A_1'B_3'-A_3'B_1')Y_2+(A_2'B_1'-A_1'B_2')Z_2=0.$$

We write formulae (25) in the form (34), namely

$$\frac{X_2r^2}{n}=A_1'\cos\tau+B_1'\sin\tau,\quad \frac{Y_2r^2}{n}=A_2'\cos\tau+B_2'\sin\tau,\quad \frac{Z_2r^2}{n}=A_3'\cos\tau+B_3'\sin\tau,$$

multiply the first of these equations by A_2', the second by $-A_1'$, add, and we have

$$(A_2'X_2-A_1'Y_2)\frac{r^2}{n}=(A_2'B_1'-A_1'B_2')\sin\tau,$$

and similarly, $$(A_3'Y_2-A_2'Z_2)\frac{r^2}{n}=(A_3'B_2'-A_2'B_3')\sin\tau;$$

and these equations, the first multiplied by $(A_3'B_2'-A_2'B_3')$ and the second by $-(A_2'B_1'-A_1'B_2')$ and added, give

$$(A_3'B_2'-A_2'B_3')(A_2'X_2-A_1'Y_2)\frac{r^2}{n}-(A_2'B_1'-A_1'B_2')(A_3'Y_2-A_2'Z_2)\frac{r^2}{n}=0$$

or $$A_2'(A_3'B_2'-A_2'B_3')X_2+A_2'(A_1'B_3'-A_3'B_1')Y_2+A_2'(A_2'B_1'-A_1'B_2')Z_2=0;$$

Q.E.D.

that is, the path described by any particle X_2, Y_2, Z_2 lies in a plane, that determined by this equation.

17. Show that the following linear equation holds between the component-moments X_1, Y_1, Z_1 of formulae (8):

$$(A_3B_2-A_2B_3)X_1+(A_1B_3-A_3B_1)Y_1+(A_2B_1-A_1B_2)Z_1=0;$$

that is, the primary (electric) oscillations (8) take place in planes.

18. Show, when $\delta_1 = \delta_2 = \delta_3 \gtrless 0$, that the secondary electric oscillations represented by formulae (25) take place in planes that are at right angles to the respective directions of propagation throughout the plane

$$a_1 x + a_2 y + a_3 z = 0.$$

19. The secondary electric oscillations represented by formulae (25) become (elliptically) longitudinal, that is, they take place in planes, whose normals make right angles with the respective directions of propagation, throughout the plane

$$a_1 a_2 \sin n\delta_{12}\gamma - a_1 a_3 \beta \sin n\delta_{13}\beta + a_2 a_3 \sin n\delta_{23}\alpha = 0.$$

That the normal n to any plane of oscillation make a right angle with the direction of propagation, the analytic relation must hold

$$\cos(r, n) = \cos(r, x)\cos(n, x) + \cos(r, y)\cos(n, y) + \cos(r, z)\cos(n, z) = 0$$

$$\alpha\cos(n, x) + \beta\cos(n, y) + \gamma\cos(n, z) = 0.$$

Replace here $\cos(n, x)$, $\cos(n, y)$ and $\cos(n, z)$ by their values from formulae (32), and we have

$$\begin{aligned} &a_2 a_3 \alpha(2 - 3\alpha^2)\sin n\delta_{23} - 3a_1 a_2 \alpha^2\gamma \sin n\delta_{12} + 3a_1 a_3 \alpha^2 \beta \sin n\delta_{13} \\ &- a_1 a_3 \beta(2 - 3\beta^2)\sin n\delta_{13} - 3a_2 a_3 \alpha\beta^2 \sin n\delta_{23} - 3a_1 a_2 \beta^2 \gamma \sin n\delta_{12} \\ &+ a_1 a_2 \gamma(2 - 3\gamma^2)\sin n\delta_{12} + 3a_1 a_3 \beta\gamma^2 \sin n\delta_{13} - 3a_2 a_3 \alpha\gamma^2 \sin n\delta_{23} = 0 \end{aligned}$$

or

$$- a_1 a_2 \gamma \sin n\delta_{12} + a_1 a_3 \beta \sin n\delta_{13} - a_2 a_3 \alpha \sin n\delta_{23} = 0$$

(cf. also formula (22A) and Exs. 10 and 11).

20. Equation (14) is that of an ellipse, whose principal axes make the angle

$$\omega = \frac{1}{2}\arctan\frac{2(A_1 A_2 + B_1 B_2)}{A_1^2 + B_1^2 - A_2^2 - B_2}$$

with the coordinate-axes X_1 and Y_1.

We write equation (14)

$$aX_1^2 + bX_1 Y_1 + cY_1^2 + d = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(a)$$

putting

$$\left.\begin{aligned} a &= (A_2^2 + B_2^2)\frac{r^2}{n^4}, \quad b = -2(A_1 A_2 + B_1 B_2)\frac{r^2}{n^4} \\ c &= (A_1^2 + B_1^2)\frac{r^2}{n^4}, \quad d = -(A_1 B_2 - A_2 B_1)^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots(b)$$

To transform this equation to its principal axes, which we shall denote by u and v, we make use of the following familiar relations between the given $(X_1 Y_1)$ and the new coordinates uv:

$$\left.\begin{aligned} X_1 &= u\cos\omega - v\sin\omega \\ Y_1 &= u\sin\omega + v\cos\omega \end{aligned}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots(c)$$

where ω, which denotes the angle between the two systems of rectangular coordinates, shall be determined thereby, that the term uv of the equation of the ellipse in u and v sought vanish, that is, that u and v be the principal axes.

Replace X_1 and Y_1 by their values (c) in u and v in equation (a), and we have

$$a(u\cos\omega - v\sin\omega)^2 + b(u\cos\omega - v\sin\omega)(u\sin\omega + v\cos\omega) + c(u\sin\omega + v\cos\omega)^2 + d = 0$$

or

$$\left.\begin{aligned} &(a\cos^2\omega + b\sin\omega\cos\omega + c\sin^2\omega)u^2 - [2(a - c)\sin\omega\cos\omega - b(\cos^2\omega - \sin^2\omega)]uv \\ &\qquad + (a\sin^2\omega - b\sin\omega\cos\omega + c\cos^2\omega)v^2 + d = 0 \end{aligned}\right\}. \quad (d)$$

That the term in uv vanish, its coefficient must vanish; we thus have the following equation for the determination of ω:

$$2(a-c)\sin\omega\cos\omega - b(\cos^2\omega - \sin^2\omega) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (e)$$

or

$$(a-c)\sin 2\omega - b\cos 2\omega = 0,$$

hence

$$\tan 2\omega = \frac{b}{a-c}$$

or, by formulae (b),

$$\left.\begin{aligned}\tan 2\omega &= \frac{b}{a-c}\\ \tan 2\omega &= -\frac{2(A_1A_2+B_1B_2)}{A_2^2+B_2^2-A_1^2-B_1^2}\end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots\ldots (f)$$

21. Determine the lengths of the principal axes of the ellipse represented by formula (14).

The equation of the given ellipse referred to its principal axes, u and v, is evidently

$$(a\cos^2\omega + b\sin\omega\cos\omega + c\sin^2\omega)u^2 + (a\sin^2\omega - b\sin\omega\cos\omega + c\cos^2\omega)v^2 + d = 0$$

(cf. formula (d), Ex. 20), where ω is to be replaced by its value (f), Ex. 20.

Equation (e) or (f), Ex. 20, evidently gives

$$\sin^2\omega = \frac{1}{2} \pm \frac{1}{2}\frac{a-c}{\sqrt{b^2+(a-c)^2}},$$

$$\cos^2\omega = \frac{1}{2} \mp \frac{1}{2}\frac{a-c}{\sqrt{b^2+(a-c)^2}},$$

and

$$\sin\omega\cos\omega = \mp\frac{1}{2}\frac{b}{\sqrt{b^2+(a-c)^2}};$$

by which the equation in uv can be written

$$\left[\frac{a+c}{2} \mp \frac{1}{2}\sqrt{b^2+(a-c)^2}\right]u^2 + \left[\frac{a+c}{2} \pm \frac{1}{2}\sqrt{b^2+(a-c)^2}\right]v^2 + d = 0$$

or

$$\frac{u^2}{\dfrac{-d}{\dfrac{a+c}{2} \mp \dfrac{1}{2}\sqrt{b^2+(a-c)^2}}} + \frac{v^2}{\dfrac{-d}{\dfrac{a+c}{2} \pm \dfrac{1}{2}\sqrt{b^2+(a-c)^2}}} = 1.$$

The lengths of the principal axes of this ellipse are thus given by the expression

$$2\sqrt{\frac{-2d}{a+c\pm\sqrt{b^2+(a-c)^2}}}$$

or, by formulae (b), Ex. 20,

$$\frac{2\sqrt{2}n^2(A_1B_2-A_2B_1)}{r\sqrt{A_2^2+B_2^2+A_1^2+B_1^2 \pm \sqrt{4(A_1A_2+B_1B_2)^2+(A_2^2+B_2^2-A_1^2-B_1^2)^2}}},$$

where the plus sign is to be taken for the minor axis and the minus sign for the major.

22. Show that equation (3),

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2\cos n(\delta_1-\delta_2)\frac{xy}{a_1a_2} = \sin^2 n(\delta_1-\delta_2),$$

is that of an ellipse, whose principal axes make the angle

$$\omega = \frac{1}{2}\arctan\frac{2a_1a_2\cos n(\delta_1-\delta_2)}{a_1^2-a_2^2}$$

with the coordinate-axes x, y, and the lengths of whose major and minor axes are given by the expression

$$\frac{2\sqrt{2}a_1a_2 \sin n(\delta_1 - \delta_2)}{\sqrt{a_1^2 + a_2^2 \pm \sqrt{(a_1^2 + a_2^2)^2 - 4a_1^2a_2^2 \sin^2 n(\delta_1 - \delta_2)}}},$$

where the plus sign is to be taken for the minor and the minus sign for the major axis.

23. Show that the angle ω, which the principal axes of the ellipse (23) make with the coordinate-axes X_1, Z_1, is given by

$$\omega = \frac{1}{2} \arctan \frac{2(A_1A_3 + B_1B_3)}{A_1^2 + B_1^2 - A_3^2 - B_3^2}.$$

24. Show that the lengths of the major and minor axes of the ellipse (23) are given by the expression

$$\frac{2\sqrt{2}n^2(A_1B_3 - A_3B_1)}{r\sqrt{A_3^2 + B_3^2 + A_1^2 + B_1^2 \pm \sqrt{4(A_1A_3 + B_1B_3)^2 + (A_3^2 + B_3^2 - A_1^2 - B_1^2)^2}}},$$

where the plus sign is to be taken for the minor and the minus sign for the major axis.

25. Show that the angle, which the principal axes of the ellipse (35) make with the coordinate-axes X_2, Y_2, is given by a similar expression to that (f), Ex. 20, for the angle, which the principal axes of the ellipse (14) make with the coordinate-axes X_1, Y_1 (X_2, Y_2).

26. Show that the lengths of the major and minor axes of the ellipse (35) are given by the expression

$$\frac{2\sqrt{2}n(A_1'B_2' - A_2'B_1')}{r^2\sqrt{A_2'^2 + B_2'^2 + A_1'^2 + B_1'^2 \pm \sqrt{4(A_1'A_2' + B_1'B_2')^2 + (A_2'^2 + B_2'^2 - A_1'^2 - B_1'^2)^2}}},$$

where the plus sign is to be taken for the minor and the minus sign for the major axis.

27. Determine the angle, which the major and minor axes of the elliptic cylinders (b) of Ex. 10, whose intersections determine the path of oscillation of the primary magnetic wave that accompanies the electric wave represented by formulae (7), make with the coordinate-axes, and also the lengths of those axes.

28. Determine the angle, which the major and minor axes of the elliptic cylinders (b), Ex. 11, make with the coordinate-axes, and also the lengths of those axes.

29. Show that the breadths of the cylinders (14) and (23) with regard to the x-axis are given by one and the same expression

$$\frac{2n^2}{r}\sqrt{A_1^2 + B_1^2}.$$

CHAPTER IV.

INTERFERENCE; INTERFERENCE PHENOMENA OF THE PRIMARY AND SECONDARY (ELECTROMAGNETIC) WAVES.

Doctrine of Interference—The doctrine of interference is only another form or consequence of the principle of superposition, a superposition not of the intensities but of the displacements (amplitudes) of the given single oscillations. The phenomena of interference embrace those cases, where the resultant intensity of two or more oscillations is *not the sum* of the *single intensities*,* which is the case when the given oscillations are taking place at right angles to each other (cf. Chapter III.), and include the particular case, where the resultant intensity entirely vanishes. The doctrine of interference does not require us to make any new hypotheses, it is a direct consequence of the undulatory theory of light and can readily be deduced from the properties peculiar to the same: For take two systems of waves, represented by the moments X', Y', Z' and X'', Y'', Z'', and both particular integrals of our fundamental differential equations of wave-motion (cf. formulae (16, I)); since now these equations are linear and homogeneous, it follows that the system of waves represented by the sums of their respective component-moments,

$$X = X' + X'' \quad Y = Y' + Y'' \quad Z = Z + Z'',$$

will also be particular integrals of these equations. The system of waves represented by the moments X, Y, Z is now the sum of the two given single systems, X', Y', Z' and X'', Y'', Z'', that is, the resultant component-moments of the given single waves are found by the superposition of their respective component-moments or displacements. The doctrine of interference thus teaches that the intensity of two or more oscillations is not given by the sum of the intensities of the given single oscillations, but that it is the intensity of the (resultant) oscillation represented by the sums of the respective component moments of the given single oscillations; for the actual determination of this resultant intensity see below.

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* Strictly speaking, we must exclude here the particular case, where the given oscillations differ in phase by quarter of a wave-length (cf. p. 107 and Ex. 4).

Interference of Plane-Waves.—Let us first examine the interference of two similarly linearly polarized plane-waves of the same period of oscillation or wave-length (colour) λ but of different amplitude and phase, for example the two waves

$$\left.\begin{aligned} y' &= a_2' \sin n(vt - x') \\ y'' &= a_2'' \sin n(vt - x'') \end{aligned}\right\}, \qquad \text{(1)}$$

where $n = \frac{2\pi}{\lambda}$ (cf. formulae (31, II.)) and x' and x'' denote given distances on the x-axis.

The resultant displacement y at any time t is now according to the principle of superposition

$$\begin{aligned} y = y' + y'' &= a_2' \sin n(vt - x') + a_2'' \sin n(vt - x'') \\ &= a \sin n(vt - x), \end{aligned}$$

where a and x are to be determined as functions of a_2', a_2'', x', x'' and n. To find these quantities, we write this equation between the same and the five given quantities explicitly, as follows:

$$\begin{aligned} (a_2' \cos nx' + a_2'' \cos nx'') \sin nvt - (a_2' \sin nx' + a_2'' \sin nx'') \cos nvt \\ = a \sin nvt \cos nx - a \cos nvt \sin nx; \end{aligned}$$

from which evidently follow, since this equation must hold for all values of t,

$$\begin{aligned} a \cos nx &= a_2' \cos nx' + a_2'' \cos nx'', \\ a \sin nx &= a_2' \sin nx' + a_2'' \sin nx''; \end{aligned}$$

and these equations give

$$a^2 = a_2'^2 + a_2''^2 + 2a_2' a_2'' \cos n(x' - x''), \qquad \text{(2)}$$

and

$$\tan nx = \frac{a_2' \sin nx' + a_2'' \sin nx''}{a_2' \cos nx' + a_2'' \cos nx''}. \qquad \text{(3)}$$

The resultant amplitude a is thus a function of the two given amplitudes, a_2' and a_2'', and the quantity $n(x' - x'')$; the latter is known as the *difference in phase* of the two oscillations. The resultant phase is a function of the amplitudes and phases of the given oscillations. For a geometrical interpretation of these formulae see Ex. 1 at end of chapter.

When $x' - x'' = 0, \lambda, 2\lambda, \ldots$, that is, when the given oscillations have the same phase (or differ in phase by whole wave-lengths), then

$$a^2 = (a_2' + a_2'')^2,$$

that is, the resultant amplitude (intensity) becomes a maximum.

When the given oscillations differ in phase by half a wave-length, that is, when

$$x' - x'' = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2} \ldots,$$

then
$$a^2 = (a_2' - a_2'')^2,$$

and the resultant amplitude (intensity) becomes a minimum. For $a_2' = a_2''$, a then vanishes, and we have total (destructive) interference.

Lastly, when the given oscillations differ in phase by quarter of a wave-length, that is, when

$$x' - x'' = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots,$$

then
$$a^2 = a_2'^2 + a_2''^2, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

or the intensity of the resultant oscillation is given by the sum of the intensities of the given oscillations. In this particular case the given oscillations would appear to advance quite independently of each other, that is, not to interfere the one with the other, like two linearly polarized oscillations, whose planes of oscillation stand at right angles to each other, and which compound, as we have seen in Chapter III., to an elliptically polarized oscillation. Two similarly linearly polarized oscillations that differ in phase by quarter of a wave-length would thus produce the same effect (intensity) as two similar linearly polarized oscillations, whose planes of oscillations stand at right angles to each other (cf. Ex. 4).

Two linear oscillations of the same wave-length and plane of oscillation (polarization) do not therefore, in general, advance independently of each other, like oscillations, whose planes of oscillation are at right angles, but they interfere with each other, compounding to a linear oscillation, whose amplitude (intensity) increases or decreases, or even vanishes ($a_2' = a_2''$), according to the difference in phase between the given oscillations (cf. formula (2)) and whose phase is determined as a function of the given amplitudes and phases by formula (3).

Phenomena of Interference: Bright and Dark Bands.—We have just seen that two oscillations (1) that differ only in amplitude and phase co-operate or (partially) neutralize each other according to their difference in phase. The amplitudes of waves from one and the same source are now, in general, the same, whereas the waves themselves differ in phase according to the distances they traverse. The resultant amplitude of two waves from one and the same source would, therefore, be double and hence their intensity four times that of either wave singly, when their phases were the same, and vanish entirely, when they differed in phase by half a wave-length. Let us now consider the effect produced on a screen that is illuminated by similar waves from two sources that are close together—to obtain two systems of similar waves (beams of light), we let the waves from any given source pass through a very narrow slit and then through two apertures (sources) that are close together (cf. pp. 112-113). Let the screen of observation AB be placed at right angles to the mean direction of propagation of the waves or to the perpendicular OP to the line (plane) CD joining the two apertures (sources) C and D at its middle point O, where P is a point of the screen (cf. Fig. 11). The waves from C and D will now co-operate at P, since the distances CP and DP traversed by the same are equal, and we shall have a bright spot of four times the intensity of that produced at that point by either wave singly. As we recede along the screen from the central point P upwards towards A, the distance to the one source C will decrease and that to the other D increase, until we arrive at a point P_1, where the difference in these distances becomes half a wave-length; at that point the given waves differ in phase by half a wave-length and thus neutralize each other, that is, the illumination will be zero. Similarly, as we recede downwards towards B, the distance to C will increase and that to D decrease, until we arrive at a point P_1', where the difference in these distances becomes half a wave-length and

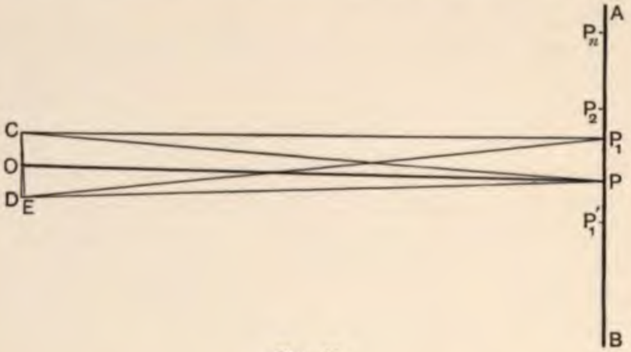

FIG. 11.

hence no illumination. The locus of the point P_1 or P_1' is evidently in the plane of the paper an hyperbola and in space an hyperboloid of revolution, that generated by the revolution of the given hyperbola round the line CD as axis. The locus of P_1 on the screen is the line (curve) intersected on the same by that hyperboloid; this line will appear on the screen as a dark line or band. As we continue to recede from the central point P, the distance to the one source will increase or decrease and that to the other decrease or increase respectively, until we arrive at a point P_2 (P_2'), where the difference in those distances becomes a whole wave-length; here the given waves co-operate again and we have a bright spot similar to that at P; similarly, the locus of P_2 is a hyperboloid of revolution, whose intersection on the screen determines the position of a (the first) bright line or band. Similarly, as we continue to recede from the central point P, we obtain alternately dark and bright lines or bands, determined by the intersections of hyperboloids of revolution on the screen.

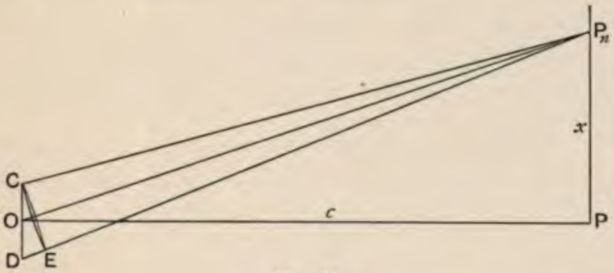

FIG. 12.

Distance of any Band from Central Point.—Let us determine the distance of any band P_n from the central point P in terms of the given quantities. The band P_n evidently corresponds to a difference in the distances traversed by the two waves or to a retardation of the one over the other of n half wave-lengths. We denote the distance of the band P_n from P by x, the distance of the screen AB from the sources C and D (the distance OP) by c and the distance CD between the sources by b (cf. Fig. 12). With P_n as centre describe an arc of radius P_nC from C to the point E of the line DP_n and draw the straight line CE, as in figure. The line CE is now perpendicular to OP_n and CD to OP, therefore the angle DCE will be equal to the angle P_nOP and hence the right-angle triangle DCE* similar to the right-angle triangle P_nOP. The similarity of these triangles gives now the following proportion between their sides:

---

* The angle DEC is only approximately a right angle.

$$PP_n : OP = DE : CE \quad \ldots\ldots\ldots\ldots (5)$$

or

$$x : c \;= \frac{n\lambda}{2} : CE \text{ approximately,*}$$

which for small values of the angle DCE, corresponding to small values of x, may evidently be written approximately

$$x : c = \frac{n\lambda}{2} : CD = \frac{n\lambda}{2} : b,$$

or

$$x = \frac{n\lambda}{2} \frac{c}{b} \dagger \quad \ldots\ldots\ldots\ldots (6)$$

The given band will, therefore, be bright or dark according as n is even or odd, whereas its distance from the central point will vary directly as the wave-length (colour) of the waves employed.

Width of Band.—By formula (6) the width of any band, from darkness to darkness, is evidently

$$x_n - x_{n-2} = \left[\frac{n\lambda}{2} - \frac{(n-2)\lambda}{2}\right]\frac{c}{b} = \lambda\frac{c}{b}; \quad \ldots\ldots\ldots\ldots (7)$$

that is, it is directly proportional to the wave-length (colour) of the waves employed. Since now this width and the distances b and c can be ascertained by measurement, we can employ this formula for the determination of the wave-length of different kinds (colours) of light. If the light-waves employed could be procured absolutely homogeneous, that is, waves of exactly one and the same wave-length (colour) λ, then the screen would be covered entirely with similar ‡ bright and dark bands; but neither is the former possible nor is the latter confirmed by experiment (see below).

Coloured Bands or Fringes.—If the light waves employed are heterogeneous or those of ordinary (white) light, we evidently get a system of coloured bands or fringes: for, each wave-length or colour represented in the given waves will give rise to a system of bands of given (but different for different colours) width, the violet bands being the narrowest and the red the broadest. Near the central point or band, which will be brightly illuminated but not coloured, the resultant (coloured) interference bands will be very distinct, but, as we recede from the same, there will be an overlapping of these numerous systems of coloured bands, and this overlapping will not only increase more and more but it will become more and more irregular, until finally, at a comparatively short distance from the central band, the total interference will become approximately one and the same at all points in regard not only to (resultant) intensity but to colour; this accounts for the rapid disappearance of the coloured interference bands as we recede from the central point and their entire obliteration at a comparatively short distance from the same. For similar reasons, together with the fact that it is quite impossible to procure absolutely homogeneous light, the interference bands of "homogeneous" waves will extend to no great distance from the central band, although, of course, to a much greater distance than the coloured bands obtained from heterogeneous or white light; this explains the empirical fact that only a small portion of the screen can be covered with interference bands in spite of the most skilful contrivances for procuring homogeneous light.

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* The angle DEC is only approximately a right angle.

† This formula holds only for small values of x; for large values of x see Ex. 12 at end of chapter.

‡ Provided the screen be small, that is, formula (7) hold (cf. Ex. 12 at end of chapter).

Conditions for the Interference of Polarized Waves.—We have assumed above given paths of oscillation for the waves treated, that is, we have examined polarized oscillations, whose amplitudes have been assumed to remain the same for finite intervals. This assumption holds now only for polarized waves that are obtained from *one* and the *same polarized* wave. Two polarized waves obtained from an ordinary (homogeneous) wave, for example, the ordinary and extraordinary waves (rays) that emerge from a doubly refracting crystal, upon the surface of which a ray of ordinary (non-polarized) light is incident, will each be linearly polarized and at right angles to each other (cf. Chapter VIII.), and each will retain its character, the same amplitude, etc., as long as the vibration in the incident wave remains one and the same. During this interval the two waves upon being brought into the same plane of polarization would interfere like the ordinary and extraordinary waves that are obtained from one and the same *polarized* wave by double refraction (see below); this interference is a consequence of the difference in phase between the two waves, due to a relative retardation of the one with respect to the other in their passage through the crystal and to the different paths traversed by the same. The interval, during which the refracted waves retain the same amplitudes or the incident wave the same direction of vibration, is now infinitely short, since the direction of oscillation in an ordinary wave changes, as we have observed on p. 72, thousands of times per second. During the succeeding interval of one and the same direction of vibration in incident wave each refracted (component) wave would retain one and the same amplitude, but these (component) amplitudes would differ from those, which the refracted waves had in the first interval, and hence also the interferences and resultant intensities during those two intervals. It is now the mean of thousands of such different intensities that is received as total effect by the retina of the eye, and that mean would evidently be approximately one and the same at every point of the field, quite regardless of the variation of the difference in phase between the two (refracted) waves from point to point, due to any difference in the distances traversed by the two waves. The field or screen would thus be uniformly illuminated, that is, there would be no perceptible or permanent interference, in which case the waves are said not to interfere permanently.

If the incident wave is polarized, the refracted waves will retain the same character or amplitudes for *finite* intervals and thus interfere permanently when brought into the same plane of polarization; for at those points of the field, where the refracted waves have the same phase, there will be permanent co-operation or maxima of intensity, and at those, where they differ in phase by half a wave-length, permanent (partial) neutralization or minima of intensity; that is, we shall have phenomena of permanent interference, and the given waves are said to interfere.

Conditions for the Interference of Ordinary Homogeneous Waves.— Two waves of ordinary homogeneous light from different sources or from different parts of the same source (flame) do not interfere (when brought to overlap). This is evident from the following: the character, both direction of oscillation and phase, of the wave from the one source will change irregularly and, as we have already observed, thousands of times per second, with regard to the character (direction of oscillation and phase) of the wave from the other source, and we shall thus have co-operation and neutralization in such rapid succession that only the mean of the same over the (finite) interval required for an impression on the retina of the eye can come into consideration; and this mean will be approximately the same at all points of the field, since any difference in phase, due to difference in the paths traversed by the two waves, can evidently be entirely neglected. From considerations similar to those on the interference of polarized waves obtained by double refraction from *one* and the *same polarized* wave, it is evident that two ordinary homogeneous waves can interfere only when they are exactly alike. To obtain two such similar waves, we let a beam of ordinary light fall on a narrow slit (in a screen), placed symmetrically near two apertures and with its length at right angles to the line joining the same; the slit must be made so narrow that it admits only a line of light. Each point of the slit being symmetrical with regard to the two apertures will now send waves to each aperture that are alike, so that the resultant wave emitted by the one will be similar to that emitted by the other. At any point of the region (screen), where the waves from these two apertures overlap, the only difference between them will be one in phase, due to the different distances traversed; and this difference in phase will be one and the same for the thousands of oscillations of different direction of oscillation arriving at the given point during the interval necessary for an impression on the retina of the eye. At those points of the field, where there is no difference in phase between the waves emitted by the two apertures, we shall have co-operation or maxima of intensity, at those points, where the difference in phase is half a wave-length, neutralization or minima of intensity, and at all intermediate points intensities that correspond to the position of the same with respect to the points of maximum and minimum intensities; that is, we shall have a (permanent) system of bright and dark bands; and the given waves are said to interfere permanently.

Conditions for Interference of Heterogeneous Waves.—For reasons similar to those on the preceding page, it is evident that two waves of white or heterogeneous light (cf. p. 72) can interfere only when they are exactly alike; such waves may be produced in a similar manner to that suggested above for the generation of similar waves of homogeneous light.

Fresnel's and Arago's Laws on Interference.—The above results on the conditions for interference can evidently be summarized in the following laws, which were first stated and empirically established by Fresnel* and Arago:

(1) Two waves (rays of light) polarized at right angles do not interfere under the same circumstances as two waves (rays) of ordinary light.

(2) Two waves (rays) of light polarized in the same plane interfere like two waves (rays) of ordinary light.

(3) Two waves (rays of light) polarized at right angles may be brought to the same plane of polarization without thereby acquiring the quality of being able to interfere (permanently) with each other.

(4) Two waves (rays of light) polarized at right angles and afterwards brought to the same plane of polarization interfere (permanently) like waves (rays) of ordinary light, if they originally belonged to the same wave (beam) of polarized light.

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*Cf. *Oeuvres*, tom. 1, p. 521.

Conditions for Interference of Elliptically Polarized Waves.—We have been considering above only the linearly polarized oscillations and the phenomena of interference to which they give rise. Let us now examine the interference of elliptically polarized oscillations and waves. Since now an elliptically polarized oscillation can be resolved into two rectangular linearly polarized ones, the interference of two elliptically polarized oscillations could be determined as follows: we resolve each elliptically polarized oscillation into any two rectangular component linearly polarized ones, that is, we resolve each along any two rectangular axes, as the y and z coordinate-axes, the x-axis being chosen as direction of propagation; the two component oscillations along either axis would now behave like two linearly polarized oscillations, that is, they would interfere permanently, if they belonged originally to *one* and the *same polarized* wave, producing a system of interference bands on a screen placed in the field, where they overlapped; but the two component oscillations along the one axis would, in general, differ from the two along the other axis not only in amplitude but also in phase, so that the system of interference bands produced by the one pair of component linear oscillations would differ both in intensity and position of the bands from that produced by the other pair; the resultant effect produced on the screen would be that due to the mutual action of these two systems of bands, which would also be a system of bands. The resultant system of bands would evidently be more or less distinct according as the given oscillations were less or more elliptically polarized respectively, the bands disappearing entirely, when the given polarization were circular, and becoming most distinct, when it approached the linear polarization.

Resultant of two Elliptically Polarized (Plane) Waves.—Let us determine the resultant of two elliptically polarized plane-waves of the same period of oscillation; let the given waves be represented by the analytical expressions

$$\left.\begin{aligned} y' &= a_2' \sin n(vt - x') \\ z' &= a_3' \sin n(vt - x' - \delta') \\ \text{and} \quad y'' &= a_2'' \sin n(vt - x'') \\ z'' &= a_3'' \sin n(vt - x'' - \delta'') \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots (8)$$

where x' and x'' denote given distances on the x-axis, the direction of propagation, and δ' and δ'' small augmentations of those distances.

By the principle of superposition the resultant of the waves (8) will be that wave, whose two components are

$$y = y' + y'' = a_2' \sin n(vt - x') + a_2'' \sin n(vt - x''),$$
$$z = z' + z'' = a_3' \sin n(vt - x' - \delta') + a_3'' \sin n(vt - x'' - \delta''),$$

or, expanded,

$$y=(a_2' \cos nx' + a_2'' \cos nx'') \sin nvt - (a_2' \sin nx' + a_2'' \sin nx'') \cos nvt,$$

$$\begin{aligned} z=[a_3' \cos n(x'+\delta') + a_3'' \cos n(x''+\delta'')] \sin nvt \\ - [a_3' \sin n(x'+\delta') + a_3'' \sin n(x''+\delta'')] \cos nvt, \end{aligned}$$

or

$$y=a_2 \sin n(vt-x), \quad z=a_3 \sin n(vt-x-\delta), \ldots\ldots\ldots\ldots\ldots (9)$$

where a_2, a_3, x, and δ are determined by the equations

$$\begin{aligned} a_2' \cos nx' + a_2'' \cos nx'' &= a_2 \cos nx, \\ a_2' \sin nx' + a_2'' \sin nx'' &= a_2 \sin nx, \\ a_3' \cos n(x+\delta') + a_3'' \cos n(x''+\delta'') &= a_3 \cos n(x+\delta), \\ a_3' \sin n(x'+\delta') + a_3'' \sin n(x''+\delta'') &= a_3 \sin n(x+\delta), \end{aligned}$$

as follows :

$$\left. \begin{aligned} a_2^2 &= a_2'^2 + a_2''^2 + 2a_2' a_2'' \cos n(x'-x'') \\ \tan nx &= \frac{a_2' \sin nx' + a_2'' \sin nx''}{a'_2 \cos nx' + a''_2 \cos n''x} \end{aligned} \right\}, \ldots\ldots\ldots\ldots\ldots (10)$$

and

$$\left. \begin{aligned} a_3^2 &= a_3'^2 + a_3''^2 + 2a_3' a_3'' \cos n(x'+\delta'-x''-\delta'') \\ \tan n(x+\delta) &= \frac{a_3' \cos n(x'+\delta') + a_3'' \cos n(x''+\delta'')}{a_3' \sin n(x'+\delta') + a_3'' \sin n(x''+\delta'')} \end{aligned} \right\}. \ldots\ldots\ldots (11)$$

The resultant oscillation will, therefore, be elliptic, that one, whose rectangular linear component oscillations (9), their amplitudes and phases, are determined by these formulae (10) and (11). Two elliptic oscillations (8) will thus compound to an elliptic oscillation.

Spherical Waves.—We have examined above waves, whose wave-fronts have been assumed to be plane; that is, either their source must be at infinite distance (the sun) or the waves themselves, upon being emitted from a source at finite distance, must be brought by means of a lens to advance along parallel lines. Strictly speaking, such waves do not exist in nature, so that the analytic expressions for the same would have no real meaning; they constitute only a particular or limiting (theoretical) case of the general one, where the source is at finite distance and the waves themselves are propagated radially or in spherical shells or wave-fronts from the same, their amplitudes decreasing as their distance from the source increases; hence the termination "spherical" waves. These more general spherical waves open up a much broader and more interesting field for research than the (theoretical) ones examined above, since they behave quite differently at different points of any region of finite dimensions (cf. below).

Resultant of two Linearly Polarized Spherical Waves.—Let us first determine the resultant of two similarly linearly polarized spherical waves, for example those represented by the analytic expressions

$$\left.\begin{aligned} y' &= \frac{a'}{x'} \sin n(vt - x') \\ y'' &= \frac{a''}{x''} \sin n(vt - x'') \end{aligned}\right\}, \dots\dots\dots\dots\dots\dots\dots (12)$$

where x' and x'' denote the distances of the waves from their respective sources.

By the principle of superposition the resultant of the waves (12) will be the wave

$$y = y' + y'' = \frac{a'}{x'} \sin n(vt - x') + \frac{a''}{x''} \sin n(vt - x'') = a \sin n(vt - x),$$

where a and nx are to be determined from the equations

$$\frac{a'}{x'} \cos nx' + \frac{a''}{x''} \cos nx'' = a \cos nx,$$

$$\frac{a'}{x'} \sin nx' + \frac{a''}{x''} \sin nx'' = a \sin nx;$$

which give

$$\left.\begin{aligned} a^2 &= \frac{a'^2}{x'^2} + \frac{a''^2}{x''^2} + \frac{2a'a''}{x'x''} \cos n(x' - x'') \\ \text{and} \qquad \tan nx &= \frac{a'x'' \sin nx' + a''x' \sin nx''}{a'x'' \cos nx' + a''x' \cos nx''} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (13)$$

These expressions differ from those already found for plane-waves (cf. formulae (2) and (3)) therein only, that they contain the distances x' and x'' of the given waves from their sources; otherwise the results obtained and the conclusions to be drawn therefrom are similar to those already stated on pp. 106-110.

Resultant of two Elliptically Polarized Spherical Waves.—Similarly, we can determine the resultant of two elliptically polarized spherical waves of the same period of oscillation, for example those represented by the expressions

$$\left.\begin{aligned} y' &= \frac{a_2'}{x'} \sin n(vt - x') \\ z' &= \frac{a_3'}{x'} \sin n(vt - x' - \delta') \\ \text{and} \qquad y'' &= \frac{a_2''}{x''} \sin n(vt - x'') \\ z'' &= \frac{a_3''}{x''} \sin n(vt - x'' - \delta'') \end{aligned}\right\}, \dots\dots\dots\dots\dots\dots (14)$$

where x' and x'' denote the distances of the waves from their respective sources. We find, namely, that the resultant wave is also an elliptically polarized spherical wave, that, to which the two rectangular linearly polarized spherical waves

$$y = a_2 \sin n(vt - x) \quad \text{and} \quad z = a_3 \sin n(vt - x - \delta),^*$$

whose amplitudes and phases are determined by the following formulae, compound:

$$\left.\begin{aligned} a_2{}^2 &= \frac{a_2'^2}{x'^2} + \frac{a_2''^2}{x''^2} + \frac{2a_2'a_2''}{x'x''} \cos n(x' - x'') \\ \tan nx &= \frac{a_2'x'' \sin nx' + a_2''x' \sin nx''}{a_2'x'' \cos nx' + a_2''x' \cos nx''} \\ \text{and} \qquad a_3{}^2 &= \frac{a_3'^2}{x'^2} + \frac{a_3''^2}{x''^2} + \frac{2a_3'a_3''}{x'x''} \cos n(x' + \delta' - x'' - \delta'') \\ \tan n(x + \delta) &= \frac{a_3'x'' \sin n(x' + \delta') + a_3''x' \sin n(x'' + \delta'')}{a_3'x'' \cos n(x' + \delta') + a_3''x' \cos n(x'' + \delta'')} \end{aligned}\right\} \ldots\ldots\ldots (15)$$

Lastly, we observe that the resultant intensity of superposed oscillations, as (14), is determined not alone by the squares of the amplitudes of the given single oscillations but also by the interference-term or terms, to which the superposition of the similar displacements (moments) in question give rise (cf. Exs. 5, 6, 10 and 11 at end of chapter).

The Electromagnetic Waves; those of Problem 3, Chapter II.—We have considered above single waves, that is, waves propagated in a given direction or along a given vector without any reference to possible disturbances along other vectors. Such waves cannot *a priori* be identical to electromagnetic waves, since the latter are, in general, propagated from their source in all directions or along all vectors. At the same time, we have made no attempt to identify the waves treated above with the electromagnetic waves, having taken quite arbitrary solutions of the general equation of wave-motion without any reference to the relations that must hold between such solutions, if the same are to represent electromagnetic waves.

The electromagnetic waves examined in Chapter II. are all linearly polarized† spherical waves. As those of problem 3 are the most general, let us employ the same for an examination of the field traversed by two systems of linearly polarized† electromagnetic waves, that are similar with regard to their radial distribution of energy from source into space (see below).

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* a_2 and a_3 are here not constant but functions of the distances from the sources x' and x''.

† Cf. p. 78.

Conditions for Interference of Elliptically Polarized Waves.—We have been considering above only the linearly polarized oscillations and the phenomena of interference to which they give rise. Let us now examine the interference of elliptically polarized oscillations and waves. Since now an elliptically polarized oscillation can be resolved into two rectangular linearly polarized ones, the interference of two elliptically polarized oscillations could be determined as follows: we resolve each elliptically polarized oscillation into any two rectangular component linearly polarized ones, that is, we resolve each along any two rectangular axes, as the y and z coordinate-axes, the x-axis being chosen as direction of propagation; the two component oscillations along either axis would now behave like two linearly polarized oscillations, that is, they would interfere permanently, if they belonged originally to *one* and the *same polarized* wave, producing a system of interference bands on a screen placed in the field, where they overlapped; but the two component oscillations along the one axis would, in general, differ from the two along the other axis not only in amplitude but also in phase, so that the system of interference bands produced by the one pair of component linear oscillations would differ both in intensity and position of the bands from that produced by the other pair; the resultant effect produced on the screen would be that due to the mutual action of these two systems of bands, which would also be a system of bands. The resultant system of bands would evidently be more or less distinct according as the given oscillations were less or more elliptically polarized respectively, the bands disappearing entirely, when the given polarization were circular, and becoming most distinct, when it approached the linear polarization.

Resultant of two Elliptically Polarized (Plane) Waves.—Let us determine the resultant of two elliptically polarized plane-waves of the same period of oscillation; let the given waves be represented by the analytical expressions

$$\left.\begin{aligned} y' &= a_2' \sin n(vt - x') \\ z' &= a_3' \sin n(vt - x' - \delta') \\ \text{and} \qquad y'' &= a_2'' \sin n(vt - x'') \\ z'' &= a_3'' \sin n(vt - x'' - \delta'') \end{aligned}\right\} \text{..........................} \quad (8)$$

where x' and x'' denote given distances on the x-axis, the direction of propagation, and δ' and δ'' small augmentations of those distances.

By the principle of superposition the resultant of the waves (8) will be that wave, whose two components are

$$\begin{aligned} y &= y' + y'' = a_2' \sin n(vt - x') + a_2'' \sin n(vt - x''), \\ z &= z' + z'' = a_3' \sin n(vt - x' - \delta') + a_3'' \sin n(vt - x'' - \delta''), \end{aligned}$$

The Resultant Primary Wave; its Amplitude and Phase.—The resultant action of the two waves represented by formulae (16) at any point P of the field (cf. formulae (17)) is now given by the component-moments

$$\begin{aligned}
X_1 = X_1' + X_1'' &= \frac{n^2}{r'} l' \sin \omega' + \frac{n^2}{r''} l'' \sin \omega'' \\
&= \frac{n^2}{r'} l' \left[\sin nvt \cos n(r'+\delta') - \cos nvt \sin n(r'+\delta')\right] \\
&\quad + \frac{n^2}{r''} l'' \left[\sin nvt \cos n(r''+\delta'') - \cos nvt \sin n(r''+\delta'')\right] \\
&= n^2\left[\frac{l'}{r'} \cos n(r'+\delta') + \frac{l''}{r''} \cos n(r''+\delta'')\right] \sin nvt \\
&\quad - n^2\left[\frac{l'}{r'} \sin n(r'+\delta') + \frac{l''}{r''} \sin n(r''+\delta'')\right] \cos nvt,
\end{aligned}$$

which can be written in the form

$$X_1 = a_1 \sin n[vt - (r+\delta_1)] = a_1 \sin nvt \cos n(r+\delta_1) - a_1 \cos nvt \sin n(r+\delta_1), \quad (19)$$

where a_1 and $n(r+\delta_1)$ are determined by the equations

$$n^2\left[\frac{l'}{r'} \cos n(r'+\delta') + \frac{l''}{r''} \cos n(r''+\delta'')\right] = a_1 \cos n(r+\delta_1)$$

and
$$n^2\left[\frac{l'}{r'} \sin n(r'+\delta') + \frac{l''}{r''} \sin n(r''+\delta'')\right] = a_1 \sin n(r+\delta_1)$$

as follows:

$$\left.\begin{aligned}
a_1^2 &= n^4\left[\frac{l'^2}{r'^2} + \frac{l''^2}{r''^2} + 2\,\frac{l'l''}{r'r''} \cos n(r' - r'' + \delta' - \delta'')\right] \\
\text{and} \quad \tan n(r+\delta_1) &= \frac{l'r'' \sin n(r'+\delta') + l''r' \sin n(r''+\delta'')}{l'r'' \cos n(r'+\delta') + l''r' \cos n(r''+\delta'')}
\end{aligned}\right\}, \quad (19\text{A})$$

and similarly,

$$Y_1 = a_2 \sin n[vt - (r+\delta_2)] \quad \text{and} \quad Z_1 = a_3 \sin n[vt - (r+\delta_3)], \ldots (20)$$

$$\left.\begin{aligned}
\text{where} \quad a_2^2 &= n^4\left[\frac{m'^2}{r'^2} + \frac{m''^2}{r''^2} + 2\,\frac{m'm''}{r'r''} \cos n(r' - r'' + \delta' - \delta'')\right] \\
\tan n(r+\delta_2) &= \frac{m'r'' \sin n(r'+\delta') + m''r' \sin n(r''+\delta'')}{m'r'' \cos n(r'+\delta') + m''r' \cos n(r''+\delta'')} \\
\text{and} \quad a_3^2 &= n^4\left[\frac{p'^2}{r'^2} + \frac{p''^2}{r''^2} + 2\,\frac{p'p''}{r'r''} \cos n(r' - r'' + \delta' - \delta'')\right] \\
\tan n(r+\delta_3) &= \frac{p'r'' \sin n(r'+\delta') + p''r' \sin n(r''+\delta'')}{p'r'' \cos n(r'+\delta') + p''r' \cos n(r''+\delta'')}
\end{aligned}\right\}. \quad (20\text{A})$$

The resultant oscillation will, therefore, be that, whose rectangular linear component-oscillations are

$$X_1 = a_1 \sin n\,[vt - (r + \delta_1)],$$
$$Y_1 = a_2 \sin n\,[vt - (r + \delta_2)],$$
$$Z_1 = a_3 \sin n\,[vt - (r + \delta_3)],$$

where a_1, a_2, a_3 and δ_1, δ_2, δ_3 $(r + \delta_1,\ r + \delta_2,\ r + \delta_3)$ are determined by formulae (19A) and (20A). Since these component-oscillations differ not only in amplitude but also in phase from one another, the resultant oscillation will be elliptic, that is, the resultant of the two primary waves (16) will be an elliptically polarized spherical wave (cf. Ex. 14).

Examination of Expression for Amplitude.—The amplitude of the resultant primary wave (19) and (20) will evidently be given by the expression

$$a^2 = a_1{}^2 + a_2{}^2 + a_3{}^2 = n^4 \left[\frac{l'^2 + m'^2 + p'^2}{r'^2} + \frac{l''^2 + m''^2 + p''^2}{r''^2} + 2\frac{l'l'' + m'm'' + p'p''}{r'r''} \cos n(r' - r'' + \delta' - \delta'') \right] \quad \ldots.(21)$$

The first term of the expression in the larger brackets, times n^4, is the amplitude squared of the wave X_1', Y_1', Z_1', were it advancing alone through the medium, and the second term, times n^4, that of the wave X_1'', Y_1'', Z_1'', were it alone in the medium. The third term arises from the simultaneous presence or action of both waves at any point P; it represents the interference of those waves at that point. The quantities δ' and δ'' of this interference term can be regarded as given, they express given differences of phase *in* the two sources. The δ in either source has also been assumed (cf. formulae (16) and (18)) to have one and the same value along all vectors from that source, that is, to remain constant throughout the medium.

Resultant Primary Wave Elliptically Polarized; Conditions for Linear Polarization of Resultant Wave.—We have observed that the resultant oscillations X_1, Y_1, Z_1 of formulae (19) and (20) take place in elliptic paths. If now the sources of the two linearly polarized waves (16) are near together and the point of observation P is at considerable distance from those sources, the two oscillations (16) will take place along approximately the same lines (see below), provided the proportion

$$a_1' : a_2' : a_3' = a_1'' : a_2'' : a_3'' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

hold between the amplitudes a_1', a_2', a_3' and a_1'', a_2'', a_3'' (cf. formulae

(6, III.)). But, if these amplitudes are entirely arbitrary, the lines of oscillation of the two systems of waves will, in general, make finite angles with each other and the resultant oscillations X_1, Y_1, Z_1 thus be highly elliptically polarized. This general case, where no relation exist between the amplitudes a_1', a_2', a_3' and a_1'', a_2'', a_3'' would, therefore, be of little interest, since, as we have seen in Chapter III., two linearly polarized waves, whose planes of oscillation are not the same, interfere less and less, as the angle between the same approaches more and more 90°.

In the following we shall thus assume the relation (22). As the distance between the two sources decreases and the point of observation P recedes from those sources, the angle between the planes of oscillation of the two oscillations (16) will become smaller and smaller, that is, the eccentricity of the elliptic path of oscillation of the resultant oscillation X_1, Y_1, Z_1 will become greater and greater and hence also the interference between the two given oscillations (16); for, the nearer the sources are together and the further the point of observation is removed from the same, the more the angles, which the vectors r' and r'' make with their respective coordinate-axes, and hence the direction-cosines of those vectors, α', β', γ' and α'', β'', γ'', approach one and the same values. On the other hand, if the distance between the sources O' and O'' of the two waves (16) is of the same dimensions as the distances of the point of observation P from those sources, then the angle between the planes of oscillation of the given oscillations will be of finite dimensions and hence the resultant oscillation X_1, Y_1, Z_1 more or less highly elliptically polarized. This general case is evidently of no particular interest as far as the phenomena of interference are concerned, whereas, its examination would offer difficulties, which we do not encounter, when the given sources are very (infinitely) near together and the point of observation P is at considerable (finite) distance from the same; in the general case all distances would namely be of the same dimensions, so that the expressions in question could not be replaced by first approximations, obtained on their expansion according to any small quantity or distance (cf. below).

It should now be possible to deduce the conclusions just drawn directly from the expression (21) for the resultant amplitude; let us examine the same. The first and second terms of the given expression cannot vanish, but they will assume given (positive) values at any given point; the third term contains the two factors, $l'l''+m'm''+p'p''$, which we shall call the coefficient of that term, and $\cos n(r'-r''+\delta'-\delta'')$; the latter, which gives rise to the phenomena of interference, will vary periodically throughout any region of very (infinitely) small dimensions, but it is not a function of the relative position of that region with regard to the coordinate-axes; in this respect it differs from the other factor or its coefficient, which, like the first and second terms of the given expression, will evidently retain approximately one and the same value throughout any such region. The mean value assumed by the third term of the expression (21) in any such region will thus depend alone on the value of the coefficient of that term in that region. This coefficient can now evidently vanish; its vanishing would determine given regions (cf. Exs. 15, 16 and 18), within which there would be no interference of the given waves (16); the vanishing of this coefficient would thus correspond to the particular case, where the planes of oscillation of the two waves (16) make right angles with each other. On the other hand, maximum values of the given coefficient would correspond to the particular case, where the two oscillations (16) are taking place along one and the same lines. On the assumption that relation (22) hold and the two sources O' and O'' be very (infinitely) near together, the lines of oscillation of the two waves (16) will be, as we have seen on p. 121, approximately one and the same at any distant point, that is, the given waves will interfere throughout *all* distant regions; in formula (21) this would evidently correspond to that particular form of the same, where the coefficient

$$l'l'' + m'm'' + p'p''$$

does not vanish (cf. below).

The Sources of Disturbance near together and the Point of Observation at Great Distance.—On the assumption of relation (22) and that the sources O' and O'' be very (infinitely) near together and the point of observation P at considerable (finite) distance from the same, α', β', γ' and $\alpha'', \beta'', \gamma''$ respectively will differ only infinitesimally from one another; that is, we can put

$$\alpha'' = \alpha' = \alpha, \quad \beta'' = \beta' = \beta, \quad \gamma'' = \gamma' = \gamma,$$

and hence

$$l'' : m'' : p'' = l' : m' : p'.$$

Formulae (17) then assume the particular form

$$l' = a_1'(\beta^2 + \gamma^2) - \alpha(a_2'\beta + a_3'\gamma) = \frac{l''}{\kappa},$$

$$m' = a_2'(\alpha^2 + \gamma^2) - \beta(a_1'\alpha + a_3'\gamma) = \frac{m''}{\kappa}.$$

$$p'' = a_3'(\alpha^2 + \beta^2) - \gamma(a_1'\alpha + a_2'\beta) = \frac{p''}{\kappa},$$

where κ denotes the factor of proportionality between the a''s and a'''s.

Replace l', m', p' and $l,''$ m'', p'' by these values in formula (21), and we have

$$\begin{aligned} a^2 = n^4[&a_1'^2(\beta^2+\gamma^2)+a_2'^2(\alpha^2+\gamma^2)+a_3'^2(\alpha^2+\beta^2) \\ &-2a_1'a_2'\alpha\beta-2a_1'a_3'\alpha\gamma-2a_2'a_3'\beta\gamma] \\ &\times\left[\frac{1}{r'^2}+\frac{\kappa^2}{r''^2}+\frac{2\kappa}{r'r''}\cos n(r'-r''+\delta'-\delta'')\right], \\ = n^4[&a_1'^2+a_2'^2+a_3'^2-(a_1'\alpha+a_2'\beta+a_3'\gamma)^2] \\ &\times\left[\frac{1}{r'^2}+\frac{\kappa^2}{r''^2}+\frac{2\kappa}{r'r''}\cos n(r'-r''+\delta'-\delta'')\right], \end{aligned}$$

or, most approximately,

$$\begin{aligned} = \frac{n^4}{r^2}[&a_1'^2+a_2'^2+a_3'^2-(a_1'\alpha+a'_2\beta+a_3'\gamma)^2] \\ &\times[1+\kappa^2+2\kappa\cos n(r'-r''+\delta'-\delta'')], \quad \ldots\ldots\ldots\ldots(23) \end{aligned}$$

where r denotes the mean distance of the point of observation from the given sources.

Examination of Expression for Resultant Amplitude; its Behaviour for Light Waves.—The expression in the first pair of large brackets of the expression (23) for the resultant amplitude is a function only of the direction-cosines α, β, γ and the a's; it can thus be regarded as constant in any region, whose dimensions are very (infinitely) small in comparison to the distance of that region from the given sources. A region of such dimensions is now one of the dimensions of the wave-length λ of light waves. In such a region the factor in the second or last pair of large brackets of the expression (23) will not, however, remain constant, since its last or interference-term will evidently vary rapidly, as $r'-r''$ increases or decreases by a quantity of the dimensions of that wave-length; the first two terms of this factor are constants. The behaviour of the resultant amplitude or intensity throughout the given region will thus depend alone on that of the interference-term. The value assumed by the interference-term will now vary as that of its factor $\cos n(r'-r''+\delta'-\delta'')$, which oscillates between the values $+1$ and -1; for the former value the resultant amplitude becomes a maximum and for the latter a minimum. For light waves or electromagnetic waves of very short wave-length, these maxima and minima will succeed one another rapidly, as we recede from any point; the breadth and distribution of these maxima and minima of intensity or bright and dark bands will evidently be determined by formulae similar to those of (6) and (7) above (cf. also below).

Behaviour of Expression for Resultant Amplitude for Electromagnetic Waves Proper.—If we employ the Hertzian or electromagnetic waves of wave-length of the dimensions of the meter, the interference-bands will be very far apart (cf. formulae (7) and (26)), in fact, their width could be of greater dimensions than those of the region, where the resultant amplitude (21) may be determined alone by the value of the interference-term; this would evidently correspond to a greater irregularity not only in the distribution but also in the intensity of the bands. The detection of the interference-bands of electromagnetic waves proper, at least their *laws* of distribution and intensity, would thus be more difficult than that of those of light-waves.

The Interference-Term; Evaluation of same for given case.—Let us, next, examine the interference-bands of electromagnetic light-waves, whose intensity at any point is determined by formula (23), in any region P of the dimensions of the wave-length λ of those waves. We choose the line, on which the two sources O' and O'' lie, as x-axis and the point O half-way between the same as origin of a system of rectangular coordinates x, y, z, denoting the distance between the sources by 2ϵ, as indicated in the annexed figure.

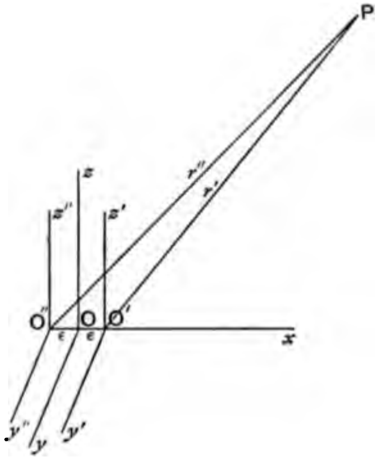

FIG. 14.

The distances r' and r'' of any point P of the given region from the sources O' and O'' will then be given by the expressions

$$r'^2 = (x+\epsilon)^2 + y^2 + z^2, \quad r''^2 = (x-\epsilon)^2 + y^2 + z^2.$$

Since now, by assumption, ϵ is very small in comparison to x, y, z, we can thus write

$$r'^2 = x^2 + y^2 + z^2 + 2\epsilon x = (x^2 + y^2 + z^2)\left(1 + \frac{2\epsilon x}{x^2 + y^2 + z^2}\right),$$

$$r''^2 = x^2 + y^2 + z^2 - 2\epsilon x = (x^2 + y^2 + z^2)\left(1 - \frac{2\epsilon x}{x^2 + y^2 + z^2}\right),$$

or, by the binomial theorem, as first approximation

$$r' = \sqrt{x^2+y^2+z^2}\left(1 + \frac{\epsilon x}{x^2+y^2+z^2}\right),$$

$$r'' = \sqrt{x^2+y^2+z^2}\left(1 - \frac{\epsilon x}{x^2+y^2+z^2}\right);$$

hence

$$r' - r'' = \frac{2\epsilon x}{\sqrt{x^2+y^2+z^2}}. \quad \text{......................................}(24)$$

The factor $\cos n(r' - r'' + \delta' - \delta'')$ of our interference-term will thus assume here the form

$$\cos n(r' - r'' + \delta' - \delta'') = \cos n\left[\frac{2\epsilon x}{\sqrt{x^2+y^2+z^2}} + \delta' - \delta''\right]; \quad \text{......}(25)$$

that is, maxima of intensity will appear, where

$$\frac{2\epsilon x}{\sqrt{x^2+y^2+z^2}} + \delta' - \delta'' = 0,\ \lambda,\ 2\lambda,\ \ldots,$$

and minima, where

$$\frac{2\epsilon x}{\sqrt{x^2+y^2+z^2}} + \delta' - \delta'' = \frac{\lambda}{2},\ \frac{3\lambda}{2},\ \frac{5\lambda}{2},\ \ldots.$$

Breadth of Interference-Bands.—Let now the point P move parallel to the x-axis and let us denote any two such points, whose distance apart is of the dimensions of the quantity ϵ, by P_1 and P_2; the factors $\cos n(r_1' - r_1'' + \delta' - \delta'')$ and $\cos n(r_2' - r_2'' + \delta' - \delta'')$ of the interference-terms at these two points can then, by formula (25), be written in the form

$$\cos n(r_1' - r_1'' + \delta' - \delta'') = \cos n\left(\frac{2\epsilon x_1}{\sqrt{x_1^2+y^2+z^2}}\right)$$

and

$$\cos n(r_2' - r_2'' + \delta' - \delta'') = \cos n\left(\frac{2\epsilon x_2}{\sqrt{x_2^2+y^2+z^2}}\right).$$

Since now x_1 and x_2 differ from x by a quantity of the dimensions of ϵ and the latter has been assumed to be very (infinitely) small in comparison to x, y, z, we may evidently interchange x_1 and x_2 with x in the expressions $x_1^2+y^2+z^2$ and $x_2^2+y^2+z^2$ without altering except infinitesimally the values of the expressions for the given interference-terms; we can thus write the given factors

$$\cos n(r_1' - r_1'' + \delta' - \delta'') = \cos n\left(\frac{2\epsilon x_1}{\sqrt{x^2+y^2+z^2}}\right)$$

and

$$\cos n(r_2' + r_2'' + \delta' - \delta'') = \cos n\left(\frac{2\epsilon x_2}{\sqrt{x^2+y^2+z^2}}\right).$$

Two such points P_1 and P_2 will evidently determine consecutive bands of one and the same intensity, when

$$\frac{2\epsilon(x_1 - x_2)}{\sqrt{x^2+y^2+z^2}} = \lambda,$$

hence

$$x_1 - x_2 = \frac{\lambda\sqrt{x^2+y^2+z^2}}{2\epsilon} \qquad \text{..................(26)}$$

(cf. also formula (7)); that is, when the two points P_1 and P_2 are so chosen that their distance apart is given by this expression (26), they will be points of equal intensity (maxima, minima, etc.).

Summary: Laws of Interference.—From formula (26) follow: (1) The further the region P is from the sources O' and O'' and the nearer these sources are to each other, the broader are the interference-bands; by a suitable choice of the former we could, therefore, always obtain a measurable distance for the latter $(x_1 - x_2)$.

(2) The longer the wave-length λ of the waves employed, the broader the interference-bands; for example, the bands obtained from the red rays would be broader than those produced by the blue ones, whereas for electromagnetic disturbances proper, as the Hertzian waves, the distance between consecutive bands would be of quite different dimensions from those for light waves (see above).

(3) Conversely, we can determine by formula (26) the wave-length λ of the waves employed, on measuring the distance between consecutive bands.

These results are similar to those already deduced above (cf. p. 110).

Superposition of two Secondary Waves.—Let us next examine the resultant action of the two secondary waves that accompany the primary waves represented by formulae (16) at any point P of the field; these waves are represented by the moments

$$\left.\begin{aligned} &X_2' = \frac{n}{r'^2} l' \cos\omega', \quad Y_2' = \frac{n}{r'^2} m' \cos\omega', \quad Z_2' = \frac{n}{r'^2} p' \cos\omega' \\ \text{and}\quad &X_2'' = \frac{n}{r''^2} l'' \cos\omega'', \quad Y_2'' = \frac{n}{r''^2} m'' \cos\omega'', \quad Z_2'' = \frac{n}{r''^2} p'' \cos\omega'' \end{aligned}\right\}, \text{..(27)}$$

$$\left.\begin{aligned} \text{where}\quad l' &= 2a_1' - 3a_1'(\beta'^2+\gamma'^2) + 3\alpha'(a_2'\beta' + a_3'\gamma') \\ m' &= 2a_2' - 3a_2'(\alpha'^2+\gamma'^2) + 3\beta'(a_1'\alpha' + a_3'\gamma') \\ p' &= 2a_3' - 3a_3'(\alpha'^2+\beta'^2) + 3\gamma'(a_1'\alpha' + a_2'\beta'), \\ l'' &= 2a_1'' - 3a_1''(\beta''^2+\gamma''^2) + 3a''(a_2''\beta'' + a_3''\gamma'') \\ m'' &= 2a_2'' - 3a_2''(\alpha''^2+\gamma''^2) + 3\beta''(a_1''\alpha'' + a_3''\gamma'') \\ p'' &= 2a_3'' - 3a_3''(\alpha''^2+\beta''^2) + 3\gamma''(a_1''\alpha'' + a_2''\beta''), \\ \text{and}\quad \omega' &= n[vt - (r'+\delta')], \quad \omega'' = n[vt - (r''+\delta'')] \end{aligned}\right\} \text{...........(28)}$$

(cf. formulae (36, II.)); the systems of coordinates x', y', z' and x'', y'', z'' are those employed above, on p. 118 (cf. also figure 13).

The resultant action of the two waves (27) at any point P of the field (cf. figure 13) will now, by the principle of superposition, be given by the component-moments

$$\begin{aligned} X_2 = X_2' + X_2'' &= \frac{n}{r'^2} l' \cos \omega' + \frac{n}{r''^2} l'' \cos \omega'' \\ &= \frac{n}{r'^2} l' [\cos nvt \cos n(r'+\delta') + \sin nvt \sin n(r'+\delta')] \\ &\quad + \frac{n}{r''^2} l'' [\cos nvt \cos n(r''+\delta'') + \sin nvt \sin n(r''+\delta'')] \\ &= n\left\{\left[\frac{l'}{r'^2} \sin n(r'+\delta') + \frac{l''}{r''^2} \sin n(r''+\delta'')\right] \sin nvt \right. \\ &\quad \left. + \left[\frac{l'}{r'^2} \cos n(r'+\delta') + \frac{l''}{r''^2} \cos n(r''+\delta'')\right] \cos nvt\right\}, \end{aligned}$$

which can be written in the form

$$\begin{aligned} X_2 &= a_1 \cos n[vt - (r+\delta_1)] \\ &= a_1 [\cos nvt \cos n(r+\delta_1) + \sin nvt \sin n(r+\delta_1)], \quad \ldots\ldots(29) \end{aligned}$$

where a_1 and $(r+\delta_1)$ are evidently determined by the equations

$$a_1 \cos n(r+\delta_1) = n\left[\frac{l'}{r'^2} \cos n(r'+\delta') + \frac{l''}{r''^2} \cos n(r''+\delta'')\right]$$

and $$a_1 \sin n(r+\delta_1) = n\left[\frac{l'}{r'^2} \sin n(r'+\delta') + \frac{l''}{r''^2} \sin n(r''+\delta'')\right]$$

as follows:

$$\left.\begin{aligned} a_1^2 &= n^2\left[\frac{l'^2}{r'^4} + \frac{l''^2}{r''^4} + \frac{2l'l''}{r'^2 r''^2} \cos n(r'-r''+\delta'-\delta'')\right] \\ \text{and} \quad \tan n(r+\delta_1) &= \frac{l' r''^2 \sin n(r'+\delta') + l'' r'^2 \sin n(r''+\delta'')}{l' r''^2 \cos n(r'+\delta') + l'' r'^2 \cos n(r''+\delta'')} \end{aligned}\right\}, \quad \ldots(29\text{A})$$

and similarly,

$$Y_2 = a_2 \cos n[vt - (r+\delta_2)] \text{ and } Z_2 = a_3 \cos n[vt - (r+\delta_3)], \ldots\ldots(30)$$

where a_2, $n(r+\delta_2)$ and a_3, $n(r+\delta_3)$ are determined by similar formulae to (29A).

The Resultant Secondary Wave Elliptically Polarized.—The resultant oscillation X_2, Y_2, Z_2 represented by formulae (29) and (30) is evidently elliptically polarized (cf. Ex. 20) like the primary oscillation, which it is accompanying.

The amplitude of the resultant oscillation at any point P is evidently

$$a^2 = a_1{}^2 + a_2{}^2 + a_3{}^2 = n^2 \left[\frac{l'^2 + m'^2 + p'^2}{r'^4} + \frac{l''^2 + m''^2 + p''^2}{r''^4} \right.$$
$$\left. + 2 \frac{l'l'' + m'm'' + p'p''}{r'^2 r''^2} \cos n(r' - r'' + \delta' - \delta'') \right] \text{............} (31)$$

Sources of Disturbance near together and Point of Observation at Great Distance.—The expression (31) is similar to the one (21) found for the amplitude of the primary waves; the same conclusions as those drawn from the latter (cf. pp. 120-122) will, therefore, hold for the given expression. For similar reasons to those set forth above we shall also restrict ourselves here to an examination of the particular case, where the relation (22) holds between the given amplitudes, the sources O' and O'' are very (infinitely) near together and the point of observation P is at considerable (finite) distance from the same; in which case α'', β'', γ'' and α', β', γ' respectively will differ only infinitesimally from one another and may thus be interchanged.

Expression for Resultant Amplitude.—In the given particular case formula (31) can evidently be written

$$a^2 = \frac{n^2}{r^4}(l^2 + m^2 + p^2)[1 + \kappa^2 + 2\kappa \cos n(r' - r'' + \delta' - \delta'')],$$

where $\qquad l'' = \kappa l' = \kappa l \quad m'' = \kappa m' = \kappa m \quad p'' = \kappa p' = \kappa p$

(cf. p. 122); or, if we replace here l, m, p (l', m', p') by their values (28),

$$a^2 = \frac{n^2}{r^2}\{4(a_1'^2 + a_2'^2 + a_3'^2) - 3[a_1'^2(\beta^2 + \gamma^2) + a_2'^2(\alpha^2 + \gamma^2) + a_3'^2(\alpha^2 + \beta^2)]$$
$$+ 6(a_1'a_2'\alpha\beta + a_1'a_3'\alpha\gamma + a_2'a_3'\beta\gamma)\}$$
$$\times [1 + \kappa^2 + 2\kappa \cos n(r' - r'' + \delta' - \delta'')]$$
$$= \frac{n^2}{r^4}[(a_1'^2 + a_2'^2 + a_3'^2) + 3(a_1'\alpha + a_2'\beta + a_3'\gamma)^2]$$
$$\times [1 + \kappa^2 + 2\kappa \cos n(r' \quad r'' + \delta' - \delta'')] \text{..........} (32)$$

This expression is similar to that (23) already found for the amplitude of the primary wave, being composed of two factors, the one a function of the direction-cosines α', β', γ' and the a''s and the other a function of the distances r' and r''. The interference-term has the same form in both expressions, so that the results obtained above for the interference-term of the primary wave will hold here for that of the secondary wave (cf. pp. 124-126); we cannot, however, conclude that the phenomena of interference, to which the secondary waves may give rise, will be the same as those produced by the primary waves, as an examination of the expressions (23) and (32) and the relative behaviour of the two systems of waves will show.

Behaviour of Resultant Amplitude for Light-Waves.—For light-waves the factor $\frac{n^4}{r^2}$ of formula (23) for the amplitude squared of the primary waves will be very (infinitely) large compared with the corresponding factor $\frac{n^2}{r^4}$ of formula (32) for the amplitude squared of the secondary waves except in the neighbourhood of the origin; at considerable (finite) distance from the same the phenomena of interference, to which the latter waves might give rise, would thus vanish, when compared with those produced by the former. We have now seen above, unless the point of observation P be removed to considerable distance from the sources, that the oscillations to be superposed will not take place in the same planes, and hence that the interference-formulae established will not hold even approximately; this would correspond, on the one hand, to a less marked interference between the secondary waves and, on the other hand, to a certain irregularity in the resultant intensity and distribution of the same throughout any region that is not at considerable distance from the sources. The detection of phenomena of interference between two systems of secondary waves of light would thus be difficult not only at considerable distance from the sources but in their neighbourhood.

Behaviour of Resultant Amplitude for Electromagnetic Waves Proper.—For electromagnetic waves proper (the Hertzian) the quantities n^4 of formula (23) and n^2 of formula (32) are of the same dimensions, and hence the detection of the secondary waves not only in the neighbourhood of the sources but at considerable (finite) distance from the same possible (cf. p. 53); but at such distances the above interference-formulae will hold for regions of the dimensions of the wave-length of light-waves, but only approximately for those of the dimensions of the wave-length of the waves in question. Since now the interference-phenomena sought could be observed only in regions of the latter dimensions (cf. p. 124) and throughout such regions not only the two systems of waves would be only approximately similarly polarized but also the formulae in question only approximately hold, all observations on interference-phenomena between the two systems of secondary waves would be accompanied by difficulties.

Interference-Phenomena of the Primary and the Secondary Waves; those in Regions, where the latter alone appear.—The most distinct phenomena of interference would evidently be the familiar ones produced by the primary waves (of light) at considerable distance from their sources, whereas those to which the secondary waves might give rise would have to be sought under considerable difficulties in the neighbourhood of the sources. The detection of the latter would evidently be facilitated greatly by the entire disappearance of the former. This would be the case only when the primary waves themselves disappeared entirely; that now the primary waves, those represented by formulae (16) to (18), disappear, the first factor in the larger brackets of the expression (23) for the resultant intensity must vanish;* this will be the case in those regions, where

$$(a_1'^2+a_2'^2+a_3'^2)-(a_1'\alpha+a_2'\beta+a_3'\gamma)^2=0,$$

that is, along the surface

$$\begin{aligned}(a_2'^2+a_3'^2)x^2+(a_1'^2+a_3'^2)y^2+(a_1'^2+a_2'^2)z^2\\ -2a_1'a_2'xy-2a_1'a_3'xz-2a_2'a_3'yz=0,^{*} \quad\ldots\ldots\ldots\ldots\ldots\ldots(33)\end{aligned}$$

which is the equation of a cone with apex at origin (cf. also Ex. 15).

Along this surface (the vector $\alpha:\beta:\gamma=a_1':a_2':a_3'$)* the formula (32) for the amplitude squared of the resultant secondary wave will evidently assume the form

$$a^2=\frac{n^2}{r^4}4(a_1'^2+a_2'^2+a_3'^2)[1+\kappa^2+2\kappa\cos n(r'-r''+\delta'-\delta'')]. \quad\ldots.(34)$$

Since this expression does not contain the direction-cosines, the resultant amplitude will have one and the same value at all points on the surface (33) (the vector $\alpha:\beta:\gamma=a_1':a_2':a_3'$)* that are equidistant from the given source. The entire disappearance of the primary waves along this surface (33) (the vector $\alpha:\beta:\gamma=a_1':a_2':a_3'$)* would facilitate the detection of the interference-phenomena, to which the secondary waves might give rise, along the same. We recall here the important property of the secondary waves in those regions, where the primary waves disappear, namely their longitudinality (cf. p. 61).

Point of Observation near Sources of Disturbance; Expression for Amplitude of Resultant Secondary Wave.—Lastly, let us examine the case, where the distance between the sources O' and O'' is of the

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* On the assumption that the amplitudes l', m', p' and l'', m'', p'' of formulae (17) be real and not imaginary quantities, not only the given factor, $l^2+m^2+p^2$, but also the different terms of the same must vanish; this is, for *real* values of the amplitudes of the component-oscillations (16)—only such values would evidently come into consideration here—the relations

$$l^2=m^2=p^2=0$$

must also hold. These relations determine the vector $\alpha:\beta:\gamma=a_1':a_2':a_3'$ (cf. p. 61)—one solution of equation (33)—which will evidently lie on the cone represented by that equation. The vanishing of the two primary waves along other vectors of the cone (33) would evidently correspond to *imaginary* values of the amplitudes l', m', p', and l'', m'', p'' of the component-oscillations of those waves.

same dimensions as the distances of the point of observation P from the same, and, for simplicity, when $a_1'=a_2'=a_3'=a_1''=a_2''=a_3''=1$, that is, the two sources shall emit similar waves. This particular case is of importance, since, as we have just seen, the marked phenomena of interference of the secondary waves must be sought in the neighbourhood of the given sources, that is, the point of observation P must be taken at a distance from the sources, that is, of the same dimensions as the distance between the sources themselves. For the given case (region) the general formula (31) for the resultant amplitude of the secondary wave will evidently assume the form

$$\left.\begin{aligned} a^2 = 3n^2\Big\{ & \frac{1+(\alpha'+\beta'+\gamma')^2}{r'^4} + \frac{1+(\alpha''+\beta''+\gamma'')^2}{r''^4} \\ & + \frac{\begin{aligned} 2[1-(\alpha'+\beta'+\gamma')^2-(\alpha''+\beta''+\gamma'')^2 \\ +3(\alpha'+\beta'+\gamma')(\alpha''+\beta''+\gamma'')(\alpha'\alpha''+\beta'\beta''+\gamma'\gamma'')]\end{aligned}}{r'^2r''^2} \\ & \times \cos n(r'-r''+\delta'-\delta'')\Big\} \end{aligned}\right\} \text{.......}(35)$$

The Sources of Disturbance on x-Axis; Coefficient of Interference Term.—Let us, next, assume that the sources O' and O'' lie on one and the same axis x at the distance b apart; the following relations will then hold between the coordinates x', y', z' and x'', y'', z'' of any point P:

$$\left.\begin{aligned} x''=x'+b \quad y''=y' \quad z''=z', \\ \text{hence} \qquad r''^2=(x'+b)^2+y'^2+z'^2 \end{aligned}\right\}\text{......................}(36)$$

For the given particular case (36) we can evidently write the coefficient of the interference-term of the above expression (35) as follows:

$$\begin{aligned} l'l''+m'm''+p'p''=3\Big\{ & 1-\left(\frac{x'+y'+z'}{r'}\right)^2-\left(\frac{x'+b+y'+z'}{r''}\right)^2 \\ & +3\left(\frac{x'+y'+z'}{r'}\right)\left(\frac{x'+b+y'+z'}{r''}\right)\left[\frac{x'(x'+b)+y'^2+z'^2}{r'r''}\right]\Big\} \end{aligned}$$

(cf. formula (31))

$$\begin{aligned} =\frac{3}{r'^2r''^2}\{ & (x'^2+y'^2+z'^2)[(x'+b)^2+y'^2+z'^2] \\ & -(x'+y'+z')^2[(x'+b)^2+y'^2+z'^2]-(x'+b+y'+z')^2(x'^2+y^2+z'^2) \\ & +3(x'+y'+z')(x'+b+y'+z')[x'(x'+b)+y'^2+z'^2]\} \end{aligned}$$

$$\left.\begin{aligned} =\frac{3}{r'^2r''^2}\{ & (x'^2+y'^2+z'^2)[2(x'^2+y'^2+z'^2+x'y'+x'z'+y'z') \\ & +b(3x'+y'+z')] \\ & +b(x'+y'+z')[x'(x'+y'+z')+b(2x'-y'-z')]\} \end{aligned}\right\}\text{..........}(37)$$

Coefficient of Interference-Term on Screen $\parallel$ yz-Plane.—On a screen placed parallel to the yz-plane at the distance c from the source O' the coefficient (37) of the given interference-term will assume the form

$$\left.\begin{aligned} & l'l'' + m'm'' + p'p'' \\ & = \frac{3}{r'^2 r''^2}\{(c^2+y'^2+z'^2)[2(c^2+y'^2+z'^2+cy'+cz'+y'z')+b(3c+y'+z')] \\ & \qquad + b(c+y'+z')[c(c+y'+z')+b(2c-y'-z')]\} \end{aligned}\right\}\ldots(37\text{A})$$

It is now evident that the interference-bands will vanish in those regions on the given screen, for which this coefficient* vanishes, that is, along and in the neighbourhood of the curve

$$\left.\begin{aligned} & (c^2+y'^2+z'^2)[2(c^2+y'^2+z'^2+cy'+cz'+y'z')+b(3c'+y'+z')] \\ & \qquad + b(c+y'+z')[c(c+y'+z')+b(2c-y'-z')] = 0^* \end{aligned}\right\}\ldots(38)$$

Regions of greatest Interference.—On the other hand, the interference will evidently be greatest at those points on the given screen, where the coefficient (37A) of the interference-term becomes a maximum; and this will evidently be the case in those regions, where the value of this coefficient approaches the particular values assumed by the coefficients of the first two terms of the expression (35) for the resultant amplitude. Let us now determine the region, in which the coefficient of the interference-term and that of the first term of expression (35) are the same for the given particular case; the region sought will evidently be that determined by the relation

$$\begin{aligned} & \frac{1}{r'^2 r''^2}\{(c^2+y'^2+z'^2)[2(c^2+y'^2+z'^2+cy'+cz'+y'z')+b(3c+y'+z')] \\ & \qquad + b(c+y'+z')[c(c+y'+z')+b(2c-y'-z')]\} = 1+(\alpha'+\beta'+\gamma')^2 \end{aligned}$$

(cf. formulae (35) and (37A)), or, if we replace here α', β', γ' by their particular values (cf. formula (36)),

$$= \frac{c^2+y'^2+z'^2+(c+y'+z')^2}{r'^2};$$

which relation is evidently satisfied, when $y'=z'=0$,† the point in which the x-axis meets the given screen; a result which we might have anticipated.

Similarly, it is easy to show that for $y'=z'=0$ the coefficient of the second term of expression (35) assumes here the same value as that of the third or interference term (cf. Ex. 21).

The neighbourhood of the point $y'=z'=0$ on the given screen would thus be suited best for an observation of the interference-phenomena produced by the given secondary waves.

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* *Cf.* foot-note to p. 130.

† Then $r''^2=(c+b)^2$ (cf. formulae (36)).

Expression for Resultant Amplitude on Screen; approximate Expressions for same in Region of greatest Interference.—The resultant amplitude of the given secondary waves at any point on the screen $x=c$ will evidently be given by the expression

$$\left.\begin{aligned}
a^2 = 3n^2\Bigg\{&\frac{c^2+y'^2+z'^2+(c+y'+z')^2}{r'^2}\,\frac{1}{r'^4} \\
&+\frac{(c+b)^2+y'^2+z'^2+(c+b+y'+z')^2}{r''^2}\,\frac{1}{r''^4} \\
&+\frac{\begin{gathered}2(c^2+y'^2+z'^2)[2(c^2+y'^2+z'^2+cy'+cz'+y'z')+b(3c+y'+z')] \\ +2b(c+y'+z')[c(c+y'+z')+b(2c-y'-z')]\end{gathered}}{r'^2r''^2} \\
&\times\frac{1}{r'^2r''^2}\cos n(r'-r''+\delta'-\delta'')\Bigg\}
\end{aligned}\right\} ; \ldots. \quad (39)$$

which in the neighbourhood of the point $y'=z'=0$ will assume the approximate value

$$\left.\begin{aligned}
a^2 = 6n^2\Bigg\{&\frac{c+y'+z'}{c}\,\frac{1}{r'^4}+\frac{c+b+y'+z'}{c+b}\,\frac{1}{r''^4} \\
&+\frac{2c(c+b)+(2c+b)(y'+z')}{c(c+b)}\,\frac{1}{r'^2r''^2}\cos n(r'-r''+\delta'-\delta'')\Bigg\}
\end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (39\text{A})$$

(cf. Ex. 22); or, if we replace r' and r'' by their approximate values,

$$\left.\begin{aligned}
a^2 = 6n^2\Bigg\{&\frac{c+y'+z'}{c^5}+\frac{c+b+y'+z'}{(c+b)^5}+\frac{2c(c+b)+(2c+b)(y'+z')}{c^3(c+b)^3} \\
&\times\cos n(r'+r''+\delta'-\delta'')\Bigg\}
\end{aligned}\right\}. \quad (39\text{B})$$

Formula (39) holds for both light-waves and the electromagnetic waves proper; formula (39A) would hold also for both types of waves, whereas formula (39B) would hold for the former most approximately but for the latter only approximately. We obtain a more approximate expression for the electromagnetic waves proper on replacing r'^4, r''^4 and $r'^2r''^2$ of formula (39A) by their more approximate values

$$r'^4 = c^4+2c^2(y'^2+z'^2),$$

$$r''^4 = (c+b)^4+2(c+b)^2(y'^2+z'^2),$$

and

$$r^2+r''^2 = c^2(c+b)^2+[c^2+(c+b)^2](y^2+z'^2).$$

The Electromagnetic Waves of the General Problem of Chapter III. and Phenomena of Interference.—The electromagnetic waves treated in the more general problem of Chapter III. and represented by formulae (7, III.) would be of little interest for us here, since they are more or less highly elliptically polarized and would thus give rise only to very small and irregular variations of intensity when superposed (cf. p. 114).

EXAMPLES.

1. Show that the equations

$$a^2 = a'^2 + a''^2 + 2a'a'' \cos n(x' - x'')$$

and
$$\tan nx = \frac{a' \sin nx' + a'' \sin nx''}{a' \cos nx' + a'' \cos nx''},$$

which may also be written in the form

$$a' \sin n(x - x') + a'' \sin n(x - x'') = 0$$

(cf. formulae (2) and (3)), can be represented graphically as follows:

Take any fixed line OX and draw OP equal to a', and making the angle nx' with OX, similarly draw OQ equal to a'', and making the angle $nx'' = QOX$, then, if we complete the parallelogram $OPRQ$, its diagonal OR will be equal to a and the angle ROX which it makes with OX will be equal to nx, while POQ will be equal to the difference in phase $n(x' - x'')$ (cf. the annexed figure). The amplitudes a' and a'' will, therefore, compound like forces.*

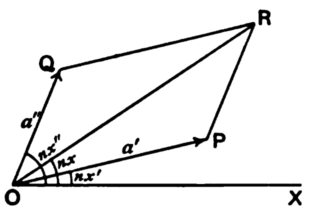

FIG. 15.

2. Show that a right-handed circular oscillation, for example

$$y' = a_2' \sin n(vt - x'),$$
$$z' = -a_2' \sin n\left(vt - x' - \frac{\lambda}{4}\right) = a_2' \cos n(vt - x'),$$

and a similar left-handed circular one, namely

$$y'' = a_2' \sin n(vt - x'),$$
$$z'' = +a_2' \sin n\left(vt - x' - \frac{\lambda}{4}\right) = -a_2' \cos n(vt - x'),$$

compound to a linear oscillation, namely

$$y = 2a_2' \sin n(vt - x').$$

3. Show that the elliptic oscillation

$$y = a_2 \sin n(vt - x), \qquad z = -a_3 \sin n\left(vt - x - \frac{\lambda}{4}\right) = a_3 \cos n(vt - x)$$

may be regarded as the resultant of the two oppositely directed circular oscillations

$$y' = \frac{1}{2}(a_2 + a_3) \sin n(vt - x), \qquad z' = \frac{1}{2}(a_2 + a_3) \cos n(vt - x),$$

and
$$y'' = \frac{1}{2}(a_2 - a_3) \sin n(vt - x), \qquad z'' = -\frac{1}{2}(a_2 - a_3) \cos n(vt - x).$$

4. The intensity of two similarly linearly polarized oscillations that differ in phase by quarter of a wave length is the same as that of two similar linearly polarized oscillations, whose planes of oscillation are at right angles to each other.

Two such pairs of oscillations are

$$\left.\begin{aligned} y' &= a' \sin n(vt - x') \\ y'' &= a'' \sin n(vt - x'') \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

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*Cf. Preston, *Theory of Light*, pp. 48, 49.

where $n(x'-x'')=\frac{\pi}{2}$, and

$$\left.\begin{array}{l} y'=a'\sin n(vt-x') \\ z''=a''\sin n(vt-x'') \end{array}\right\}, \qquad \text{(B)}$$

where x' and x'' are entirely arbitrary.

The resultant (oscillation) of the former pair (A) is, by the principle of superposition (cf. p. 13),

$$y=a\sin n(vt-x), \qquad \text{(C)}$$

where

$$a=\sqrt{a'^2+a''^2} \qquad \text{(D)}$$

and

$$x=\frac{1}{n}\arctan\frac{a'\sin nx'-a''\cos nx'}{a'\cos nx'+a''\sin nx'},$$

the particular form assumed by formulae (2) and (3) respectively, when $n(x'-x'')=\frac{\pi}{2}$.

The intensity I (generated by a complete oscillation) of the resultant oscillation (C) is

$$I=\frac{1}{T}\int_0^T\frac{m}{2}\left(\frac{dy}{dt}\right)^2 dt=\frac{ma^2n^2v^2}{2T}\int_0^T\cos^2 n(vt-x)\,dt$$

$$=\frac{ma^2n^2v^2}{4T}\int_0^T[1+\cos 2n(vt-x)]\,dt$$

$$=\frac{ma^2n^2v^2}{4T}\left|\, T+\frac{1}{2nv}\sin 2n(vT-x)\,\right|_0^T=\frac{ma^2n^2v^2}{4},$$

or, by formula (D), $=\frac{m(a'^2+a''^2)n^2v^2}{4}$.

The intensity of the elliptic oscillation (B) (cf. formulae (2) and (3), III.) is now, since its component-oscillations take place entirely independently of each other, given by the sum of the intensities of those component-oscillations, that is,

$$I=I_1+I_2=\frac{1}{T}\int_0^T\frac{m}{2}\left(\frac{dy'}{dt}\right)^2 dt+\frac{1}{T}\int_0^T\frac{m}{2}\left(\frac{dz''}{dt}\right)^2 dt$$

$$=\frac{ma'^2n^2v^2}{2T}\int_0^T\cos^2 n(vt-x')\,dt+\frac{ma''^2n^2v^2}{2T}\int_0^T\cos^2 n(vt-x'')\,dt$$

$$=\frac{ma'^2n^2v^2}{4}+\frac{ma''^2n^2v^2}{4}=\frac{m(a'^2+a''^2)n^2v^2}{4}. \qquad \text{Q.E.D.}$$

5. The resultant intensity of the two similarly linearly polarized oscillations

$$y'=a'\sin n(vt-x'),$$
$$y''=a''\sin n(vt-x'')$$

(cf. formulae (1)) is given by the expression

$$I=\frac{mn^2v^2}{4}[a'^2+a''^2+2a'a''\cos n(x'-x'')],$$

where m denotes the mass of the oscillating element.

The intensity I of any linear oscillation y is given by the integral

$$I=\frac{1}{T}\int^T\frac{m}{2}\left(\frac{dy}{dt}\right)^2 dt.$$

Here $y=y'+y''=a'\sin n(vt-x')+a''\sin n(vt-x'')$;

hence
$$I=\frac{m}{2T}\int_0^T n^2v^2[a'\cos n(vt-x')+a''\cos n(vt-x'')]^2dt$$
$$=\frac{mn^2v^2}{2T}\Big\{a'^2\int_0^T\cos^2 n(vt-x')dt+a''^2\int_0^T\cos^2 n(vt-x'')dt$$
$$+2a'a''\int_0^T\cos n(vt-x')\cos n(vt-x'')dt\Big\}$$
$$=\frac{mn^2v^2}{2T}\Big\{\frac{a'^2}{2}\int_0^T[1+\cos 2n(vt-x')]dt+\frac{a''^2}{2}\int_0^T[1+\cos 2n(vt-x'')]dt$$
$$+2a'a''\int_0^T[\cos nx'\cos nx''\cos^2 nvt+\sin nx'\sin nx''\sin^2 nvt$$
$$+\sin n(x'+x'')\cos nvt\sin nvt]dt\Big\}$$
$$=\frac{mn^2v^2}{2T}\Big\{\frac{a'^2}{2}\Big|t+\frac{\sin 2n(vt-x')}{2nv}\Big|_0^T+\frac{a''^2}{2}\Big|t+\frac{\sin 2n(vt-x'')}{2nv}\Big|_0^T$$
$$+a'a''\cos nx'\cos nx''\int_0^T(1+\cos 2nvt)dt$$
$$+a'a''\sin nx'\sin nx''\int_0^T(1-\cos 2nvt)dt$$
$$+a'a''\sin n(x'+x'')\int_0^T\sin 2nvt\,dt\Big\}$$
$$=\frac{mn^2v^2}{2T}\Big[\frac{a'^2}{2}T+\frac{a''^2}{2}T+a'a''\cos nx'\cos nx''\Big|t+\frac{\sin 2nvt}{2nv}\Big|_0^T$$
$$+a'a''\sin nx'\sin nx''\Big|t-\frac{\sin 2nvt}{2nv}\Big|_0^T+a'a''\sin n(x'+x'')\Big|-\frac{\cos 2nvt}{2nv}\Big|_0^T\Big]$$
$$=\frac{mn^2v^2}{2T}\Big[\frac{a'^2+a''^2}{2}T+a'a''\cos nx'\cos nx''T+a'a''\sin nx'\sin nx''T\Big]$$
$$=\frac{mn^2v^2}{4}[a'^2+a''^2+2a'a''\cos n(x'-x'')].\qquad\text{Q.E.D.}$$

Derive this expression also directly from the resultant oscillation
$$y=a\sin n(vt-x),$$
where a and x are determined by formulae (2) and (3).

6. Show that the resultant intensity of the two elliptic oscillations represented by formulae (8) is given by the expression
$$I=\frac{mn^2v^2}{4}[a_2'^2+a_2''^2+a_3'^2+a_3''^2+2a_2'a_2''\cos n(x'-x'')$$
$$+2a_3'a_3''\cos n(x'+\delta'-x''-\delta'')],$$
where m denotes the mass of the oscillating particle.

7. Determine the difference in phase $n\delta$ between the components (9) of the resultant elliptically polarized plane-wave examined on p. 115.

The formula (11) for $\tan n(x+\delta)$ can be written
$$\tan n(x+\delta)=\frac{\tan nx+\tan n\delta}{1-\tan nx\tan n\delta}=A,$$

where
$$A=\frac{a_3'\sin n(x'+\delta')+a_3''\sin n(x''+\delta'')}{a_3'\cos n(x'+\delta')+a_3''\cos n(x''+\delta'')}; \quad\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

which gives the following value for $\tan n\delta$:

$$\tan n\delta=\frac{A+\tan nx}{1+A\tan nx}.$$

Replace here A by its value (a) and $\tan nx$ by its value from formulae (10), and we have

$$\tan n\delta$$
$$=\frac{\begin{array}{l}[a_3'\sin n(x'+\delta')+a_3''\sin n(x''+\delta'')][a_2'\cos nx'+a_2''\cos nx'']\\ \qquad +[a_3'\cos n(x'+\delta')+a_3''\cos n(x''+\delta'')][a_2'\sin nx'+a_2''\sin nx'']\end{array}}{\begin{array}{l}[a_3'\cos n(x'+\delta')+a_3''\cos n(x''+\delta'')][a_2'\cos nx'+a_2''\cos nx'']\\ \qquad +[a_3'\sin n(x'+\delta')+a_3''\sin n(x''+\delta'')][a_2'\sin nx'+a_2''\sin nx'']\end{array}}$$

$$=\frac{\begin{array}{l}a_2'a_3'\sin n(2x'+\delta')+a_2''a_3'\sin n(x'+x''+\delta')\\ \qquad +a_2'a_3''\sin n(x'+x''+\delta'')+a_2''a_3''\sin n(2x''+\delta'')\end{array}}{a_2'a_3'\cos n\delta'+a_2''a_3'\cos n(x'-x''+\delta')+a_2'a_3''\cos n(x'-x''-\delta'')+a_2''a_3''\cos n\delta''}.$$

8. Determine the resultant of the two elliptically polarized spherical waves

$$y'=\frac{a_2'}{x'}\sin n(vt-x'),$$
$$z'=\frac{a_3'}{x'}\sin n\left(vt-x'-\frac{\lambda}{4}\right)=-\frac{a'_3}{x'}\cos n(vt-x'),$$

and
$$y''=\frac{a_2''}{x''}\sin n(vt-x''),$$
$$z''=\frac{a_3''}{x''}\sin n\left(vt-x''-\frac{\lambda}{4}\right)=-\frac{a_3''}{x''}\cos n(vt-x'').$$

The components of the resultant wave are

$$y=y'+y''=\frac{a_2'}{x'}\sin n(vt-x')+\frac{a_2''}{x''}\sin n(vt-x'')=a_2\sin n(vt-x),*$$

where a_2 and nx are given by formulae (13), and

$$z=z'+z''=-\frac{a_3'}{x'}\cos n(vt-x')-\frac{a_3''}{x''}\cos n(vt-x'')=-a_3\cos n(vt-x-\delta),*$$

where a_3 and $n(x+\delta)$ are determined by the equations

$$\frac{a_3'}{x'}\cos nx'+\frac{a_3''}{x''}\cos nx''=a_3\cos n(x+\delta),$$
$$\frac{a_3'}{x'}\sin nx'+\frac{a_3''}{x''}\sin nx''=a_3\sin n(x+\delta),$$

as follows:

$$a_3^2=\frac{a_3'^2}{x'^2}+\frac{a_3''^2}{x''^2}+2\frac{a_3'a_3''}{x'x''}\cos n(x'-x''),$$

and
$$\tan n(x+\delta\dagger)=\frac{a_3'x''\sin nx'+a_3''x'\sin nx''}{a_3'x''\cos nx'+a_3''x'\cos nx''}$$

(cf. formulae (15)).

Show, when $a_2'=a_3'$ and $a_2''=a_3''$, that is, when the given oscillations are circularly polarized and similarly directed, that $\delta=0$, that is, that the components of the resultant oscillation have the same phase.

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* Cf. footnote to p. 117.

† This δ is evidently not identical to the δ of formulae (15).

9. Show that the difference in phase $n\delta$ between the components of the resultant elliptically polarized spherical wave represented by formulae (14) is given by the expression

$$\tan n\delta = \frac{\begin{array}{l} a_2'a_3'x''^2 \sin n(2x'+\delta') + a_2''a_3'x'x'' \sin n(x'+x''+\delta') \\ \qquad + a_2'a_3''x'x'' \sin n(x'+x''+\delta'') + a_2''a_3''x'^2 \sin n(2x''+\delta'') \end{array}}{\begin{array}{l} a_2'a_3'x''^2 \cos n\delta' + a_2''a_3'x'x'' \cos n(x'-x''+\delta') \\ \qquad + a_2'a_3''x'x'' \cos n(x'-x''-\delta'') + a_2''a_3''x'^2 \cos n\delta'' \end{array}}$$

10. Determine the resultant intensity of the two similarly linearly polarized spherical waves represented by formulae (12).

The intensity I in question is given by the integral

$$I = \frac{1}{T}\int_0^T \frac{m}{2}\left(\frac{dy}{dt}\right)^2 dt,$$

where $y = a \sin n(vt - x)$, and a and nx are determined by formulae (13).

We thus have

$$I = \frac{mn^2v^2a^2}{2T}\int_0^T \cos^2 n(vt-x)\,dt$$

$$= \frac{mn^2v^2a^2}{4T}\int_0^T [1 + \cos 2n(vt-x)]\,dt = \frac{mn^2v^2a^2}{4},$$

or, on replacing a^2 by its value,

$$I = \frac{mn^2v^2}{4}\left[\frac{a'^2}{x'^2} + \frac{a''^2}{x''^2} + 2\frac{a'a''}{x'x''}\cos n(x'-x'')\right].$$

11. Show that the intensity of the resultant elliptically polarized spherical wave represented by formulae (14) is given by the expression

$$I = \frac{mn^2v^2}{4}\left\{\frac{a_2'^2 + a_3'^2}{x'^2} + \frac{a_2''^2 + a_3''^2}{x''^2} + \frac{2}{x'x''}\begin{array}{l}[a_2'a_2'' \cos n(x'-x'') \\ \qquad + a_3'a_3'' \cos n(x'+\delta'-x''-\delta'')]\end{array}\right\}.$$

12. The distance x of a dark or bright point or band at considerable distance from the central band is given by the expression

$$\frac{n\lambda a}{\sqrt{4b^2 - n^2\lambda^2}} = x. \qquad \text{(A)}$$

For large values of x, that is, large values of the angle DCE of Fig. 12, the distance CE cannot evidently be replaced by the distance CD between the sources. We must, therefore, employ the following proportion instead of that (5) chosen on p. 110:

$$PP_n : OP_n = DE : CD,$$

or

$$x : \sqrt{a^2 + x^2} = \frac{n\lambda}{2} : b,$$

which gives

$$x^2 = \frac{\frac{n^2\lambda^2}{4}a^2}{b^2 - \frac{n^2\lambda^2}{4}}. \qquad \text{Q.E.D.}$$

This expression shows that the number of bands cannot exceed $\frac{2b}{\lambda}$.

13. The width of a bright band at considerable distance from the central band is given (approximately) by the expression

$$\frac{2\lambda a}{\sqrt{4b^2 - n^2\lambda^2}} = x_n - x_{n-2}. \qquad \text{(A)}$$

By example 12 (cf. formula (A)) the exacter expression for the distance between two consecutive (dark) bands that are at considerable distance from the central band is

$$x_n - x_{n-2} = \frac{n\lambda a}{\sqrt{4b^2 - n^2\lambda^2}} - \frac{(n-2)\lambda a}{\sqrt{4b^2 - (n-2)^2\lambda^2}}.$$

Since now $4b^2$ is large in comparison to $n^2\lambda^2$ even at considerable distance from the central band and n^2 is very large compared with n—interference has been observed with retardations of over 200,000, even 500,000, wave-lengths—the terms $4n\lambda^2$ and $-4\lambda^2$ under the square-root-sign of the second term of this expression for the width of band may evidently be neglected in comparison to the two other terms, $4b^2$ and $-n^2\lambda^2$, and thus rejected. The given expression can thus be written most approximately

$$x_n - x_{n-2} = \frac{n\lambda a}{\sqrt{4b^2 - n^2\lambda^2}} - \frac{(n-2)\lambda a}{\sqrt{4b^2 - n^2\lambda^2}} = \frac{2\lambda a}{\sqrt{4b^2 - n^2\lambda^2}}. \qquad \text{Q.E.D.}$$

Observe that this expression reduces to that of formula (7), the one in general use, for small values of n.

A comparison of the expression (A) for the width of any distant interference band with that (7), which holds only for bands near the centre, shows that the bands do not retain one and the same width, as we recede from the centre, but that they increase in width. Take, for example, the 10,000th interference band of waves of wave-length 2000×10^{-6} mm., that is, $n = 10^4$ and $\lambda = 2 \times 10^{-3}$. By formula (A) the width of this band is $\frac{\lambda a}{\sqrt{b^2 - (10\text{ mm.})^2}}$. Show then that for $b = 10$ cm. the 10,000th band would be about 5·4 % broader than those near the centre, and that for $b = 10$ mm. the 10,000th band could not appear.

14. Show that the resultant oscillations X_1, Y_1, Z_1 represented by formulae (19) and (20) take place in plane elliptic paths (cf. p. 120).

15. Show, when the sources O' and O'' of the waves represented by formulae (16) are very (infinitely) near together and the point of observation P is at considerable (finite) distance from the same, that the coefficient $l'l'' + m'm'' + p'p''$ of the interference term of the expression (21) for the amplitude of the resultant (primary) wave assumes the form

$$\begin{aligned} l'l'' + m'm'' + p'p'' = {} & a_1'a_1''(\beta^2 + \gamma^2) + a_2'a_2''(\alpha^2 + \gamma^2) + a_3'a_3''(\alpha^2 + \beta^2) \\ & - (a_1'a_2'' + a_2'a_1'')\alpha\beta - (a_1'a_3'' + a_3'a_1'')\alpha\gamma - (a_2'a_3'' + a_3'a_2'')\beta\gamma, \end{aligned}$$

where $\alpha = \alpha' = \alpha''$, $\beta = \beta' = \beta''$, $\gamma = \gamma' = \gamma''$;

that is, the given coefficient vanishes along the surface

$$\left.\begin{aligned} & (a_2'a_2'' + a_3'a_3'')x^2 + (a_1'a_1'' + a_3'a_3'')y^2 + (a_1'a_1'' + a_2'a_2'')z^2 \\ & \quad - (a_1'a_2'' + a_2'a_1'')xy - (a_1'a_3'' + a_3'a_1'')xz - (a_2'a_3'' + a_3'a_2'')yz = 0 \end{aligned}\right\}, \quad \ldots\ldots(\text{A})$$

the equation of a cone with apex at origin.

16. On a screen placed parallel to the yz-plane at the distance c from the origin formula (A) of Ex. 15 assumes the form

$$\begin{aligned} & (a_1'a_1'' + a_3'a_3'')y^2 + (a_1'a_1'' + a_2'a_2'')z^2 - (a_2'a_3'' + a_3'a_2'')yz \\ & \quad - (a_1'a_2'' + a_2'a_1'')cy - (a_1'a_3'' + a_3'a_1'')cz + (a_2'a_2'' + a_3'a_3'')c^2 = 0, \end{aligned}$$

the equation of a conic.

Show, when $a_1'=a_1''=0$, that the coefficient $l'l''+m'm''+p'p''$ of the given interference-term vanishes at the points

$$y=z=\pm c\sqrt{\frac{a_2'a_2''+a_3'a_3''}{(a_2'-a_3')(a_2''-a_3'')}}$$

on the given screen ($x=c$), whereas the coefficients $l'^2+m'^2+p'^2$ and $l''^2+m''^2+p''^2$ of the first and second terms of the given expression for the resultant amplitude do not vanish at those points.

17. Show, when the distance between the sources O' and O'' is of the same dimensions as the distances of the point of observation P from the same, that the coefficients of the different terms of the expression (21) for the amplitude of the resultant (primary) wave assume the form

$$\begin{aligned}
l'^2+m'^2+p'^2&=3-(\alpha'+\beta'+\gamma')^2,\\
l''^2+m''^2+p''^2&=3-(\alpha''+\beta''+\gamma'')^2,\\
l'l''+m'm''+p'p''&=3-(\alpha'+\beta'+\gamma')^2-(\alpha''+\beta''+\gamma'')^2\\
&\qquad+(\alpha'+\beta'+\gamma')(\alpha''+\beta''+\gamma'')(\alpha'\alpha''+\beta'\beta''\gamma'\gamma'').
\end{aligned}$$

On the assumption that the sources O' and O'' lie on one and the same axis x (cf. formulae (36)), these coefficients assume the form

$$l'^2+m'^2+p'^2=3-\left(\frac{x'+y'+z'}{r'}\right)^2=\frac{2(x'^2+y'^2+z'^2-x'y'-x'z'-y'z')}{x'^2+y'^2+z'^2},$$

$$l''^2+m''^2+p''^2=3-\left(\frac{x'+b+y'+z'}{r''}\right)^2=\frac{2[(x'+b)^2+y'^2+z'^2-(x'+b)y'-(x'+b)z'-y'z']}{(x'+b)^2+y'^2+z'^2},$$

$$\begin{aligned}
l'l''+m'm''+p'p''=\frac{1}{r'^2r''^2}\Big\{&3(x'^2+y'^2+z'^2)[(x'+b)^2+y'^2+z'^2]\\
&-(x'+y'+z')^2[(x'+b)^2+y'^2+z'^2]-(x'+b+y'+z')^2(x'^2+y'^2+z'^2)\\
&+(x'+y'+z')(x'+b+y'+z')[x'(x'+b)+y'^2+z'^2]\Big\}\\
=\frac{1}{r'^2r''^2}\Big\{&(x'^2+y'^2+z'^2)[2(x'^2+y'^2+z'^2-x'y'-x'z'-y'z')\\
&+b(5x'-y'-z')+2b^2]-(x'+y'+z')^2[b(x'+b)]+(x'+y'+z')b^2x'\Big\}.
\end{aligned}$$

18. On a screen placed parallel to the yz-plane at the distance c from the source O' the expression for the coefficient $l'l''+m'm''+p'p''$ of Ex. 17 assumes the form

$$\left.\begin{aligned}
l'l''+m'm''+p'p''=\frac{1}{r'^2r''^2}\Big\{&(c^2+y'^2+z'^2)[2(c^2+y'^2+z'^2-cy'-cz'-y'z')\\
&+b(5c-y'-z')+2b^2]-(c+y'+z')^2[b(c+b)]+(c+y'+z')b^2c\Big\}
\end{aligned}\right\}. \quad \text{(A)}$$

The interference-bands will thus vanish at those points on the given screen, where this expression for the coefficient vanishes.

The interference-bands will be most distinct at those points on the given screen, where the coefficient (A) becomes a maximum, that is, where the value of this coefficient approaches those assumed by the coefficients of the first and second terms of the given expression for the resultant amplitude.

Show that the coefficients of the different terms of the given expression for the tant amplitude all assume the same value at the point $y'=z'=0$ on the screen.

19. Show that the amplitude of the resultant primary wave of Ex. 17 is given approximately by the following expression in the neighbourhood of the point $y'=z'=0$ on the screen $x=c$:

$$a^2=2n^4\left\{\frac{c-y'-z'}{c}\frac{1}{r'^2}+\frac{c+b-y'-z'}{c+b}\frac{1}{r''^2}\right.$$
$$\left.+\frac{2c(c+b)-(2c+b)(y'+z')}{c(c+b)}\frac{1}{r'r''}\cos n(r'-r''+\delta'-\delta'')\right\}$$

(cf. also p. 133).

20. Show that the resultant secondary oscillations represented by formulae (29) and (30) take place in plane elliptic paths.

21. Show that for $x''=c+b$, $x'=b$, $y''=y'=z''=z'=0$ (cf. formulae (36)) the coefficient of the second term of expression (35) assumes the same value as that of the interference-term.

22. Determine the approximate value of the expression (39) in the neighbourhood of the point $y'=z'=0$.

In the neighbourhood of the given point y'^2 and z'^2 will be very small in comparison to y', z' and the given quantities c and b, and they may thus be rejected in the determination of the coefficients of the different terms of the given expression; we thus have

$$l'^2+m'^2+p'^2=\frac{3[c^2+c^2+2c(y'+z')]}{c^2}=\frac{6(c+y'+z')}{c},$$

$$l''^2+m''^2+p''^2=\frac{3[(c+b)^2+(c+b)^2+2(c+b)(y'+z')]}{(c+b)^2}=\frac{6(c+b+y'+z')}{c+b},$$

and $l'l''+m'm''+p'p''$

$$=\frac{3c^2[2c^2+2c(y'+z')+b(3c+y'+z')]+3b\{c[c^2+2c(y'+z')]+b[2c^2+c(y'+z')]\}}{c^2(c+b)^2}$$

$$=\frac{3[c^2(2c^2+4cb+2b^2)+c(2c^2+3cb+b^2)(y'+z')]}{c^2(c+b)^2}$$

$$=\frac{3[2c^2(c+b)^2+c(2c+b)(c+b)(y'+z')]}{c^2(c+b)^2}=\frac{3[2c(c+b)+(2c+b)(y'+z')]}{c(c+b)}.$$

The given expression can thus be written

$$a^2=6n^2\left\{\frac{c+y'+z'}{c}\frac{1}{r'^4}+\frac{c+b+y'+z'}{c+b}\frac{1}{r''^4}\right.$$
$$\left.+\frac{2c(c+b)+(2c+b)(y'+z')}{c(c+b)}\frac{1}{r'^2r''^2}\cos n(r'-r''+\delta'-\delta'')\right\}.$$

23. Determine and examine, as in text, the expressions for the amplitudes of the resultant primary and secondary waves obtained by the superposition of two (similar) electromagnetic (Hertzian) waves of the type represented by formulae (9) of Chapter II.

24. Determine and examine the expressions for the amplitudes of the magnetic waves that accompany the resultant (electric) primary and secondary waves examined in the text, those represented by formulae (19) and (20), and (29) and (30) respectively.

CHAPTER V.

HUYGENS'S PRINCIPLE.

Rectilinear Propagation of Light.—In Chapter I. we have observed that the wave-theory, as first postulated by Fresnel, accounts for the rectilinear propagation of light and thus furnishes another argument for its universal acceptance. At first sight a rectilinear propagation would appear to be explained better by the emission than by the wave-theory, which would argue in favour of the former. On the other hand, a closer examination of light phenomena shows that light is propagated only approximately in straight lines and that, like sound, although in a much less degree, it bends round the edges of obstacles placed in its course; for example, the actual shadow cast by a small body that is illuminated through a narrow slit is smaller than the geometrical shadow. The rectilinear propagation of light is, therefore, only an apparently or approximately rectilinear one. It is evident that the emission theory would fail to account for any but a strictly rectilinear propagation, whereas, as we have maintained above and shall confirm below, the approximately rectilinear propagation of light in an homogeneous medium and the bending of its rays round the edges of obstacles placed in its course are direct consequences of the wave-theory.

Huygens's Principle.—Huygens's attempt to explain the rectilinear propagation of light was founded on his so-called "principle," which can be stated as follows: Every point of any wave-front of a system of light waves is conceived as the source of a system of elementary or secondary * waves that are propagated radially from that point with the same velocity as that of the light waves themselves. The envelope of the elementary waves emitted from the various points or sources on any given wave-front will, after the elapse of any given time, evidently coincide with the wave-front of the given wave at that time. Huygens now *assumes* that the effective parts of these elementary waves in generating the new wave-front are confined alone to those portions of them that touch the given envelope. In this manner any and all subsequent wave-fronts are supposed to be generated and the wave itself thus propagated.

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* Not to be confounded with the secondary waves of Chapters II. and III.

We may illustrate Huygens's principle as follows: Let O be the source of a system of (spherical) waves, AB a screen with aperture CD placed in their course, and EF, an arc of radius r, any given wave-front of the pencil of waves, that are passing through the given aperture, at any time t_0, as indicated in the annexed figure. According to Huygens's principle every point of the wave-front EF is to be conceived as a source of elementary (spherical) waves, after the lapse of any time t_1 each source will have emitted a (spherical) wave, all the wave-front elements of which have advanced to one and the same distance, t_1v, from that source, where v is the velocity of propagation of light. The wave-fronts of the elementary waves, emitted from the various points or sources on the given wave-front EF, at the time t, will thus be spheres of one and the same radius, t_1v, described about those points as centres, as indicated in the above figure. The envelope of these spheres is now the two arcs E_1F_1 and E_2F_2 of radii $r-t_1v$ and $r+t_1v$ respectively, and not, as Huygens assumes, the latter alone, which evidently represents the wave-front of the given wave at the time t_1. The arc E_1F_1 would correspond to a wave

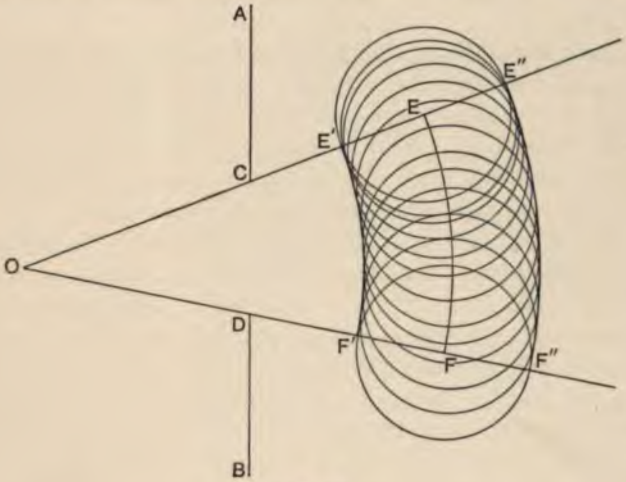

FIG. 16.

that is propagated backwards from the given wave-front *EF* towards the centre of disturbance *O*; the presence of such reflected waves in an homogeneous medium is not, however, confirmed by facts.

Difficulties encountered in Huygens's Principle.—Apart from the difficulty encountered in ridding Huygens's principle of the reflected waves, just mentioned, it is evident, if we assume, as Huygens does, that the effective parts of the elementary waves be only those portions of them that touch the envelope of the waves, that the wave-front *EF* would be propagated radially from (and towards) the centre of disturbance *O*, that is, light would be propagated according to Huygens's principle always in straight lines; this would now exclude the possibility of accounting for the slight bending of light rays (waves) round edges, as those *C* and *D* of Figure 16, or upon their passage through narrow slits.

Huygens's Principle and the Laws of Refraction and Reflection.—It is easy to show that the laws of reflection and refraction of rays (waves) on their passage from one medium into another can be explained on the assumption of Huygens's principle;* but this argues little in favour of the above form of presentation of the same, since these laws are only direct consequences of the simplest assumptions.†

Fresnel's Modifications of Huygens's Principle.—Fresnel rejected Huygens's purely arbitrary assumption that the effective parts of the elementary waves be only these portions of them that touch the envelopes of their wave-fronts and attempted to calculate on the theory of interference the oscillatory state due to the resultant action of those waves at any point. These calculations not only give only an approximately rectilinear propagation of light-waves through an homogeneous medium and a slight bending of the same round edges, but they also—on the assumption of a suitable law for the action of the elementary wave with regard to its obliquity (cf. below)—rid Huygens's principle of the reflected waves mentioned above.

Determination of Light-Vector by Fresnel's Method.—Let us next calculate by Fresnel's method the oscillatory state at any point due to the resultant action of the elementary waves emitted from any given (spherical) wave-front. Let *O* be the source of disturbance, *ABCP*, a sphere of radius *r*, with centre at *O*, the wave-front of any given wave emitted from that source, and *Q* a point outside the given sphere, the point, at which the oscillatory state due to the resultant action of the elementary waves, emitted from the numerous (elementary) sources on the given wave-front, at that point, is sought (cf. Figure 17).

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* Cf. Preston, *Theory of Light*, § 66.

† Cf. *ibid.*, §§ 58 and 65.

The oscillatory state or light-vector* at any point or source *M* on the given wave-front can now evidently be represented by the quantity (moment)

$$S=\frac{a}{r_1}\sin\frac{2\pi}{\lambda}(vt-r_1); \qquad \text{.........................(1)}$$

for the primary waves treated in Chapters II.-IV. this vector or moment would always lie in the given wave-front.

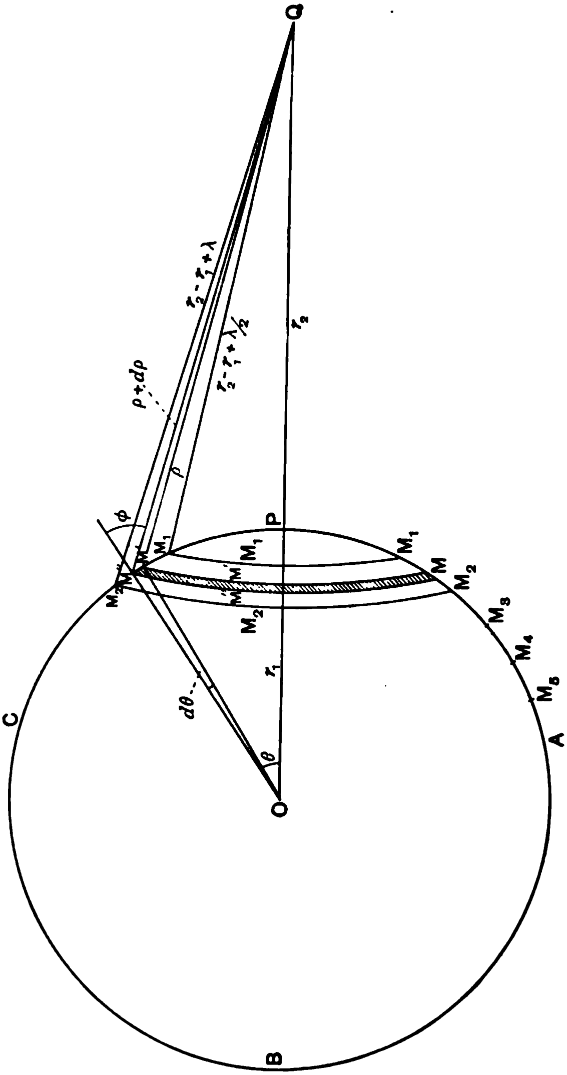

FIG. 17

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* Cf. Drude, *Lehrbuch der Optik*, p. 118.

Division of Wave-Front into Zones or Half-Period Elements.—Fresnel now divides the surface of the given sphere or wave-front up into zones referred to their so-called "pole" P or that point on the surface of the sphere that is nearest to the point of observation Q (cf. Figure 17). The region (on the surface of the given sphere) round this pole extending as far as the circle, whose distance from the point Q is $r_2 - r_1 + \frac{\lambda}{2}$, where r_2 denotes the distance of that point from the source O (cf. Figure 17), is termed the first "zone" or "half period element"; let us denote the circle bounding this region by M_1. The second zone extends from the circle M_1 to that circle M_2 on the surface of the given sphere, whose distance from Q is $r_2 - r_1 + \lambda$. Similarly, by describing on the surface of the given sphere circles M_3, M_4, etc., whose distances from the point Q, are $r_2 - r_1 + \frac{3\lambda}{2}$, $r_2 - r_1 + 2\lambda$, etc., we obtain the *3rd*, *4th*, etc., zones or half-period elements. These zones are evidently not of the same width, but they decrease in width, as we recede from the pole P towards the circle AC (cf. Figure 17), from which circle they increase in width, as we approach the point B; AC is here that circle on the surface of the given sphere, for which the vectors from the point Q are tangents to the same, and B the point diametrically opposite the pole P.

Determination of Area of any Zone.—Let us now consider the action of the elementary waves emitted from any zone, for example the second, at the point Q. For this purpose we divide the given zone up into an infinite number of concentric circular zones or zonal elements of infinitesimal width; let this width be so chosen that, if ρ denote the distance from the point Q to any circle M' forming the boundary between any two such zonal elements, the next such bounding circle M'' be taken at the distance $\rho + d\rho$ from Q; let θ denote the angle the vector OM' makes with the vector OP at O and $\theta + d\theta$ the angle between the vectors OM'' and OP (cf. Figure 17). The area do of the zonal element $M'M''$ is evidently

$$do = 2\pi r_1 \sin\theta\, r_1 d\theta = 2\pi r_1^2 \sin\theta\, d\theta. \quad \text{.................. (2)}$$

The following analytic relation now holds between the quantities ρ, r_1, r_2, and θ:

$$\rho^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta; \quad \text{........................ (3)}$$

which differentiated gives the following relation between the differentials $d\rho$ and $d\theta$:

$$\rho d\rho = r_1 r_2 \sin\theta\, d\theta. \quad \text{.............................. (4)}$$

By this formula we can write the expression (2) for the area *do*,

$$do = 2\pi \frac{r_1}{r_2} \rho d\rho. \quad \text{.................................(5)}$$

Expression for the Light-Vector.—We have seen above that the light-vector s at any point M can be represented by the expression

$$s = \frac{a}{r_1} \sin \frac{2\pi}{\lambda}(vt - r_1).$$

The light-vector s' at an external point Q due to the action of the elementary wave emitted from any such source M on the given wave-front would now be inversely proportional to the distance of the point Q from that source; it would also be a function of the obliquity or the angle ϕ between the normal to the given wave-front at the point M and the vector from that element to the point Q (cf. Figure 17). The light-vector s' at the external point Q due to the action of the elementary wave emitted from the source M could, therefore, be represented by the expression

$$s' = \frac{a}{r_1} \frac{f(\phi)}{\rho} \sin \frac{2\pi}{\lambda}[vt - (r_1 + \rho)], \quad \text{.....................(6)}$$

where $f(\phi)$ expresses the law of variation of s' with regard to the obliquity ϕ.

Light-Vector produced by Elementary Waves of any Zonal Element.—The light-vector at Q due to the action of the waves emitted from all the points or elementary sources on any zonal element, for example $M'M''$, is now assumed (cf. p. 164) to be proportional to the number of those sources, that is, to the area *do* of that zonal element; if we denote that vector by dS', we should thus have

$$dS' = s'do = \frac{a}{r_1} \frac{f(\phi)}{\rho} \sin \frac{2\pi}{\lambda}[vt - (r_1 + \rho)]do, \quad \text{...............(7)}$$

or, on replacing *do* by its value (5),

$$dS' = 2\pi a \frac{f(\phi)}{r_2} \sin \frac{2\pi}{\lambda}[vt - (r_1 + \rho)]\, d\rho. \quad \text{.............(7A)}$$

Laws of Obliquity; **Natural Law.**—The light-vector s' at an external point evidently depends upon the law of variation of the light-vector over the wave-front of the elementary wave, that is, its law of variation with regard to the angle the vector from the source (M) of the elementary wave to any wave-front element of the same makes with the normal to the wave-front proper at that source. It is now natural to assume that this light-vector vanishes at all points behind the wave-front proper; that is, we suppose each elementary source (M) to emit only a hemispherical wave, that in front of the wave-front proper (cf. Figure 17), and we thus exclude all reflected waves (cf. p. 144) from the medium. Moreover, it is natural to assume that the law of variation of the light-vector over this hemisphere be according to the cosine of the obliquity or the angle ϕ between the vector from the source (M) to the given element and the normal to the wave-front proper at that source, the vector thus varying from zero at the base of the hemisphere or equator to a maximum at its pole. Since now the light-vector s' at an external point Q is determined by the action of that wave-element of the elementary wave in question that passes through that point, it would thus be proportional to the cosine of the obliquity ϕ of the given element. Let us express this law, which we shall designate as the "Natural Law of Obliquity," in the form

$$s' \text{ proportional to } \cos\phi = f(\phi). \qquad \text{(8)}$$

Stokes's Law of Obliquity.—Sir G. G. Stokes* has now found that the light-vector s' at an external point varies as $1+\cos\phi$; let us express this law in the form

$$s' \text{ proportional to } (1+\cos\phi) = f(\phi). \qquad \text{(9)}$$

$f(\phi)$ becomes a maximum, 2, for the elementary wave emitted from the pole P and decreases in value, as we recede from the pole, assuming the value unity for the waves emitted from the circle (AC), for which the vectors from the external point (Q) are tangent to the wave-front proper ($ABCP$), towards the point B diametrically opposite the pole P, at which point $f(\phi)$ vanishes (cf. also p. 175). According to this law of Stokes the wave-front of the elementary wave must evidently be regarded as a complete sphere and not as an hemisphere, as assumed above, and hence the presence of reflected waves granted.

Laws of Obliquity in Terms of ρ.—The obliquity ϕ can now be expressed as a function of the given quantities r_1 and r_2 and the variable ρ (cf. Ex. 1). We could thus express the law of variation of the light-vector s' with regard to ϕ in the form

$$s' \text{ proportional to } F(r_1, r_2, \rho),$$

or, for given r_1 and r_2,

$$s' \text{ proportional to } F(\rho). \qquad \text{(10)}$$

Throughout any given zone $M_n - M_{n+1}$ ρ increases from

$$\rho_n = r_2 - r_1 + \frac{n\lambda}{2} \text{ for the circle } M_n$$

to

$$\rho_{n+1} = r_2 - r_1 + \frac{n+1}{2}\lambda \text{ for the circle } M_{n+1},$$

---

* "On the Dynamical Theory of Diffraction," *Math. and Phys. Papers*, vol. ii., p. 243.

that is, it increases by the quantity $\frac{\lambda}{2}$. For light-waves λ is now so small compared with r_1 and r_2 that the function $F(\rho)$ could be regarded as approximately constant throughout any zone. In determining the light-vector S_n' at an external point Q, due to the action of the waves emitted from any zone $M_n - M_{n+1}$ or n, we could, therefore, set $F(\rho)$ before the sign of integration.

Determination of Light-Vector produced by Elementary Waves of any Zone.—For light-waves the vector S_n' due to the action of the elementary waves emitted from any zone n would, therefore, by formula (7A), be given by the integral

$$
\begin{aligned}
S_n' &= 2\pi a \frac{F(\rho)}{r_2} \int_{r_2 - r_1 + \frac{n+1}{2}\lambda}^{r_2 - r_1 + \frac{n}{2}\lambda} \sin \frac{2\pi}{\lambda} [vt - (r_1 + \rho)] d\rho \\
&= a\lambda \frac{F(\rho)}{r_2} \left\{ \cos \frac{2\pi}{\lambda} \left[vt - \left(r_2 + \frac{n\lambda}{2} \right) \right] - \cos \frac{2\pi}{\lambda} \left[vt - \left(r_2 + \frac{n+1}{2} \lambda \right) \right] \right\} \\
&= a\lambda \frac{F(\rho)}{r_2} \left\{ \cos \frac{2\pi}{\lambda} (vt - r_2) \cos n\pi - \cos \frac{2\pi}{\lambda} (vt - r_2) \cos (n+1) \pi \right\} \\
&= (-1)^n 2a\lambda \frac{F(\rho)}{r_2} \cos \frac{2\pi}{\lambda} (vt - r_2). \qquad \ldots\ldots(11)
\end{aligned}
$$

Light-Vector produced by Elementary Waves of two Consecutive Zones.—Let us now consider the light-vector at an external point Q due to the mutual action of the elementary waves emitted from two consecutive zones. It is evident from formula (11) (cf. also formula (5)) that the waves emitted from the one zone, being opposite in phase to those emitted from the other, would neutralize each other,* provided their variation in obliquity could be entirely neglected (cf. also p. 164). The only effect that could be produced at an external

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* The waves emitted from the zonal element included between the vectors

$$\rho = r_2 - r_1 + \frac{n\lambda}{2} \text{ and } \rho + d\rho = r_2 - r_1 + \frac{n\lambda}{2} + d\rho$$

of the one zone would differ in phase from those emitted from the zonal element included between the vectors

$$\rho = r_2 - r_1 + \frac{n+1}{2}\lambda \text{ and } \rho + d\rho = r_2 - r_1 + \frac{n+1}{2}\lambda + d\rho$$

of the other zone by half a wave-length; since now the elements of area included between these two pairs of vectors are the same (cf. formulae (5) and (11)), the waves from the one zone would neutralize those from the other. Similarly, the waves from the next and all following consecutive zonal elements of the one zone would neutralize those from the corresponding elements of the other zone.

point would, therefore, be that due to the variation in the (mean) obliquity between the waves emitted from the one zone and those emitted from the other (cf. also p. 165). The total resultant effect or light-vector at an external point would thus be given by a sum of such effects $\Sigma\frac{1}{2}(S'_n+S'_{n+1})$ arising from the variations in obliquity between the waves emitted from consecutive zones. Let us determine the expression for such an effect.

For two consecutive zones, n and $n+1$, the law of variation of s' with regard to obliquity could be expressed by the functions

$$F\left(r_2-r_1+\frac{n\lambda}{2}\right) \text{ and } F\left(r_2-r_1+\frac{n+1}{2}\lambda\right)$$

(cf. formula (10)) or for light-waves, since λ is then infinitely small compared with r_2-r_1, approximately by the functions

$$F\left(r_2-r_1+\frac{n\lambda}{2}\right) \text{ and } F\left(r_2-r_1+\frac{n\lambda}{2}\right)+\frac{1}{2}\lambda F'\left(r_2-r_1+\frac{n\lambda}{2}\right), \quad (12)$$

where F' denotes the derivative of F with regard to λ.

By formula (11), the resultant effect or light-vector, $S'_n+S'_{n+1}$, due to the mutual action of the elementary waves emitted from any two consecutive zones n and $n+1$ could thus be written

$$S'_n+S'_{n+1}=(-1)^n 2\frac{a\lambda}{r_2}F\left(r_2-r_1+\frac{n\lambda}{2}\right)\cos\frac{2\pi}{\lambda}(vt-r_2)$$

$$+(-1)^{n+1}2\frac{a\lambda}{r_2}\left[F\left(r_2-r_1+\frac{n\lambda}{2}\right)+\frac{1}{2}\lambda F'\left(r_2-r_1+\frac{n\lambda}{2}\right)\right]\cos\frac{2\pi}{\lambda}(vt-r_2)$$

$$=(-1)^{n+1}\frac{a\lambda^2}{r_2}F'\left(r_2-r_1+\frac{n\lambda}{2}\right)\cos\frac{2\pi}{\lambda}(vt-r_2). \quad \dots\dots\dots\dots\dots(13)$$

Expression for Total Effect or Light-Vector.—The resultant effect or light-vector (13) is now small, proportional to λ, compared with that (11) produced by the waves emitted from either zone. It follows, moreover, from formula (11) that the light-vector S_n due to the action of the elementary waves emitted from any zone differs in sign from that S'_{n+1} due to the action of the waves from the adjacent zone. If we denote the absolute value of any light-vector S_n' by S_n, the total resultant effect or light-vector S at an external point Q due to the mutual action at that point of all the elementary waves emitted from any (spherical) wave-front $ABCP$ (cf. Figure 17) will, by the principle of superposition, be given by the series

$$S=S_0-S_1+S_2-S_3+-\dots(-1)^m S_m \quad \dots\dots\dots\dots\dots\dots(14)$$

(cf. formula (11)), where $m+1$ is the number of zones, into which the given wave-front can be divided.

Whatever law of variation of light-vector with regard to obliquity be assumed, it must evidently be such that the greater the obliquity the smaller the light-vector; since now the variation of S_n is alone due to a variation of the obliquity (cf. p. 150), it would follow that, as n increased, S_n would decrease in value; that is

$$S_n > S_{n+1}.$$

For m even, $m = 2\kappa$, where κ is an integer, the above series (14) for S can evidently be written

$$\left.\begin{aligned} S = \frac{S_0}{2} + \left(\frac{S_0}{2} - S_1 + \frac{S_2}{2}\right) + \left(\frac{S_2}{2} - S_3 + \frac{S_4}{2}\right) + \ldots \\ + \left(\frac{S_{m-2}}{2} - S_{m-1} + \frac{S_m}{2}\right) + \frac{S_m}{2} \end{aligned}\right\} \ldots\ldots\ldots\ldots(15)$$

Relation between the Light-Vectors of Consecutive Zones.—Let us now examine any bracket term

$$\left(\frac{S_{n-1}}{2} - S_n + \frac{S_{n+1}}{2}\right)$$

of the series (15); here n must evidently be taken odd.

By formula (13), we have

$$-S_n + S_{n+1} = S''_n + S''_{n+1} = (-1)^{n+1}\frac{a\lambda^2}{r_2}F'\left(r_2 - r_1 + \frac{n\lambda}{2}\right)\cos\frac{2\pi}{\lambda}(vt - r_2),$$

which for n odd becomes

$$= \frac{a\lambda^2}{r_2}F'\left(r_2 - r_1 + \frac{n\lambda}{2}\right)\cos\frac{2\pi}{\lambda}(vt - r_2);$$

and similarly,

$$-S_n + S_{n-1} = S''_n + S''_{n-1} = (-1)^n\frac{2a\lambda}{r^2}F\left(r_2 - r_1 + \frac{n\lambda}{2}\right)\cos\frac{2\pi}{\lambda}(vt - r_2)$$

$$+ (-1)^{n-1}\frac{2a\lambda}{r_2}\left[F\left(r_2 - r_1 + \frac{n\lambda}{2}\right) - \frac{1}{2}\lambda F'\left(r_2 - r_1 + \frac{n\lambda}{2}\right)\right]\cos\frac{2\pi}{\lambda}(vt - r_2),$$

which for n odd becomes

$$= -\frac{a\lambda^2}{r_2}F'\left(r_2 - r_1 + \frac{n\lambda}{2}\right)\cos\frac{2\pi}{\lambda}(vt - r_2).$$

Add these two expressions, and we have

$$S_{n-1} - 2S_n + S_{n-1} = 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

In obtaining this relation between the S's, we have rejected only terms of the third and higher orders of magnitude in λ, those arising from the development of the function F with regard to that quantity (cf. formula (12)); the given expression between the S's can, therefore, differ from zero by a quantity of only the third or higher orders of magnitude in λ, that is, by a quantity, whose order of magnitude in λ is at least two higher than that of the expression (11) for S_n. It is evident that such quantities can be rejected, when compared with S_n (S_0), even when the number of terms of the given series is of the same order of magnitude as $\frac{1}{\lambda}$. The series (15) thus reduces to the simple expression

$$S=\frac{S_0}{2}+\frac{S_m}{2}.$$

Similarly, it is easy to show, when m is odd, that the series (14) for S reduces to

$$S=\frac{S_0}{2}-\frac{S_m}{2}$$

(cf. Ex. 2 at end of chapter).

The light-vector S at an external point (Q) would thus be given by the expression

$$S=\frac{S_0}{2}\pm\frac{S_m}{2}.^{*} \quad \text{.............................(17)}$$

Replace here S_0 and S_m by their values from formula (11), and we have

$$S=\frac{a\lambda}{r_2}\left[F(r_2-r_1)\pm F\left(r_2-r_1+\frac{m\lambda}{2}\right)\right]\cos\frac{2\pi}{\lambda}(vt-r_2), \quad \text{......(17A)}$$

where $m+1$ is the number of zones that contribute to the total effect at the given point.

Laws of Obliquity.—To evaluate the expression (17A) for S, we must now know the law of variation of the light-vector s' with regard to the obliquity ϕ, that is, the function $F(\rho)$ (cf. formula (10)). For Stokes's law (cf. formula (9)), we find (cf. Ex. 1 at end of chapter)

$$F(\rho)=\frac{r_2^2-(r_1-\rho)^2}{2r_1\rho}. \quad \text{..........................(18)}$$

We have seen on p. 148 that this law gives reflected waves. That no reflected waves may appear, it is evident that the law of variation of the light-vector s' with regard to the obliquity ϕ must be such that the elementary waves emitted from the zones, for which ϕ is greater than 90°, will have no effect at the point in question. On the other hand, the law sought must evidently express the empirical fact that the larger ϕ, between 0° and 90°, the smaller the light-vector s'. The simplest such law is now the natural law proposed on p. 148, namely,

$$s' \text{ proportional to } \cos\phi \quad (0 \gtreqless \phi \leqq 90^\circ)$$

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[*] For another proof of this formula see: A. Schuster, *Philosophical Magazine*, vol. 31, p. 85.

(cf. formula (8)); which gives the following value for the function $F(\rho)$:

$$F(\rho)=\frac{r_2^2-r_1^2-\rho^2}{2r_1\rho} \quad\text{..........................(18A)}.$$

(cf. Ex. 1 at end of chapter).

Total Light-Vector for Natural Law.—On the assumption of the natural law (18A) the last term of the expression (17A) for S evidently vanishes, and S is then given by the expression

$$S=\frac{a\lambda}{r_2}\cos\frac{2\pi}{\lambda}(vt-r_2). \quad\text{..........................(19)}$$

Total Light-Vector for Stokes's Law.—On the assumption of Stokes's law (18) S is evidently given by the expression

$$S=\frac{2a\lambda}{r_2}\cos\frac{2\pi}{\lambda}(vt-r_2). \quad\text{..........................(20)}$$

Approximately Rectilinear Propagation of Light-Waves. Effect of Small Circular Screen at Pole: Bending of Waves.—On the assumption of either of the above laws (18) or (18A) the following relation evidently holds between the total light-vector S and the light-vector S_0 due to the action of the elementary waves emitted from the central zone only:

$$S=\frac{S_0}{2} \quad\text{..................................(21)}$$

(cf. formula (17)); that is, the total light-vector S at an external point Q could be conceived as produced alone by the action of the elementary waves emitted from the first half of the central zone, or the effective portion of the given wave-front could be regarded as confined to a very small area (of the dimensions of the wave-length λ) around the pole P; in other words, the light received at Q could be conceived as propagated from the source O in approximately straight lines. This approximately rectilinear propagation of light is evidently a consequence of the extremely short wave-length of the light waves; for waves of long wave-length, as the electromagnetic waves proper (Hertzian) or those of sound, the propagation would deviate considerably from the rectilinear. If the first half of the central zone be intercepted at the pole P by an opaque screen, the elementary waves emitted from the other portions of the given wave-front would have apparently no effect at the point Q, that is, that point would receive no light. This conclusion is now neither correct nor is it confirmed by experiment. If we screen off the first half of the central zone by means of a small circular screen placed at P, the first zone will then extend from the edge of that screen to the circle on the given wave-front, whose distance from the point Q is $r_2 - r_1 + \frac{3\lambda}{4}$; the second zone from this circle to that whose distance from Q is $r_2 - r_1 + \frac{5\lambda}{4}$, etc.; the total effect at the point Q, therefore, would not be zero, as concluded above, but it would be given by a series similar to that (14) already found, which, like the latter, would assume most approximately the value $\frac{S_1}{2}$, where S_1 denotes the light-vector produced by the elementary waves emitted from the (first) zone bounded by the edge of the screen and the circle on the given wave-front, whose distance from the point Q is $r_2 - r_1 + \frac{3\lambda}{4}$. The point Q or the line PQ would, therefore, be not dark but illuminated, and evidently (cf. formulae (11) and (18A)) only to an infinitesimally less degree than in the case, where no obstacle were placed in the course of the waves; this would be interpreted according to the Emission Theory as a bending of the waves around the edge of the screen or obstacles placed in their course, a result that is confirmed by experiment. If the screen be large, of dimensions not of the wave-length but of the distance of the point Q from the screen, then the effect at that point would be small compared with that, where no obstacle were placed in the course of the waves (cf. formula (18A)).

Effect of Small Screen of Irregular Contour: Great Diminution in Intensity.—A case, where the total effect or light-vector at the point Q will be found to be small compared with that where no obstacle is placed in the course of the waves, is that where the screen is comparatively large—large compared with λ^2—and either not exactly circular or not placed with its centre at the pole P; these conditions would evidently correspond somewhat better to the actual facts of experiment than those assumed above. To determine the resultant effect or light-vector at the point Q, we imagine the edge or contour of the given screen replaced by a great (infinite) number of very (infinitely) short circular arcs of varying radius with common centre at the pole P. The light-vector dS produced by the elementary waves emitted from that unscreened portion of the given wave-front that lies between any two such consecutive vectors extended will then, by formula (19), be given by the expression

$$dS = \frac{dS_1}{2} = \frac{a\lambda}{r_1 + \rho_1} \frac{d\phi_1}{2\pi} \cos \frac{2\pi}{\lambda}[vt - (r_1 + \rho_1)],$$

where $d\phi_1$ denotes the angle subtended at the pole P by those two vectors and ρ_1 the radius of the corresponding circular arc (given portion of edge of screen).

Similarly, the light-vector dS, to which the unscreened portion of the next circular sector of the given wave-front would give rise, will be given by

$$dS = \frac{dS_2}{2} = \frac{a\lambda}{r_1+\rho_2}\,\frac{d\phi_2}{2\pi}\cos\frac{2\pi}{\lambda}\left[vt-(r_1+\rho_2)\right],\ \text{etc.}$$

The total effect or light-vector S at the point Q produced by the action of the elementary waves emitted from the entire unscreened portion of the given wave-front will, therefore, be given by the sum of all the light-vectors $dS = \frac{dS_\kappa}{2}$; we thus have

$$\left.\begin{aligned} S = \Sigma\frac{dS_\kappa}{2} = \frac{a\lambda}{2\pi}\Bigl\{&\frac{d\phi_1}{r_1+\rho_1}\cos\frac{2\pi}{\lambda}\left[vt-(r_1+\rho_1)\right]+\frac{d\phi_2}{r_1+\rho_2}\cos\frac{2\pi}{\lambda}\\ &\times\left[vt-(r_1+\rho_2)\right]+\ldots+\frac{d\phi_\kappa}{r_1+\rho_\kappa}\cos\frac{2\pi}{\lambda}\left[vt-(r_1+\rho_\kappa)\right]\Bigr\}\end{aligned}\right\}, \quad\ldots\ldots\ldots(22)$$

where κ denotes the number of circular arcs, by which the contour of the given screen has been replaced.

Since now the contour of the given screen is assumed to vary only very little from that of an exact circle with centre at P, $\rho_1, \rho_2, \ldots$ will vary only very little from one another, and they may thus be replaced by any mean value of the same, ρ, in the *coefficients* of the cosines in the expression (22) for S, but they cannot evidently be replaced by any such mean value in the *arguments* of those cosines, since $r_1+\rho_1$, $r_1+\rho_2, \ldots$ are divided here by the small quantity λ. The above expression (22) for S can thus be written most approximately

$$\left.\begin{aligned} S = \frac{a\lambda}{2\pi(r_1+\rho)}\Bigl\{&d\phi_1\cos\frac{2\pi}{\lambda}\left[vt-(r_1+\rho_1)\right]+d\phi_2\cos\frac{2\pi}{\lambda}\\ &\times\left[vt-(r_1+\rho_2)\right]+\ldots+d\phi_\kappa\cos\frac{2\pi}{\lambda}\left[vt-(r_1+\rho_\kappa)\right]\Bigr\}\end{aligned}\right\}. \quad\ldots\ldots\ldots\ldots(22\text{A})$$

The smallest deviations in the contour of the given screen from the exact circle with centre at P would now, in general, at least for light-waves, correspond to variations of several waves-lengths in the quantities $\rho_1, \rho_2 \ldots$, and hence to a great irregularity in the values assumed by the different cosines in this expression (22A) for S; some would be positive, others negative, and others vanish entirely. The value assumed by the series in the largest pair of brackets would, therefore, be small here compared with that of the series for the case, where the ρ's and hence each term of the series have one and the same value. The total effect or light-vector S at the point Q screened off from the source O by a small (of the dimensions of the millimetre) opaque body would, in general (for light-waves), thus be small compared with the natural light-vector; that is, the points directly behind the screen or the line PQ would be illuminated only weakly.

Effect of Large Screen with Small Circular Aperture at Pole: Maxima and Minima of Intensity.—If we replace the small opaque screen employed on p. 153 by a large one with a small circular aperture at P, we obtain quite different results from those above for the quasi-complementary case. If the given aperture admit the waves of only the first half of the central zone, the light-vector at Q will evidently be exactly the same as when the screen is removed, that is, it will be the normal or natural light-vector or intensity. If the aperture be increased to admit the waves of the whole central zone only, the light-vector will be double the natural light-vector or the intensity four times the natural one. If the aperture admit all the waves from the central and first zones only, the resultant light-vector and hence the intensity at Q will approximately vanish. As we increase the opening in the screen, the resultant intensity at P will thus vary periodically between maxima and minima, but these maxima and minima will evidently become less and less pronounced, so that after the opening has attained a certain size, there will be no appreciable variation in the illumination at Q. These results, all of which are also confirmed by experiment, differ materially from those obtained by means of a small opaque screen at the pole P; here the intensity varies periodically between maxima and minima, as the aperture is increased in size, approaching a given uniform intensity, after the aperture has attained a certain size, whereas in the quasi-complementary case the intensity decreased from a given finite maximum continuously but rapidly to a small value, as the dimensions of the screen approached those of the distance (squared) of the point Q from the same. Instead of varying the size of the small intercepting screen or that of the aperture in the large screen, we can evidently obtain the same results by taking the point of observation Q at different points on the line PQ.

Effect of Large Screen with Small Irregular Aperture: Natural Intensity.—When the aperture in the intercepting screen is not exactly circular (with respect to pole), it is evident from considerations similar to those on the preceding page that the maxima and minima of intensity at the point Q will be much less marked than when the aperture is exactly circular (with respect to pole); the resultant intensity will then approach one and the same, the natural intensity, for all sizes of aperture (cf. Ex. 3 at end of chapter). For the quasi-complementary case, where the aperture was replaced by a small screen, we found only a weak illumination along the line PQ.

The Electromagnetic Vector.—The above formulae, from (11) on, have been deduced on the assumption that the wave-length λ were very (infinitely) small compared with the distances r_1 and r_2; they would thus hold for light-waves but not for electromagnetic waves proper, as the Hertzian. Let us now examine the resultant effect or electromagnetic vector at an external point Q (cf. Figure 17) due to the action of the elementary waves emitted from any given wave-front $ABCP$ of a system of (spherical) waves of long wave-length, as the electromagnetic waves proper.

Determination of Electromagnetic Vector produced by Elementary Waves from any Zone.—For waves of long wave-length the electromagnetic vector S_n' due to the action of the elementary waves emitted from any zone n would evidently be given by the integral.

$$S_n' = \frac{2\pi a}{r_2}\int F(\rho)\sin\frac{2\pi}{\lambda}[vt-(r_1+\rho)]\,d\rho \text{ ....(23)}$$

(cf. formulae (7A) and (10)), where $F(\rho)$ cannot be regarded as constant throughout the given zone, as on p. 149, and has thus been retained under the sign of integration. In order to evaluate this integral we must now assume some law of variation for the electromagnetic vector s' with regard to the obliquity ϕ, that is, we must know the function $F(\rho)$ (cf. p. 152). Let us assume here the natural law of obliquity, expressed by formula (8) or (18A).

Replace $F(\rho)$ by its value (18A) in the expression (23) for S_n', and we have

$$\left.\begin{aligned} S_n' &= \frac{\pi a(r_2^2-r_1^2)}{r_2r_1}\int\sin\frac{2\pi}{\lambda}[vt-(r_1+\rho)]\frac{d\rho}{\rho} \\ &\quad\frac{\pi a}{r_2r_1}\int\sin\frac{2\pi}{\lambda}[vt-(r_1+\rho)]\rho\,d\rho \end{aligned}\right\};\text{ ....(24)}$$

which we write $$S'_n = AS'_{1_n} + BS'_{2_n}, \text{ ....(24A)}$$

where $$A = \frac{\pi a(r_2^2-r_1^2)}{r_2r_1},\quad B = -\frac{\pi a}{r_2r_1}, \text{ ....(25)}$$

$$S'_{1_n} = \int\sin\omega\frac{d\rho}{\rho},\qquad S'_{2_n} = \int\sin\omega\,\rho\,d\rho, \text{ ....(26)}$$

where $$\omega = \frac{2\pi}{\lambda}[vt-(r_1+\rho)]. \text{ ....(26A)}$$

Let us first evaluate the integral S'_{1_n}. By the reduction-formulae

$$\int\frac{\sin\omega\,d\rho}{\rho^\kappa} = \frac{\lambda}{2\pi}\int\frac{d(\cos\omega)}{\rho^\kappa} = \frac{\lambda}{2\pi}\,\frac{\cos\omega}{\rho^\kappa} + \frac{1}{\kappa}\,\frac{\lambda}{2\pi}\int\frac{\cos\omega\,d\rho}{\rho^{\kappa+1}}$$

and $$\int\frac{\cos\omega\,d\rho}{\rho^\kappa} = -\frac{\lambda}{2\pi}\int\frac{d(\sin\omega)}{\rho^\kappa} = -\frac{\lambda}{2\pi}\,\frac{\sin\omega}{\rho^\kappa} - \frac{1}{\kappa}\,\frac{\lambda}{2\pi}\int\frac{\sin\omega\,d\rho}{\rho^{\kappa+1}},$$

the repeated integration by parts of the given integral will evidently lead to the following series for the same:

$$\begin{aligned}S'_{1_n}&=\frac{\lambda}{2\pi}\frac{\cos\omega}{\rho}+\frac{\lambda}{2\pi}\Big\{-\frac{\lambda}{2\pi}\frac{\sin\omega}{\rho^2}-\frac{1}{2}\frac{\lambda}{2\pi}\Big\{\frac{\lambda}{2\pi}\frac{\cos\omega}{\rho^3}+\frac{1}{3}\frac{\lambda}{2\pi}\Big\{-\frac{\lambda}{2\pi}\frac{\sin\omega}{\rho^4}\\&\quad-\frac{1}{4}\frac{\lambda}{2\pi}\Big\{\frac{\lambda}{2\pi}\frac{\cos\omega}{\rho^5}+\frac{1}{5}\frac{\lambda}{2\pi}\Big\{-\frac{\lambda}{2\pi}\frac{\sin\omega}{\rho^6}-\frac{1}{6}\frac{\lambda}{2\pi}\Big\{\frac{\lambda}{2\pi}\frac{\cos\omega}{\rho^7}+\ldots\Big\}\Big\}\Big\}\ldots\\&=\frac{\lambda}{2\pi}\frac{\cos\omega}{\rho}-\left(\frac{\lambda}{2\pi}\right)^2\frac{\sin\omega}{\rho^2}-\frac{1}{2!}\left(\frac{\lambda}{2\pi}\right)^3\frac{\cos\omega}{\rho^3}+\frac{1}{3!}\left(\frac{\lambda}{2\pi}\right)^4\frac{\sin\omega}{\rho^4}\\&\quad+\frac{1}{4!}\left(\frac{\lambda}{2\pi}\right)^5\frac{\cos\omega}{\rho^5}-\frac{1}{5!}\left(\frac{\lambda}{2\pi}\right)^6\frac{\sin\omega}{\rho^6}-\frac{1}{6!}\left(\frac{\lambda}{2\pi}\right)^7\frac{\cos\omega}{\rho^7}++--\ldots\\&=\frac{\lambda}{2\pi\rho}\left[1-\frac{1}{2!}\left(\frac{\lambda}{2\pi\rho}\right)^2+\frac{1}{4!}\left(\frac{\lambda}{2\pi\rho}\right)^4-\frac{1}{6!}\left(\frac{\lambda}{2\pi\rho}\right)^6+-\ldots\right]\cos\omega\\&\quad-\frac{\lambda}{2\pi\rho}\left[\frac{\lambda}{2\pi\rho}-\frac{1}{3!}\left(\frac{\lambda}{2\pi\rho}\right)^3+\frac{1}{5!}\left(\frac{\lambda}{2\pi\rho}\right)^5-+\ldots\right]\sin\omega,\end{aligned}$$

or, since $\sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-+\ldots$

and $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+-\ldots,$

$$S'_{1_n}=\frac{\lambda}{2\pi\rho}\left[\cos\frac{\lambda}{2\pi\rho}\cos\omega-\sin\frac{\lambda}{2\pi\rho}\sin\omega\right]=\frac{\lambda}{2\pi\rho}\cos\left(\frac{\lambda}{2\pi\rho}+\omega\right),\ \ldots(27)$$

where the limits of integration are $r_2-r_1+\frac{n\lambda}{2}$ and $r_2-r_1+\frac{n+1}{2}\lambda$. Replace here ρ by these limits, and we have

$$\left.\begin{aligned}S'_{1_n}&=\frac{\lambda}{2\pi\left(r_2-r_1+\frac{n\lambda}{2}\right)}\cos\left[\frac{\lambda}{2\pi\left(r_2-r_1+\frac{n\lambda}{2}\right)}+\frac{2\pi}{\lambda}\left(vt-r_2-\frac{n\lambda}{2}\right)\right]\\&\quad-\frac{\lambda}{2\pi\left(r_2-r_1+\frac{n+1}{2}\lambda\right)}\\&\quad\times\cos\left[\frac{\lambda}{2\pi\left(r_2-r_1+\frac{n+1}{2}\lambda\right)}+\frac{2\pi}{\lambda}\left(vt-r_2-\frac{n+1}{2}\lambda\right)\right]\end{aligned}\right\}.\ (27\text{A})$$

We next evaluate the integral S'_{2_n}; we evidently have

$$S'_{2_n}=\int\rho\sin\omega\,d\rho=\frac{\lambda^2}{4\pi^2}\sin\omega+\frac{\lambda}{2\pi}\rho\cos\omega,\ \ldots\ldots\ldots\ldots\ldots(28)$$

where the limits of integration are, as above, $r_2-r_1+\frac{n\lambda}{2}$ and $r_2-r_1+\frac{n+1}{2}\lambda$. On replacing here ρ by these limits, we have

$$\left.\begin{aligned} S'_{2_n} = {} & \frac{\lambda^2}{4\pi^2}\left[\sin\frac{2\pi}{\lambda}\left(vt - r_2 - \frac{n\lambda}{2}\right) - \sin\frac{2\pi}{\lambda}\left(vt - r_2 - \frac{n+1}{2}\lambda\right)\right] \\ & + \frac{\lambda}{2\pi}\left[\left(r_2 - r_1 + \frac{n\lambda}{2}\right)\cos\frac{2\pi}{\lambda}\left(vt - r_2 - \frac{n\lambda}{2}\right)\right. \\ & \left. - \left(r_2 - r_1 + \frac{n+1}{2}\lambda\right)\cos\frac{2\pi}{\lambda}\left(vt - r_2 - \frac{n+1}{2}\lambda\right)\right] \end{aligned}\right\}, \quad (28\text{A})$$

and, on developing these sines and cosines as functions, sines and cosines, of the angles $\frac{2\pi}{\lambda}(vt - r_2)$ and $n\pi$ or $(n+1)\pi$ respectively,

$$\left.\begin{aligned} S'_{2_n} = {} & \frac{\lambda^2}{4\pi^2}\left[\sin\frac{2\pi}{\lambda}(vt - r_2)\cos n\pi - \sin\frac{2\pi}{\lambda}(vt - r_2)\cos(n+1)\pi\right] \\ & + \frac{\lambda}{2\pi}\left[\left(r_2 - r_1 + \frac{n\lambda}{2}\right)\cos\frac{2\pi}{\lambda}(vt - r_2)\cos n\pi\right. \\ & \left. - \left(r_2 - r_1 + \frac{n+1}{2}\lambda\right)\cos\frac{2\pi}{\lambda}(vt - r_2)\cos(n+1)\pi\right] \\ = {} & (-1)^n\frac{\lambda^2}{2\pi^2}\sin\frac{2\pi}{\lambda}(vt - r_2) \\ & + (-1)^n\frac{\lambda}{2\pi}\left[2(r_2 - r_1) + \frac{2n+1}{2}\lambda\right]\cos\frac{2\pi}{\lambda}(vt - r_2) \end{aligned}\right\}. \quad (28\text{B})$$

Determination of Total Effect or Electromagnetic Vector.—The total effect or electromagnetic vector S produced at the point Q by the elementary waves emitted from the whole effective wave-front APC (cf. Fig. 17) is now given by the expression

$$S = A(S'_{1_0} + S'_{1_1} + S'_{1_2} + \dots S'_{1_m}) + B(S'_{2_0} + S'_{2_1} + S'_{2_2} + \dots S'_{2_m}) \quad (29)$$

(cf. formulae (24A)).

On replacing the S'_1's of the first series of this expression by their values (27A), we find, since the first term of the given expression for any S'_{1_n} and the second or last term of the corresponding expression for the preceding S'_1, $S'_{1_{n-1}}$, evidently cancel,

$$\left.\begin{aligned} & S'_{1_0} + S'_{1_1} + S'_{1_2} + \dots S'_{1_m} \\ & = \frac{\lambda}{2\pi(r_2 - r_1)}\cos\left[\frac{\lambda}{2\pi(r_2 - r_1)} + \frac{2\pi}{\lambda}(vt - r_2)\right] - \frac{\lambda}{2\pi\left(r_2 - r_1 + \frac{m+\epsilon}{2}\lambda\right)} \\ & \quad \times \cos\left[\frac{\lambda}{2\pi\left(r_2 - r_1 + \frac{m+\epsilon}{2}\lambda\right)} + \frac{2\pi}{\lambda}\left(vt - r_2 - \frac{m+\epsilon}{2}\lambda\right)\right] \end{aligned}\right\}, \quad \dots(30)$$

where $\epsilon < 1$ (cf. below).

Similarly, the first and third terms of the expression (28A) for any S'_{2_n} and the second and fourth (last) terms of the corresponding expression for the preceding S'_{2}, $S'_{2_{n-1}}$, will evidently cancel, so that the second series in the expression (29) for S will reduce to

$$\left.\begin{aligned} & S'_{2_0} + S'_{2_1} + S'_{2_2} + \dots S'_{2_m} \\ & \quad = \frac{\lambda^2}{4\pi^2} \sin \frac{2\pi}{\lambda}(vt - r_2) + \frac{\lambda}{2\pi}(r_2 - r_1) \cos \frac{2\pi}{\lambda}(vt - r_2) \\ & \quad - \frac{\lambda^2}{4\pi^2} \sin \frac{2\pi}{\lambda}\left(vt - r_2 - \frac{m+\epsilon}{2}\lambda\right) - \frac{\lambda}{2\pi}\left(r_2 - r_1 + \frac{m+\epsilon}{2}\lambda\right) \\ & \quad \times \cos \frac{2\pi}{\lambda}\left(vt - r_2 - \frac{m+\epsilon}{2}\lambda\right) \end{aligned}\right\}, \quad \dots(31)$$

where $\epsilon < 1$.

The given wave-front between $\phi = 0°$ and $\phi = 90°$ could, in general, be divided up into only m *whole* zones or half period elements; that is, the $m+1$ zone would be not a whole zone but only part of one, that namely extending from the circle on the given wave-front, whose distance from the point Q is

$$\rho_m = r_2 - r_1 + \frac{m}{2}\lambda,$$

to that circle, for which the vectors ρ from Q are tangent to that wave-front, that is, for which $\phi = 90°$. The distance ρ of any point on the latter circle from Q is evidently

$$\rho = r_2 - r_1 + \frac{m+\epsilon}{2}\lambda = \sqrt{r_2^2 - r_1^2}, \quad \dots\dots\dots\dots\dots\dots\dots(32)$$

by which the quantity ϵ, which is smaller than unity, is determined as a function of the given quantities r_2, r_1, m, and λ. The particular case, where the given wave-front could be divided up into exactly $m+1$ *whole* zones, would evidently be characterized by the following conditional relation between r_2, r_1, m, and λ:

$$r_2 - r_1 + \frac{m+1}{2}\lambda = \sqrt{r_2^2 - r_1^2} \dots\dots\dots\dots\dots\dots\dots(32\text{A})$$

Replace $\frac{m+\epsilon}{2}\lambda$ by its value from formula (32) in the expressions (30) and (31) for the two series in formula (29), and we find, by the latter and formulae (25), the following expression for S:

$$\begin{aligned} S = & \frac{\pi a (r_2^2 - r_1^2)}{r_2 r_1}\left\{\frac{\lambda}{2\pi(r_2 - r_1)} \cos\left[\frac{\lambda}{2\pi(r_2 - r_1)} + \frac{2\pi}{\lambda}(vt - r_2)\right]\right. \\ & \left. - \frac{\lambda}{2\pi\sqrt{r_2^2 - r_1^2}} \cos\left[\frac{\lambda}{2\pi\sqrt{r_2^2 - r_1^2}} + \frac{2\pi}{\lambda}\left(vt - r_1 - \sqrt{r_2^2 - r_1^2}\right)\right]\right\} \\ & - \frac{\pi a}{r_2 r_1}\left\{\frac{\lambda^2}{4\pi^2} \sin \frac{2\pi}{\lambda}(vt - r_2) + \frac{\lambda}{2\pi}(r_2 - r_1) \cos \frac{2\pi}{\lambda}(vt - r_2)\right. \\ & \left. - \frac{\lambda^2}{4\pi^2} \sin \frac{2\pi}{\lambda}\left(vt - r_1 - \sqrt{r_2^2 - r_1^2}\right) - \frac{\lambda}{2\pi}\sqrt{r_2^2 - r_1^2} \cos \frac{2\pi}{\lambda}\left(vt - r_1 - \sqrt{r_2^2 - r_1^2}\right)\right\}, \end{aligned}$$

which we write in the two terms

$$\left.\begin{aligned} S=\frac{a\lambda}{2r_2r_1}\Big\{&(r_2+r_1)\cos\left[\frac{\lambda}{2\pi(r_2-r_1)}+\frac{2\pi}{\lambda}(vt-r_2)\right]\\ &-(r_2-r_1)\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{\lambda}{2\pi}\sin\frac{2\pi}{\lambda}(vt-r_2)\Big\}\\ -\frac{a\lambda}{2r_2r_1}\Big\{&\sqrt{r_2^2-r_1^2}\cos\left[\frac{\lambda}{2\pi\sqrt{r_2^2-r_1^2}}+\frac{2\pi}{\lambda}\left(vt-r_1-\sqrt{r_2^2-r_1^2}\right)\right]\\ &-\sqrt{r_2^2-r_1^2}\cos\frac{2\pi}{\lambda}\left(vt-r_1-\sqrt{r_2^2-r_1^2}\right)\\ &-\frac{\lambda}{2\pi}\sin\frac{2\pi}{\lambda}\left(vt-r_1-\sqrt{r_2^2-r_1^2}\right)\Big\} \end{aligned}\right\}\ldots.(33)$$

Expression for Electromagnetic Vector; reduces to Light-Vector for very small values of λ.—The first term of the expression (33) represents the residual electromagnetic vector produced at the point Q by the elementary waves emitted from the central zone, whose action at that point has not been annulled by the action of the waves emitted from the next (first) zone, and the second or last term the residual electromagnetic vector produced by the elementary waves emitted from the $m+1$st or last zone, whose action has not been annulled by the action of the waves emitted from the mth zone; the other elementary waves, from the first to the mth zone, produce no effect at the point Q. It is now easy to show that for light-waves the expression (33) reduces to

$$S=\frac{a\lambda}{r_2}\cos\frac{2\pi}{\lambda}(vt-r_2)$$

(cf. Ex. 6 at end of chapter), which is the expression for the light-vector of light-waves (cf. formula (19)). The expression (33) for waves of long wave-length thus differs from that for light-waves in the appearance of terms of the second and higher orders of magnitude in λ.

Approximate Expression for Electromagnetic Vector—λ small.—Let us examine the expression (33) for S, when λ is small but not very (infinitely) small in comparison to the distances r_2 and r_1, that is, let us retain terms of only the first, second and third orders of magnitude in λ in the given expression for S. On developing the cosines of the first terms in the two pairs of largest brackets in expression (33) as functions, sines and cosines, of the single angles, and replacing the sines and cosines of the angles

$$\frac{\lambda}{2\pi(r_2-r_1)}\text{ and }\frac{\lambda}{2\pi\sqrt{r_2^2-r_1^2}}$$

by the trigonometric series for the same, we can write the expression sought

$$
\begin{aligned}
S = {} & \frac{a\lambda}{2r_2r_1}\Bigg\{(r_2+r_1)\Bigg[\left(1-\frac{\lambda^2}{2!4\pi^2(r_2-r_1)^2}\right)\cos\frac{2\pi}{\lambda}(vt-r_2) \\
& -\frac{\lambda}{2\pi(r_2-r_1)}\sin\frac{2\pi}{\lambda}(vt-r_2)\Bigg]-(r_2-r_1)\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{\lambda}{2\pi}\sin\frac{2\pi}{\lambda}(vt-r_2)\Bigg\} \\
& -\frac{a\lambda}{2r_2r_1}\Bigg\{\sqrt{r_2^2-r_1^2}\Bigg[\left(1-\frac{\lambda^2}{2!4\pi^2(r_2^2-r_1^2)}\right)\cos\frac{2\pi}{\lambda}(vt-r_1-\sqrt{r_2^2-r_1^2}) \\
& -\frac{\lambda}{2\pi\sqrt{r_2^2-r_1^2}}\sin\frac{2\pi}{\lambda}(vt-r_1-\sqrt{r_2^2-r_1^2})\Bigg] \\
& -\sqrt{r_2^2-r_1^2}\cos\frac{2\pi}{\lambda}(vt-r_1-\sqrt{r_2^2-r_1^2})+\frac{\lambda}{2\pi}\sin\frac{2\pi}{\lambda}(vt-r_1-\sqrt{r_2^2-r_1^2})\Bigg\}
\end{aligned}
$$

$$
\left.
\begin{aligned}
= {} & \frac{a\lambda}{r_2}\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^2}{2\pi r_1(r_2-r_1)}\sin\frac{2\pi}{\lambda}(vt-r_2) \\
& -\frac{a\lambda^3}{16\pi^2r_2r_1}\,\frac{r_2+r_1}{(r_2-r_1)^2}\cos\frac{2\pi}{\lambda}(vt-r_2) \\
& +\frac{a\lambda^2}{2\pi r_2r_1}\sin\frac{2\pi}{\lambda}(vt-r_1-\sqrt{r_2^2-r_1^2}) \\
& +\frac{a\lambda^8}{16\pi^2r_2r_1}\sqrt{r_2^2-r_1^2}\cos\frac{2\pi}{\lambda}(vt-r_1-\sqrt{r_2^2-r_1^2})
\end{aligned}
\right\}\ldots\ldots\ldots\ldots(33\text{A})
$$

Examination of Approximate Expression for Electromagnetic Vector; Behaviour of same compared with that of Light-Vector.— The first three terms of the expression (33A) represent the residual electromagnetic vector produced at the point Q by the elementary waves emitted from the central zone and the last two terms the residual vector due to the waves emitted from the $m+1$st or last zone (cf. above). The former is represented by terms of the first, second, and third orders of magnitude in λ and the latter by such of the second and third orders only. The total effect or electromagnetic vector at Q could, therefore, be regarded as produced chiefly by the action of the waves emitted from the central zone, except for (very) large values of λ, that is, the maximum deviations in the paths of propagation of the given waves from the rectilinear would be approximately of the dimensions of the central zone. For values of λ, for which the terms of the second order of magnitude in λ of the given expression (33A) contribute materially to the total electromagnetic vector at Q, that vector would have to be regarded as the sum of two such, that produced by the waves emitted from the central zone, and that due to those emitted from the $m+1$st or last zone; and here the former would be given not alone by the first term of expression (33A), the light-vector proper, but by the sum of two or more terms containing λ in not only the first but also the second (and third) power; at the same time the total effect at Q could not be regarded as produced alone or chiefly by the action of the waves emitted from the central zone, for the waves from the last zone then contribute materially to the total effect; in other words the given propagation would then be said according to the Emission Theory to deviate materially from the rectilinear. If we intercept the given waves at the pole P by a circular screen, the effect at Q will evidently diminish only very little, as the size of the screen is increased, that is, figuratively speaking, there will be a marked bending of the waves round the edges of obstacles placed in their course. It is also easy to show (cf. Ex. 7) that slight deviations in the contour of the intercepting screen from the exact circle with centre at pole will have little effect at the point Q; in this respect the light-waves and the electromagnetic waves of long wave-length will differ materially from each other (cf. p. 155). It is also possible to show (cf. Ex. 8) that the electromagnetic vector, like the light-vector, will pass through maxima and minima, as the circular aperture in the large intercepting screen is increased in size; but these maxima and minima of intensity will not be so pronounced as those produced by light-waves on account of the appearance of terms of the second and higher orders of magnitude in λ—these vanish for light waves—which will tend to diminish the maxima and to increase the minima (cf. Ex. 8 at end of chapter). For reasons similar to those stated in Ex. 7 at end of chapter, it is evident that slight deviations in the contour of the aperture from the exact circle with centre at pole will have little (infinitesimal) effect on the value assumed by the given electromagnetic vector.

Shortcomings of Huygens's Principle as postulated by Fresnel; Necessary Modifications. Another Method.—Although the (approximately) rectilinear propagation of light (electromagnetic) waves through homogeneous media, the apparent bending of the same around the edges of obstacles placed in their course, and their behaviour, as confirmed by experiment, where they are intercepted by small screens or pass through small apertures in large opaque screens, can be deduced from Huygens's principle as modified by Fresnel, there are several serious shortcomings embodied in the same. The light-vector S at an external point (Q) is evidently

$$S=\frac{a}{r_2}\sin\frac{2\pi}{\lambda}(vt-r_2) \qquad\text{.............................(34)}$$

and not

$$S=\frac{a\lambda}{r_2}\cos\frac{2\pi}{\lambda}(vt-r_2),$$

as found above (cf. formula (19)); that is, Fresnel's method gives not only a false phase for the light-vector at Q, one that differs from the actual phase by quarter of a wave length, but also a false amplitude, one that is λ times the actual amplitude. We observe that the expression (33A) found for the electromagnetic vector at Q differs from the given one (34) still more than that (19) which has been deduced for the light-vector does. How are now these incorrect expressions for the vectors to be accounted for? Let us examine the above development, as postulated by Fresnel, in detail. The light-vector s' at the point Q due to the action of the elementary waves emitted from any source M on the wave-front $ABCP$ (cf. Fig. 17) was, by formula (6),

$$s' = \frac{a}{r_1}\frac{f(\phi)}{\rho}\sin\frac{2\pi}{\lambda}[vt - (r_1 + \rho)],$$

and the light-vector dS' at Q due to the action of all the elementary waves emitted from any zonal element was then *assumed* to be s' times the area do of that element. This assumption is now evidently not justified. The elementary waves emitted from any such zonal element may be assumed to have one and the same phase, but not one and the same direction or plane of oscillation; take, for example, the primary waves treated in Chapters II. and III. or any system of primary waves, that is, waves whose oscillations are taking place at right angles to their directions of propagation, whereby their directions or paths of oscillation (in planes of oscillation) may change thousands of times per second (cf. p. 72); two such waves emitted from different parts of any zonal element will now have different planes of oscillation at the point Q, so that the resultant effect at Q would not be given by the sum of the single effects or light-vectors, without any reference to the nature of the same, as assumed above, but it would have to be obtained from the superposition of the single effects, according to the doctrine of interference. The effect at Q due to the action of all the elementary waves emitted from any zonal element would, therefore, be not $s'do$, as assumed by Fresnel, but that obtained from the superposition of all the single effects at Q, according to the doctrine of interference, and the latter would evidently be less than $s'do$, since many of the different component moments both in the plane at right angles to the line OQ and along that line would neutralize one another or interfere destructively.

Again, on p. 149, it is taken for granted that the only effect that could be produced at the point Q due to the joint action of the elementary waves emitted from two consecutive zones would be that due to the variation in the mean obliquity between the waves emitted from the one zone and those emitted from the other; this assumption is also incorrect, for there will evidently be a certain effect produced at the point Q by the variation in the mean angle, which the planes of oscillation of the waves from the one zone and those from the other make with the line OQ. An attempt to calculate the total light (electromagnetic) vector at Q on the introduction of this and the foregoing modifications would evidently prove fruitless, for, in the first place, we really know nothing about the behaviour (direction or path of oscillation, etc.) of the light (electromagnetic) vector along the surface of the given wave-front—we could only treat given particular cases as the problems of Chapters II. and III.—and secondly, if we did, the actual calculations would present unsurmountable difficulties. For this reason we shall abandon the above method of treatment of Huygens's principle and seek to confirm the same from an entirely different standpoint, where no knowledge of the behaviour of the light (electromagnetic) vector throughout the region in question will be required, except that it be a particular integral of the general equation of wave-motion; this will enable us to treat not only light-waves and electromagnetic waves, whose directions of oscillation are always at right angles to their directions of propagation, but apparently electromagnetic waves, as the secondary, whose planes of oscillation are not at right angles to their directions of propagation; the rigorous treatment of the latter according to Fresnel's method would evidently present even more serious difficulties than those encountered in the treatment of the former.

Rigorous Proof of Huygens's Principle; Derivation of Formula for Function at any Point in Terms of Surface and Volume-Integrals of same throughout any Closed Region.—In the rigorous proof of Huygens's principle we shall start from a formula between the volume-integral of a given function of a certain function U and r and the surface-integral of another function of the same (see below). Let us first derive the formula in question. Let the function U contain x, y, z and also r explicitly, where r denotes the distance of the point in question from the origin of the coordinates x, y, z, that is, $r^2=x^2+y^2+z^2$; further, let $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}$ or $\frac{\partial U}{\partial r}$ denote the change in U due alone to a change in the variable x, y, z or r respectively, whereby the three other variables contained in U shall remain constant; and, lastly, let $\frac{dU}{dx}, \frac{dU}{dy}$ or $\frac{dU}{dz}$ denote the change in U due to the increment dx, dy, or dz along the x, y or z-axis respectively, whereby r will evidently undergo a change. We have then

$$\frac{dU}{dx}=\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial r}{\partial x}=\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\frac{x}{r}=\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\cos(r, x),$$

or, if we write here $\frac{1}{r}\frac{\partial U}{\partial x}$ in place of U,

$$\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial r}\right)=\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial U}{\partial r}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)\cos(r,x),$$

or, since r is to be regarded as constant by the differentiation $\frac{\partial}{\partial x}$,

$$\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)=\frac{1}{r}\frac{\partial^2 U}{\partial x^2}-\frac{1}{r^2}\frac{\partial U}{\partial x}\cos(r,x)+\frac{1}{r}\frac{\partial^2 U}{\partial r\,\partial x}\cos(r,x)$$

and similarly

$$\frac{d}{dy}\left(\frac{1}{r}\frac{\partial U}{\partial y}\right)=\frac{1}{r}\frac{\partial^2 U}{\partial y^2}-\frac{1}{r^2}\frac{\partial U}{\partial y}\cos(r,y)+\frac{1}{r}\frac{\partial^2 U}{\partial r\,\partial y}\cos(r,y)\quad\ldots\ldots(35)$$

and

$$\frac{d}{dz}\left(\frac{1}{r}\frac{\partial U}{\partial z}\right)=\frac{1}{r}\frac{\partial^2 U}{\partial z^2}-\frac{1}{r^2}\frac{\partial U}{\partial z}\cos(r,z)+\frac{1}{r}\frac{\partial^2 U}{\partial r\,\partial z}\cos(r,z)$$

Let now $\frac{dU}{dr}$ denote the change in U due to the increment dr along the vector r; the total change in U due to this increment will evidently consist of four (partial) changes, that in U due alone to the change in r and those due alone to the changes in x, y and z singly; that is, the total change in U will evidently be given by the expression

$$\frac{dU}{dr}=\frac{\partial U}{\partial r}+\frac{\partial U}{\partial x}\cos(r,x)+\frac{\partial U}{\partial y}\cos(r,y)+\frac{\partial U}{\partial z}\cos(r,z);\quad\ldots\ldots(36)$$

if we write here $\frac{\partial U}{\partial r}$ in place of U, we have

$$\frac{d}{dr}\left(\frac{\partial U}{\partial r}\right)=\frac{\partial^2 U}{\partial r^2}+\frac{\partial^2 U}{\partial r\,\partial x}\cos(r,x)+\frac{\partial^2 U}{\partial r\,\partial y}\cos(r,y)+\frac{\partial^2 U}{\partial r\,\partial z}\cos(r,z).\quad(37)$$

Add the three equations (35), and we have

$$\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)+\frac{d}{dy}\left(\frac{1}{r}\frac{\partial U}{\partial y}\right)+\frac{d}{dz}\left(\frac{1}{r}\frac{\partial U}{dz}\right)=\frac{1}{r}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)$$
$$-\frac{1}{r^2}\left[\frac{\partial U}{\partial x}\cos(r,x)+\frac{\partial U}{\partial y}\cos(r,y)+\frac{\partial U}{\partial z}\cos(r,z)\right]$$
$$+\frac{1}{r}\left[\frac{\partial^2 U}{\partial r\,\partial x}\cos(r,x)+\frac{\partial^2 U}{\partial r\,\partial y}\cos(r,y)+\frac{\partial^2 U}{\partial r\,\partial z}\cos(r,z)\right],$$

which, by formulae (36) and (37), can be written

$$\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)+\frac{d}{dy}\left(\frac{1}{r}\frac{\partial U}{\partial y}\right)+\frac{d}{dz}\left(\frac{1}{r}\frac{\partial U}{\partial z}\right)=\frac{1}{r}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)$$
$$-\frac{1}{r^2}\left(\frac{dU}{dr}-\frac{\partial U}{\partial r}\right)+\frac{1}{r}\left[\frac{d}{dr}\left(\frac{\partial U}{\partial r}\right)-\frac{\partial^2 U}{\partial r^2}\right],$$

or. since $\frac{1}{r}\frac{d}{dr}\left(\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial U}{\partial r}=\frac{1}{r^2}\frac{d}{dr}\left(r\frac{\partial U}{\partial r}\right),$

$$=\frac{1}{r}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}-\frac{\partial^2 U}{\partial r^2}\right)+\frac{1}{r^2}\frac{d}{dr}\left(r\frac{\partial U}{\partial r}-U\right).$$

Multiply both sides of this equation by $dxdydz$ and integrate the same through the region bounded by any closed surface S, and we have

$$\left.\begin{aligned}&\iiint\left[\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)+\frac{d}{dy}\left(\frac{1}{r}\frac{\partial U}{\partial y}\right)+\frac{d}{dz}\left(\frac{1}{r}\frac{\partial U}{dz}\right)\right]dxdydz\\&\quad=\iiint\frac{1}{r^2}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}-\frac{\partial^2 U}{\partial r^2}\right)dxdydz\\&\qquad+\iiint\frac{1}{r^2}\frac{d}{dr}\left(r\frac{\partial U}{\partial r}-U\right)dxdydz\end{aligned}\right\}\text{. ...(38)}$$

Let us now assume that the functions $\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)$, $\left(\frac{1}{r}\frac{\partial U}{\partial y}\right)$ and $\left(\frac{1}{r}\frac{\partial U}{\partial z}\right)$ and their derivatives with regard to x, y, z be single-valued, finite and continuous throughout the region bounded by the surface S. We can then integrate the different terms of the first integral of formula (38) by parts, the first with regard to x, the second to y and the third to z, and we have

$$\iiint\frac{d^*}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)dxdydz=\iint dydz\left(\frac{1}{r}\frac{\partial U}{\partial x}\right),$$

the limits of integration to be taken at those values a_1, a_2, $a_3\ldots$ of x on the surface S, where the cylindrical element $dydz$ (as base) parallel to the x-axis enters and leaves the given region (cf. the annexed figure); we thus have

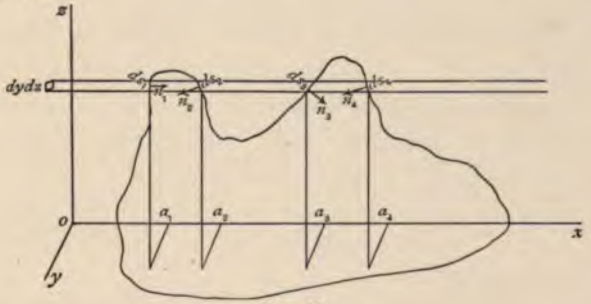

FIG. 18.

$$\iiint\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)dxdydz$$
$$=\iint dydz\left[-\left(\frac{1}{r}\frac{dU}{\partial x}\right)_{x=a_1}+\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)_{x=a_2}-\left(\frac{1}{r}\frac{\partial U}{\partial x}\right)_{x=a_3}+-\ldots\right].$$

---

*This $\frac{d}{dx}$ is evidently the *partial* differential.

If ds denotes the area of the surface S intercepted by the cylindrical element $dydz$ at the different points a_1, a_2, a_3, ..., then

$$dxdy = ds_1 \cos(n_1, x) = -ds_2 \cos(n_2, x) = ds_3 \cos(n_3, x) \ldots$$

(cf. Fig. 18); and the given integral can be written

$$\iiint \frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right) dxdydz = -\int \frac{1}{r}\frac{\partial U}{\partial x}\cos(n, x)ds.$$

Similarly, the other two volume-integrals of the left-hand member of formula (38) can be replaced by surface-integrals, namely,

$$\iiint \frac{d}{dy}\left(\frac{1}{r}\frac{\partial U}{\partial y}\right) dxdydz = -\int \frac{1}{r}\frac{\partial U}{\partial y}\cos(n, y)ds$$

and

$$\iiint \frac{d}{dz}\left(\frac{1}{r}\frac{\partial U}{\partial z}\right) dxdydz = -\int \frac{1}{r}\frac{\partial U}{\partial z}\cos(n, z)ds.$$

Formula (38) can thus be written

$$\left.\begin{aligned} -\int \frac{1}{r}\left[\frac{\partial U}{\partial x}\cos(n, x) + \frac{\partial U}{\partial y}\cos(n, y) + \frac{\partial U}{\partial z}\cos(n, z)\right] ds \\ = \iiint \frac{1}{r}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{\partial^2 U}{\partial r^2}\right) dxdydz \\ + \iiint \frac{1}{r^2}\frac{d}{dr}\left(r\frac{\partial U}{\partial r} - U\right) dxdydz \end{aligned}\right\} \ldots\ldots(39)$$

or, since

$$\frac{\partial U}{\partial x}\cos(n, x) + \frac{\partial U}{\partial y}\cos(n, y) + \frac{\partial U}{\partial z}\cos(n, z) = \frac{\partial U}{\partial n},$$

where $\frac{\partial U}{\partial n}$ denotes the change in U produced alone by the increments in the variables x, y, z as we advance the distance dn along the inner normal to the surface S—that is, r is to be regarded here as constant—

$$\left.\begin{aligned} -\int \frac{1}{r}\frac{\partial U}{\partial n} ds = \iiint \frac{1}{r}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{\partial^2 U}{\partial r^2}\right) dxdydz \\ + \iiint \frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{r}\frac{\partial U}{\partial r} - U\right) dxdydz, \end{aligned}\right\} \ldots\ldots(40)$$

the formula sought (cf. above).

In formula (40) we must now discriminate between the two cases:

I. The region of integration, enclosed by the surface S, contains the origin of our system of coordinates x, y, z; in which case the given formulae would not hold in their above form, since $1/r$ then becomes infinite at the origin; and

II. The origin of our system of coordinates lies outside the region of integration.

Case I.: The Point Q lies inside the Surface S.—In order to be able to apply formula (40) to this case, we must evidently exclude the origin of our system of coordinates from the region of integration; for this purpose we describe a sphere of very small radius ρ with centre at origin around the same. The region of integration will then be bounded by the given outer surface S and the surface of the small sphere, as inner surface. The value assumed by the first or surface-integral of formula (40) on the surface of the given sphere will now be very (infinitely) small, since the area of that surface is proportional to ρ^2, whereas $1/\rho$ alone appears in the expression to be integrated; the value assumed by this integral on the surface of the sphere may, therefore, be neglected compared with that assumed by the same on the surface S. Similarly, since the volume of the given sphere is proportional to ρ^3 and the expression under the integration-signs of the first volume-integral of formula (40) contains ρ in the first power only in the denominator, the value assumed by the given integral throughout that sphere will be very (infinitely) small compared with that assumed by the same throughout the given region; we could, therefore, extend the given integration throughout the whole region bounded by the surface S instead of throughout the given region or that bounded by S, as outer surface, and the surface of the given sphere, as inner surface, without effecting the value sought.

Lastly, let us examine the last integral of formula (40); for this purpose we imagine the given region divided up into elements formed by the intersections of cones with apices of solid angular aperture $d\phi$ at the origin and of spheres with common centre at the origin and whose radii differ from one another by dr. The volume of any such element will evidently be

$$r^2 d\phi dr.$$

Replace the rectangular volume-element $dxdydz$ by this new one in the given integral, and we have

$$\left.\begin{aligned}\iiint \frac{1}{r^2}\frac{d}{dr}\left(r\frac{\partial U}{\partial r}-U\right)dxdydz &= \int d\phi \int_\rho^r dr \frac{d}{dr}\left(r\frac{\partial U}{\partial r}-U\right)\\ &= \int d\phi \left| r\frac{\partial U}{\partial r}-U\right|_\rho^r\end{aligned}\right\} \quad \ldots(41)$$

At the lower limit $r\frac{\partial U}{\partial r}$ becomes infinitely small as ρ approaches zero and can thus be rejected, whereas, as we know from the theory of the potential,

$$\int d\phi\, U_{\rho=0} = 4\pi U_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(42)$$

where U_0 denotes the value of U at the centre of the given sphere or origin.

At the upper limit, the surface S, the following relation will evidently hold between the surface-element $r^2 d\phi$ and the surface-element ds of the surface S, intercepted by the cone $d\phi$:

$$r^2 d\phi = -ds \cos(n, r),$$

where n denotes the inner normal to the surface.

At the upper limit the given integral can thus be written

$$\left.\begin{aligned}\int d\phi \left| r\frac{\partial U}{\partial r} - U \right|^r &= -\int ds \cos(n, r) \left| \frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2} \right|^r \\ &= -\int ds \cos(n, r) \frac{\partial}{\partial r}\left(\frac{U}{r}\right)\end{aligned}\right\}, \quad \text{.........(43)}$$

where the integration is to be extended over the whole surface S.

In the given case formula (40) will thus assume the form

$$\begin{aligned}-\int \frac{1}{r}\frac{\partial U}{\partial n} ds &= \iiint \frac{1}{r}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{\partial^2 U}{\partial r^2}\right) dxdydz \\ &\quad - \int \cos(n, r)\frac{\partial}{\partial r}\left(\frac{U}{r}\right) ds + 4\pi U_0,\end{aligned}$$

or

$$\left.\begin{aligned}&\int \left[\cos(n, r)\frac{\partial}{\partial r}\left(\frac{U}{r}\right) - \frac{1}{r}\frac{\partial U}{\partial n}\right] ds \\ &= \iiint \frac{1}{r}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{\partial^2 U}{\partial r^2}\right) dxdydz + 4\pi U_0\end{aligned}\right\}, \quad \text{......(44)}$$

where the surface-integration is to be extended over the whole surface S only and the volume-integration over the whole region enclosed by that surface.

Case II. The Point Q lies outside the Surface S.—This case differs from the foregoing in that the origin of our system of coordinates lies outside the region of integration, so that the considerations pertaining to the sphere employed in the latter will not have to be introduced here. In the given case the third or last integral of formula (40) can now be brought into another form; for this purpose we first replace there the volume element $dxdydz$ by $r^2 d\phi dr$ introduced above, and we have

$$\left.\begin{aligned}\iiint \frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{r}\frac{\partial U}{\partial r} - U\right) dxdydz &= \int d\phi \int_{r_e}^{r_i} dr \frac{d}{dr}\left(r\frac{\partial U}{\partial r} - U\right) \\ &= \int d\phi \left| r\frac{\partial U}{\partial r} - U \right|_{r_e}^{r_i}\end{aligned}\right\}, \quad \text{...(45)}$$

where r_e denotes the distance from the origin of the point on the surface S, at which the cone of solid angular aperture $d\phi$ with apex at origin enters that surface, and r_i the distance from origin of the point on given surface, at which that cone leaves the same.

We next replace the surface-element $r^2 d\phi$ by its value in terms of the surface-element ds intercepted on the surface S by the given cone; at the point, where this cone enters the surface S, we evidently have

$$r_e^2 d\phi = +ds_e \cos(n_e, r),$$

and at that, where it leaves the same,

$$r_i^2 d\phi = -ds_i \cos(n_i, r),$$

where n denotes the inner normal to surface.

We can then write the integral (45)

$$\begin{aligned}&\iiint \frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{r}\frac{\partial U}{\partial r} - U\right) dx\,dy\,dz \\ &\quad = \int ds_i \cos(n_i, r)\left[\frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2}\right]_{r_i} - \int ds_e \cos(n_e, r)\left[\frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2}\right]_{r_e} \\ &\quad = -\int ds_i \cos(n_i, r)\left[\frac{\partial}{\partial r}\left(\frac{U}{r}\right)\right]_{r_i} - \int ds_e \cos(n_e, r)\left[\frac{\partial}{\partial r}\left(\frac{U}{r}\right)\right]_{r_e};\end{aligned}$$

these last two integrals are evidently equivalent to the single integral

$$-\int ds \cos(n, r)\frac{\partial}{\partial r}\left(\frac{U}{r}\right),$$

where the integration is to be extended over the whole surface S.

In case II. formula (40) will thus assume the form

$$\left.\begin{aligned}&\int\left[\cos(n, r)\frac{\partial}{\partial r}\left(\frac{U}{r}\right) - \frac{1}{r}\frac{\partial U}{\partial n}\right] ds \\ &\qquad = \iiint \frac{1}{r}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{\partial^2 U}{\partial r^2}\right) dx\,dy\,dz.\end{aligned}\right\} \quad \text{......(46)}$$

Application of Formula (44) to Huygens's Principle.—We can now employ formula (44) for an examination of Huygens's principle as follows: Let the function U be the light-vector s of argument $(t - r/v)$, where r denotes the distance of any point of the region enclosed by the surface S from the point of observation Q, which shall lie within that region and which we shall choose as origin of our system of coordinates; that is, we put

$$U = s(t - r/v), \quad \text{..............................(47)}$$

At the point Q, U will then assume the value

$$U_0 = s_0(t). \quad \text{................ ....(47A)}$$

Since now any light-vector s is thereby defined, that it is an integral of the partial differential equation

$$\frac{\partial^2 s}{\partial t^2} = v^2 \nabla^2 s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48)$$

(cf. formulae (16 and 27, I.)), it follows that the light-vector U will also be a particular integral of the same equation or

$$\frac{\partial^2 U}{\partial t^2} = v^2 \nabla^2 U. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48\text{A})$$

The Light-Vector s a Purely Spherical Wave-Function.—Let us now examine the case, where the light-vector s is a purely spherical wave-function, that is, a function of r (and t) alone and not of x, y, z singly. The general differential equation (48) will then assume the simple form

$$\frac{\partial^2 s}{dt^2} = v^2 \frac{\partial^2 s}{\partial r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(49)$$

(cf. pp. 17 and 18). Since now U is also a function of r (and t) only, it will likewise be defined by the same equation or

$$\frac{\partial^2 U}{\partial t^2} = v^2 \frac{\partial^2 U}{\partial r^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(49\text{A})$$

By formulae (48A) and (49A) the following relation will, therefore, hold here:

$$\frac{\partial^2 U}{\partial t^2} = v^2 \nabla^2 U = v^2 \frac{\partial^2 U}{\partial r^2},$$

hence

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial r^2}. \quad \ldots\ldots\ldots\ldots\ldots (50)$$

Replace now U by its value (47), etc., in formula (44), and we have, since by (50) the volume-integral of the same vanishes,

$$\iint \left[\cos(n, r) \frac{\partial}{\partial r} \frac{s(t - r/v)}{r} - \frac{1}{r} \frac{\partial}{\partial n} s(t - r/v) \right] ds = 4\pi s_0(t). \quad \ldots\ldots(51)$$

We can now interpret this formula as follows: The light-vector s_0 at any point Q, which we choose as origin of our system of coordinates, and at any time t can be conceived as produced by elementary disturbances emitted at the times $(t - r/v)$ by any surface S enclosing that point, where r denotes the distance of any point or elementary source M on that surface from the point Q and v the common velocity of propagation, with which those disturbances are approaching that point. After the elapse of the times r/v we shall have the same phase along the surface S, as we had at Q, when the given disturbances left that surface. We observe that these disturbances are of a much more complicated nature than those emitted by Fresnel's elementary sources, the latter having been assumed to be proportional merely to the light-vector s (cf. p. 147).

$\mathbf{s}=\frac{\mathbf{a}}{\mathbf{r}}\sin\frac{2\pi}{\lambda}(\mathbf{vt}-\mathbf{r})$.—By formula (51) we can evidently determine the light-vector $s_0(t)$ at any point Q, provided we know the light-vector s and $\frac{\partial s}{\partial n}$ along any closed surface S enclosing that point. Let us assume that the light-vector $s(t)$ at any point of the region enclosed by the surface S be given by the expression

$$s(t)=\frac{a}{\rho}\sin\frac{2\pi}{\lambda}(vt-\rho), \qquad (52)$$

where ρ denotes the distance of that point from the source of disturbance O—$s(t)$ is here a purely spherical wave-function, as assumed above. The function U will then assume the form

$$U=s(t-r/v)=\frac{a}{\rho}\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]. \qquad (53)$$

This function U must now remain finite throughout the region of integration (cf. p. 167); this condition will be satisfied, if the source of disturbance O lies outside the region of integration, enclosed by the surface S. The relative positions of the points O, M, and Q to the surface S could then be represented as in the annexed figure.

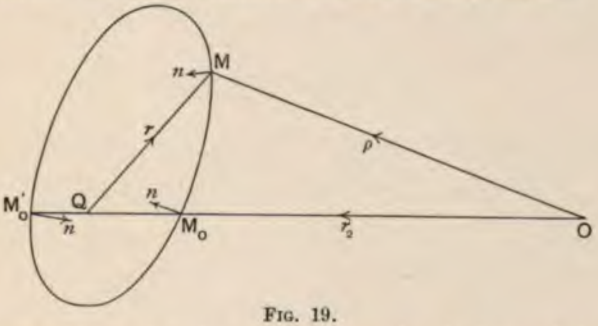

FIG. 19.

Let us now determine the expressions under the integral-sign of the integral (51) for the given case; we have, by formula (53),

$$\frac{\partial}{\partial r}\left[\frac{s(t-r/v)}{r}\right]=\frac{\partial}{\partial r}\left\{\frac{a}{\rho r}\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]\right\}$$

$$=-\frac{2\pi a}{\lambda\rho r}\cos\frac{2\pi}{\lambda}[vt-(\rho+r)]-\frac{a}{\rho r^2}\sin\frac{2\pi}{\lambda}[vt-(\rho+r)],$$

and $\frac{\partial}{\partial n}[s(t-r/v)]=\frac{\partial}{\partial n}\left\{\frac{a}{\rho}\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]\right\}$

$$=\frac{\partial}{\partial \rho}\left\{\frac{a}{\rho}\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]\right\}\cos(n,\rho)$$

$$=-\left\{\frac{2\pi a}{\lambda\rho}\cos\frac{2\pi}{\lambda}[vt-(\rho+r)]+\frac{a}{\rho^2}\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]\right\}\cos(n,\rho);$$

we can thus write the integral (51) for $s_0(t)$ here

$$\left.\begin{aligned}&\frac{2\pi a}{\lambda}\int\frac{1}{\rho r}[\cos(n,\rho)-\cos(n,r)]\cos\frac{2\pi}{\lambda}[vt-(\rho+r)]ds\\&\quad+a\int\frac{1}{\rho r}\left[-\frac{\cos(n,\rho)}{\rho}+\frac{\cos(n,r)}{r}\right]\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]ds=4\pi s_0(t)\end{aligned}\right\}. \quad \ldots(54)$$

Approximate Expression for $s_0(t)$ for Light-Waves.—For waves of small wave-length λ, as those of light, the second or last integral of the expression (54) for $s_0(t)$, being of a higher order of magnitude in λ than the first integral, would be very (infinitely) small compared with the latter and could thus be rejected. The light-vector $s_0(t)$ would then be given most approximately by the integral

$$s_0(t)=\frac{a}{2\lambda}\int\frac{1}{\rho r}[\cos(n,\rho)-\cos(n,r)]\cos\frac{2\pi}{\lambda}[vt-(\rho+r)]ds. \quad \ldots(55)$$

Approximate Expression for $s_0(t)$ compared with that obtained by Fresnel's Method.—Let us compare the expression (55) for $s_0(t)$ with that obtained above according to Fresnel's method. For this purpose we compare the two expressions for any surface-element do (ds) of the surface S. The light-vector due to any such element is, according to Fresnel's method,

$$\frac{a}{r_1}\frac{f(\phi)}{\rho}\sin\frac{2\pi}{\lambda}[vt-(r_1+\rho)]do.$$

(cf. formula (7)), and by formula (55)

$$\frac{a}{2\lambda}\frac{1}{\rho r}[\cos(n,\rho)-\cos(n,r)]\cos\frac{2\pi}{\lambda}[vt-(\rho+r)]ds.$$

Since now the r_1 of the former expression is the r of the latter, we must evidently put

$$f(\phi)=\frac{\cos(n,\rho)-\cos(n,r)}{2\lambda}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(56)$$

if we neglect the difference in phase between the two expressions.

For the surface-element at the point M_0, where the line OQ of Fig. 19 enters the surface S—this element corresponds to the element at the pole P of Fresnel's construction (cf. Fig. 17),

$$\cos(n,\rho)=-\cos(n,r)$$

(cf. Fig. 19), and formula (56) will assume the simple form

$$f_0(\phi)=\frac{\cos(n,\rho)}{\lambda}.$$

If the element at the point M_0 stands at right angles to the line OQ, as in Fresnel's construction, then $\cos(n,\rho)=1$ (cf. Fig. 19), and hence

$$f_0(\phi)=\frac{1}{\lambda},$$

the value for $f(\phi)$ obtained from a comparison of the expression for the total light-vector at any point according to Fresnel's method with the actual expression for the same (cf. p. 164).

For the surface-element at the point M_0', where the line OQ leaves the surface S (cf. Fig. 19), $\cos(n, \rho) = \cos(n, r)$, and hence the effect contributed to the total effect at Q by this element zero; that is, by formula (55), no elementary waves will be propagated directly backwards towards the source, from which the given wave is advancing, as postulated by Stokes's law of obliquity (cf. p. 148).

Confirmation of Formula (55).—Lastly, it is evident that formula (55) will give the correct phase for $s_0(t)$, since the integration by parts of the given integral always gives $\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]$ and not $\cos\frac{2\pi}{\lambda}[vt-(\rho+r)]$ as factor in the term of the lowest order of magnitude in λ, which term alone is to be retained for waves of short wave-length, as the light-waves; for a confirmation of this statement see Ex. 9 at end of chapter, where by a suitable choice of the surface S the light-vector $s_0(t)$ will, in fact, be found to be given, as determined by formula (55), by

$$s_0(t) = \frac{a}{r_2}\sin\frac{2\pi}{\lambda}(vt - r_2),$$

the correct expression for the same. Formula (55) thus differs from Fresnel's formula also in that it gives the correct phase for the light-vector (cf. p. 164).

General Expression for $s_0(t)$ for Waves of any Wave-Length; Evaluation and Confirmation of Validity of same for a Sphere S with centre at Q.—Lastly, let us evaluate the integral (54) for $s_0(t)$ for waves of long wave-length, when the surface S is a sphere with centre at point of observation Q. For this purpose we divide the surface of the given sphere up into surface-elements ds similar to the zonal elements do employed on p. 146—the construction is that represented in Fig. 17 with the points O and Q interchanged. The area of any such surface-element ds can now by formula (5) be written in the form

$$ds = 2\pi\frac{r}{r_2}\rho d\rho,$$

where r, the r_1 of Fig. 17, denotes the radius of the given sphere and r_2 the distance of the point of observation Q from the source of disturbance O.

For the given surface the inner normal n—the inner normal is always to be taken (cf. p. 171)—at any point M will evidently coincide with the vector $-r$ (cf. Fig. 19); we thus have

$$\cos(n,\ \rho)=\cos(-r,\ \rho),$$

and

$$\cos(n,\ r)=\cos(-r,\ r)=-1.$$

The angle $(-r,\ \rho)$ is now the angle ϕ employed on p. 147 (cf. also Fig. 17); it can, therefore, be expressed as a function of the distances r, r_2, and ρ (cf. Ex. 1 at end of chapter); we have, namely,

$$\cos\phi=\frac{{r_2}^2-r^2-\rho^2}{2r\rho}=\cos(-r,\ \rho)=\cos(n,\ \rho). \quad\text{............(57)}$$

By this and the above relations that hold for the given surface, the integral (54) for $s_0(t)$ can be written

$$\left.\begin{aligned}4\pi s_0(t)&=\frac{4\pi^2a}{\lambda r_2}\int\left[\frac{{r_2}^2-r^2-\rho^2}{2r\rho}+1\right]\cos\frac{2\pi}{\lambda}[vt-(\rho+r)]\,d\rho\\&\quad+\frac{2\pi a}{r_2}\int\left[\frac{{r_2}^2-r^2-\rho^2}{2r\rho^2}+\frac{1}{r}\right]\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]\,d\rho\\&=\frac{4\pi^2a}{\lambda r_2}\left\{\frac{{r_2}^2-r^2}{2r}\int\frac{\cos\omega\,d\rho}{\rho}-\frac{1}{2r}\int\rho\cos\omega\,d\rho+\int\cos\omega\,d\rho\right\}\\&\quad+\frac{\pi a}{r_2r}\left\{({r_2}^2-r^2)\int\frac{\sin\omega\,d\rho}{\rho^2}+\int\sin\omega\,d\rho\right\}\end{aligned}\right\}, \quad\text{.....(58)}$$

where

$$\omega=\frac{2\pi}{\lambda}[vt-(\rho+r)].$$

By the reduction formulae on p. 157 the first and fourth integrals of this expression for s_0 can now be integrated as follows:

$$\begin{aligned}\int\frac{\cos\omega\,d\rho}{\rho}&=-\frac{\lambda}{2\pi}\,\frac{\sin\omega}{\rho}-\frac{\lambda}{2\pi}\left\{\frac{\lambda}{2\pi}\,\frac{\cos\omega}{\rho^2}+\frac{1}{2}\,\frac{\lambda}{2\pi}\left\{-\frac{\lambda}{2\pi}\,\frac{\sin\omega}{\rho^3}\right.\right.\\&\qquad\left.\left.-\frac{1}{3}\left\{\frac{\lambda}{2\pi}\,\frac{\cos\omega}{\rho^4}+\dots\right\}\right\}\right\}\\&=-\frac{\lambda}{2\pi}\,\frac{\sin\omega}{\rho}-\left(\frac{\lambda}{2\pi}\right)^2\frac{\cos\omega}{\rho}+\frac{1}{2!}\left(\frac{\lambda}{2\pi}\right)^3\frac{\sin\omega}{\rho^3}+\frac{1}{3!}\left(\frac{\lambda}{2\pi}\right)^4\frac{\cos\omega}{\rho}--++\dots\\&=-\left[1-\frac{1}{2!}\left(\frac{\lambda}{2\pi\rho}\right)^2+\frac{1}{4!}\left(\frac{\lambda}{2\pi\rho}\right)^4-+\dots\right]\frac{\lambda}{2\pi}\,\frac{\sin\omega}{\rho}\\&\quad-\left[\frac{\lambda}{2\pi\rho}-\frac{1}{3!}\left(\frac{\lambda}{2\pi\rho}\right)^3+\frac{1}{5!}\left(\frac{\lambda}{2\pi\rho}\right)^5-+\dots\right]\frac{\lambda}{2\pi}\,\frac{\cos\omega}{\rho}\\&=-\frac{\lambda}{2\pi}\left(\cos\frac{\lambda}{2\pi\rho}\,\frac{\sin\omega}{\rho}+\sin\frac{\lambda}{2\pi\rho}\,\frac{\cos\omega}{\rho}\right)\end{aligned}$$

(cf. series for $\sin x$ and $\cos x$ on p. 158); and, similarly,

$$\int\frac{\sin\omega\,d\rho}{\rho^2}=\frac{\lambda}{2\pi}\frac{\cos\omega}{\rho^2}+\frac{1}{2}\frac{\lambda}{2\pi}\left\{-\frac{\lambda}{2\pi}\frac{\sin\omega}{\rho^3}-\frac{1}{3}\frac{\lambda}{2\pi}\left\{\frac{\lambda}{2\pi}\frac{\cos\omega}{\rho^4}\right.\right.$$

$$\left.\left.+\frac{1}{4}\frac{\lambda}{2\pi}\left\{-\frac{\lambda}{2\pi}\frac{\sin\omega}{\rho^5}-\dots\right\}\right\}\right\}$$

$$=\frac{\lambda}{2\pi}\frac{\cos\omega}{\rho^2}-\frac{1}{2!}\left(\frac{\lambda}{2\pi}\right)^2\frac{\sin\omega}{\rho^3}-\frac{1}{3!}\left(\frac{\lambda}{2\pi}\right)^3\frac{\cos\omega}{\rho^4}+\frac{1}{4!}\left(\frac{\lambda}{2\pi}\right)^4\frac{\sin\omega}{\rho^5}+--\dots$$

$$=\left[\frac{\lambda}{2\pi\rho}-\frac{1}{3!}\left(\frac{\lambda}{2\pi\rho}\right)^3+-\dots\right]\frac{\cos\omega}{\rho}$$

$$+\left[1-\frac{1}{2!}\left(\frac{\lambda}{2\pi\rho}\right)^2+\frac{1}{4!}\left(\frac{\lambda}{2\pi\rho}\right)^4-+\dots\right]\frac{\sin\omega}{\rho}-\frac{\sin\omega}{\rho}$$

$$=\sin\frac{\lambda}{2\pi\rho}\frac{\cos\omega}{\rho}+\cos\frac{\lambda}{2\pi\rho}\frac{\sin\omega}{\rho}-\frac{\sin\omega}{\rho}.$$

The values of the other integrals of the expression (58) are evidently

$$\int\rho\cos\omega\,d\rho=-\frac{\lambda}{2\pi}\rho\sin\omega+\left(\frac{\lambda}{2\pi}\right)^2\cos\omega,$$

$$\int\cos\omega\,d\rho=-\frac{\lambda}{2\pi}\sin\omega,\ \text{ and }\ \int\sin\omega\,d\rho=\frac{\lambda}{2\pi}\cos\omega.$$

Replace these integrals by their values in the above expression (58) for $s_0(t)$, and we find

$$4\pi s_0(t)=-\pi a\frac{r_2^2-r^2}{r_2r}\left(\cos\frac{\lambda}{2\pi\rho}\frac{\sin\omega}{\rho}+\sin\frac{\lambda}{2\pi\rho}\frac{\cos\omega}{\rho}\right)$$

$$+\frac{\pi a}{r_2r}\left(\rho\sin\omega-\frac{\lambda}{2\pi}\cos\omega\right)-\frac{2\pi a}{r_2}\sin\omega$$

$$+\pi a\frac{r_2^2-r^2}{r_2r}\left(\sin\frac{\lambda}{2\pi\rho}\frac{\cos\omega}{\rho}+\cos\frac{\lambda}{2\pi\rho}\frac{\sin\omega}{\rho}-\frac{\sin\omega}{\rho}\right)$$

$$+\frac{\pi a}{r_2r}\frac{\lambda}{2\pi}\cos\omega$$

$$=\frac{\pi a}{r_2}\left[\frac{\rho}{r}-2-\frac{r_2^2-r^2}{r\rho}\right]\sin\omega,\ \dots\dots\dots\dots(59)$$

where the integration is to be extended from

$$\rho_1=r_2-r\ \text{ to }\ \rho_2=r_2+r.$$

Replace here ρ by these limits, and we have

$$4s_0(t)=\frac{a}{r_2}\left[\frac{r_2+r}{r}-2-\frac{r_2^2-r^2}{r(r_2+r)}\right]\sin\frac{2\pi}{\lambda}[vt-(r_2+2r)]$$

$$-\frac{a}{r_2}\left[\frac{r_2-r}{r}-2-\frac{r_2^2-r^2}{r(r_2-r)}\right]\sin\frac{2\pi}{\lambda}(vt-r_2),$$

or, since the coefficient of $\sin\frac{2\pi}{\lambda}[vt-(r_2+2r)]$ evidently vanishes,

$$s_0(t)=\frac{a}{r_2}\sin\frac{2\pi}{\lambda}(vt-r_2),$$

the correct expression for the given vector at the point Q. It thus follows that not only the approximate formula (55), but also the exacter one (54), which holds for waves of any wave-length, will give the correct expression for the light (electromagnetic) vector, both amplitude and phase.

The Light or Electromagnetic Vector a Spherical Function and Huygens's Principle; the Primary and Secondary Waves.—Formula (54) evidently holds for both light and electromagnetic waves of any wave-length, but provided only the vector s employed be a *purely* spherical wave-function, that is, a function of r (and t) only and not of x, y, z singly. For the general spherical wave-function the differential equation (49), which defines the purely spherical function, does not now hold, and hence the relation (50) also not; in which case, the volume integral of formula (44) will not vanish; this would evidently complicate the treatment of the given case, since it would then require a knowledge of s and its derivatives not only along the surface S, but also throughout the whole region of integration. On the other hand, the given treatment would also be complicated greatly, since not only the general spherical wave-function itself always consists of two or more terms, neither of which is a particular integral of our general equation of wave-motion (48) (cf. p. 35), but also its value is a function not alone of r, the distance from the source, but also of x, y, z singly. Whether it would be possible to evaluate $s_0(t)$ for such a function is a question that could be decided only by investigation.

EXAMPLES.

1. Confirm the expressions for $F(\rho)$ of formulae (18) and (18A), Stokes's and the natural laws of obliquity.

The annexed figure (cf. also Fig. 17) gives the following relations between the obliquity ϕ, the angles θ and θ', and the distances ρ, r_1, and r_2:

$$\phi=\theta+\theta',$$

$$\rho^2=r_2^2+r_1^2-2r_2r_1\cos\theta,$$

and

$$r_1^2=r_2^2+\rho^2-2r_2\rho\cos\theta',$$

by which equations ϕ can evidently be determined as a function of ρ, r_1, and r_2.

The last two equations give

$$\cos\theta=\frac{r_2^2+r_1^2-\rho^2}{2r_2r_1},\quad\text{hence}\quad\sin\theta=\pm\frac{\sqrt{-(r_2^4+r_1^4+\rho^4)+2(r_2^2r_1^2+r_2^2\rho^2+r_1^2\rho^2)}}{2r_2r_1},$$

and

$$\cos\theta'=\frac{r_2^2-r_1^2+\rho^2}{2r_2\rho},\quad\text{hence}\quad\sin\theta'=\pm\frac{\sqrt{-(r_2^4+r_1^4+\rho^4)+2(r_2^2r_1^2+r_2^2\rho^2+r_1^2\rho^2)}}{2r_2\rho}.$$

Since now $0 \gtreqless \theta \leqq \pi$ and $0 \gtreqless \theta' < \pi/2$, $\sin\theta$ and $\sin\theta'$ will always be positive, that is, the plus signs must be taken before the expressions for $\sin\theta$ and $\sin\theta'$.

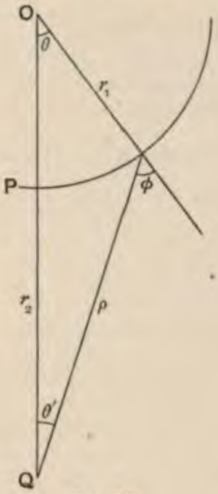

FIG. 20.

By the first of the above relations the obliquity ϕ can thus be written

$$\begin{aligned}\cos\phi &= \cos(\theta+\theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta' \\ &= \frac{r_2^4 - r_1^4 - \rho^4 + 2r_1^2\rho^2}{4r_2^2 r_1\rho} + \frac{r_2^4 + r_1^4 + \rho^4 - 2(r_2^2 r_1^2 + r_2^2\rho^2 + r_1^2\rho^2)}{4r_2^2 r_1\rho} \\ &= \frac{r_2^2 - r_1^2 - \rho^2}{2r_1\rho},\end{aligned}$$

which is the natural law of obliquity in terms of ρ, r_1, and r_2.

Stokes's law of obliquity can thus be written

$$s' \text{ proportional to } (1+\cos\phi) = 1 + \frac{r_2^2 - r_1^2 - \rho^2}{2r_1\rho} = \frac{r_2^2 - (r_1-\rho)^2}{2r_1\rho}. \qquad \text{Q.E.D.}$$

2. Show, when m is odd, that the series (14) for S reduces to

$$S = \frac{S_0}{2} - \frac{S_m}{2}.$$

3. The (Fresnel) light-vector S assumes most approximately one and the same values, those of the natural light-vectors, along the line PQ, Fig. 17, when the small aperture in the large intercepting screen is not exactly circular.

To determine the light-vector produced at any point Q by the elementary waves that are admitted through any irregular aperture at the pole P, we divide the given aperture or unscreened portion of the given wave-front up into circular sectors, as on p. 154 the small screen, replacing the edge or contour of the given aperture by circular arcs with common centre at P; the light-vector dS produced by the elementary waves that pass through any such unscreened sector of the given aperture will then, by formulae (17) and (17A), be given by the expression

$$dS = \frac{dS_0}{2} \pm \frac{dS_{m_1}}{2} = \frac{a\lambda}{r_2}\frac{d\phi_1}{2\pi}[F(r_2-r_1) \pm F(\rho_1)]\cos\frac{2\pi}{\lambda}(vt - r_2), \ldots\ldots\; \ldots\ldots(a)$$

where $d\phi_1$ denotes the angle subtended by the arc of the given sector at the pole P and ρ_1 the distance of that arc (edge of aperture) from the point Q; m_1 corresponds to that value of m, for which $\rho_1 = r_2 - r_1 + \frac{m_1\lambda}{2}$.

The total effect or light-vector S produced at the point Q by the elementary waves admitted through the whole aperture will, therefore, be given by the sum of the light vectors dS; we thus have

$$S = \Sigma dS = \Sigma \frac{dS_{0_\kappa}}{2} \pm \Sigma \frac{dS_{m_\kappa}}{2},$$

where κ denotes the number of sectors, into which the given aperture is divided, or by (a),

$$S = \frac{a\lambda}{r_2} \frac{F(r_2 - r_1)}{2\pi} \Sigma d\phi_\kappa \cos \frac{2\pi}{\lambda}(vt - r_2)$$
$$+ \frac{a\lambda}{r_2} \frac{1}{2\pi} \{\pm d\phi_1 F(\rho_1) \pm d\phi_2 F(\rho_2) \pm \dots \pm d\phi_\kappa F(\rho_\kappa)\} \cos \frac{2\pi}{\lambda}(vt - r_2). \quad \text{.........}(b)$$

The series in the larger brackets in the second term of this expression will now be small compared with the value assumed by the same, when the aperture is exactly circular and with its centre at P, since approximately (in mean) one half of the terms of this series will be positive and the other half negative. The approximate value assumed by the expression (b) for S will thus be

$$S = \frac{a\lambda}{r_2} F(r_2 - r_1) \cos \frac{2\pi}{\lambda}(vt - r_2) = \frac{S_0}{2},$$

a function of r_2 only, the distance of the point of observation Q from the given source (cf. p. 153).

4. Show that the following relation holds between the S'_2's (cf. formulae (28B)) of any three consecutive zones:

$$\frac{S'_{2_{n-1}}}{2} - S'_{2_n} + \frac{S'_{2_{n+1}}}{2} = 0.$$

5. On the assumption of Stokes's law of obliquity (cf. formulae 18)), show that the total electromagnetic vector at the point Q, Fig. 17, is determined by the expression

$$S = \frac{a\lambda}{2r_2 r_1} \left\{ (r_2 + r_1) \cos \left[\frac{\lambda}{2\pi(r_2 - r_1)} + \frac{2\pi}{\lambda}(vt - r_2) \right] - (r_2 - 3r_1) \cos \frac{2\pi}{\lambda}(vt - r_2) \right.$$
$$\left. - \frac{\lambda}{2\pi} \sin \frac{2\pi}{\lambda}(vt - r_2) \right\}$$
$$- \frac{a\lambda}{2r_2 r_1} \left\{ (r_2 - r_1) \cos \left[\frac{\lambda}{2\pi(r_2 + r_1)} + \frac{2\pi}{\lambda}(vt - r_2 - 2r_1) \right] - (r_2 - r_1) \cos \frac{2\pi}{\lambda}(vt - r_2 - 2r_1) \right.$$
$$\left. - \frac{\lambda}{2\pi} \sin \frac{2\pi}{\lambda}(vt - r_2 - 2r_1) \right\},$$

which for comparatively small values of λ can be written approximately in the form

$$S = \frac{2a\lambda}{r_2} \cos \frac{2\pi}{\lambda}(vt - r_2) - \frac{a\lambda^2}{2\pi r_1(r_2 - r_1)} \sin \frac{2\pi}{\lambda}(vt - r_2) - \frac{a\lambda^3}{16\pi^2 r_2 r_1} \frac{r_2 + r_1}{(r_2 - r_1)^2} \cos \frac{2\pi}{\lambda}(vt - r_2)$$
$$+ \frac{a\lambda^2}{2\pi r_1(r_2 + r_1)} \cos \frac{2\pi}{\lambda}(vt - r_2 - 2r_1) + \frac{a\lambda^3}{16\pi^2 r_2 r_1} \frac{r_2 - r_1}{(r_2 + r_1)^2} \sin \frac{2\pi}{\lambda}(vt - r_2 - 2r_1)$$

(cf. formula (33A)); it is evident that this expression reduces to that (20) for the light-vector proper for very small values of λ.

6. For light-waves the expression (33) for the electromagnetic vector reduces to that for the light-vector (cf. formula (19)). This follows directly from the approximate expression (cf. formula (33A)) for the electromagnetic vector.

7. Small deviations in the contour of the intercepting screen from the exact circle with centre at pole will have no appreciable effect on the value of the electromagnetic vector behind that screen. This is evident, since all the terms of the series for the electromagnetic vector, corresponding to the different elements of arc, by which the contour of the given screen may be replaced, will have, in general, one and the same and not, as in the case of light waves, different signs (cf. Ex. 3), so that the given series will assume a finite value.

8. Determine the behaviour of the (electromagnetic) vector at the point Q, Fig. 17, due to the action of electromagnetic waves admitted through a small circular aperture (with centre at pole) in a large intercepting screen.

The examination of the following two particular cases will suffice for the determination of the behaviour of the given vector: (1) the aperture admits the waves from the whole central zone only, and (2) the aperture admits the waves from the whole central and adjacent (first) zones only.

Case 1. By formulae (24A)–(26A), (27), and (28) the total electromagnetic vector at Q will evidently be given here by the expression

$$\begin{aligned}S=\frac{\pi a(r_2^2-r_1^2)}{r_2r_1}\,\frac{\lambda}{2\pi\rho}\cos\left\{\frac{\lambda}{2\pi\rho}+\frac{2\pi}{\lambda}[vt-(r_1+\rho)]\right\}\\ -\frac{\pi a}{r_2r_1}\left\{\frac{\lambda^2}{4\pi^2}\sin\frac{2\pi}{\lambda}[vt-(r_1+\rho)]+\frac{\lambda}{2\pi}\rho\cos\frac{2\pi}{\lambda}[vt-(r_1+\rho)]\right\},\end{aligned}$$

where the integration is to be extended from

$$\rho=r_2-r_1 \quad\text{to}\quad \rho=r_2-r_1+\lambda/2.$$

Replace here ρ by these limits, and we have

$$\begin{aligned}S=\frac{\pi a(r_2^2-r_1^2)}{r_2r_1}&\left\{\frac{\lambda}{2\pi(r_2-r_1)}\cos\left[\frac{\lambda}{2\pi(r_2-r_1)}+\frac{2\pi}{\lambda}(vt-r_2)\right]\right.\\ &\left.-\frac{\lambda}{2\pi(r_2-r_1+\lambda/2)}\cos\left[\frac{\lambda}{2\pi(r_2-r_1+\lambda/2)}+\frac{2\pi}{\lambda}(vt-r_2-\lambda/2)\right]\right\}\\ &-\frac{\pi a}{r_2r_1}\left[\frac{\lambda^2}{4\pi^2}\sin\frac{2\pi}{\lambda}(vt-r_2)+\frac{\lambda}{2\pi}(r_2-r_1)\cos\frac{2\pi}{\lambda}(vt-r_2)\right.\\ &\left.-\frac{\lambda^2}{4\pi^2}\sin\frac{2\pi}{\lambda}(vt-r_2-\lambda/2)-\frac{\lambda}{2\pi}(r_2-r_1+\lambda/2)\cos\frac{2\pi}{\lambda}(vt-r_2-\lambda/2)\right],\end{aligned}$$

or, approximately (for comparatively small values of λ),

$$\begin{aligned}S=\frac{a\lambda}{2r_2r_1}(r_2+r_1)&\left\{\left[1-\frac{\lambda^2}{8\pi^2(r_2-r_1)^2}\right]\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{\lambda}{2\pi(r_2-r_1)}\sin\frac{2\pi}{\lambda}(vt-r_2)\right\}\\ -\frac{a\lambda}{2r_2r_1}&(r_2-r_1)\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^2}{4\pi r_2r_1}\sin\frac{2\pi}{\lambda}(vt-r_2)+\frac{a\lambda(r_2^2-r_1^2)}{2r_2r_1(r_2-r_1+\lambda/2)}\\ &\times\left\{\left[1-\frac{\lambda^2}{8\pi^2(r_2-r_1+\lambda/2)^2}\right]\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{\lambda}{2\pi(r_2-r_1+\lambda/2)}\sin\frac{2\pi}{\lambda}(vt-r_2)\right\}\\ -\frac{a\lambda}{2r_2r_1}&(r_2-r_1+\lambda/2)\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^2}{4\pi r_2r_1}\sin\frac{2\pi}{\lambda}(vt-r_2)\end{aligned}$$

$$
\begin{aligned}
&=\frac{a\lambda}{r_2}\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^2}{2\pi r_1(r_2-r_1)}\sin\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^3}{16\pi^2r_2r_1}\frac{r_2+r_1}{(r_2-r_1)^2}\cos\frac{2\pi}{\lambda}(vt-r_2)\\
&+\left[\frac{a\lambda}{r_2}-\frac{a\lambda^2(r_2+\lambda/4)}{2r_2r_1(r_2-r_1+\lambda/2)}\right]\cos\frac{2\pi}{\lambda}(vt-r_2)\\
&-\left[\frac{a\lambda^2}{2\pi r_1(r_2-r_1)}-\frac{a\lambda^3}{4\pi r_2r_1}\frac{(r_2+r_1)(r_2-r_1+\lambda/4)}{(r_2-r_1)(r_2-r_1+\lambda/2)}\right]\sin\frac{2\pi}{\lambda}(vt-r_2)\\
&-\left\{\frac{a\lambda^3}{16\pi^2r_2r_1}\frac{r_2+r_1}{(r_2-r_1)^2}-\frac{a\lambda^4}{16\pi^2r_2r_1}\frac{(r_2+r_1)[\frac{3}{2}(r_2-r_1)^2+3(r_2-r_1)\lambda/4+\lambda^2/8]}{(r_2-r_1)^2(r_2-r_1+\lambda/2)^3}\right\}\\
&\quad\times\cos\frac{2\pi}{\lambda}(vt-r_2),
\end{aligned}
$$

or, if we reject all terms of higher orders of magnitude in λ than the third,

$$
\begin{aligned}
S=2&\left\{\frac{a\lambda}{r_2}\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^2}{2\pi r_1(r_2-r_1)}\sin\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^3}{16\pi^2r_2r_1}\frac{r_2+r_1}{(r_2-r_1)^2}\cos\frac{2\pi}{\lambda}(vt-r_2)\right\}\\
&-\frac{a\lambda^2(r_2+\lambda/4)}{2r_2r_1(r_2-r_1+\lambda/2)}\cos\frac{2\pi}{\lambda}(vt-r_2)+\frac{a\lambda^3}{4\pi r_2r_1}\frac{r_2+r_1}{(r_2-r_1+\lambda/2)^2}\sin\frac{2\pi}{\lambda}(vt-r_2).
\end{aligned}
$$

A comparison of this expression with that (33A) for the natural electromagnetic vector or that produced, when no obstacle is placed in the course of the given waves, shows that the given electromagnetic vector is approximately twice as large as the natural one, or, more exactly, that the former differs from the latter (doubled) by an expression that is of one higher order of magnitude in λ than that for either.

Case 2. Similarly, the electromagnetic vector would evidently be given here by the expression

$$
\begin{aligned}
S=\frac{\pi a(r_2^2-r_1^2)}{r_2r_1}&\left\{\frac{\lambda}{2\pi(r_2-r_1)}\cos\left[\frac{\lambda}{2\pi(r_2-r_1)}+\frac{2\pi}{\lambda}(vt-r_2)\right]\right.\\
&\left.-\frac{\lambda}{2\pi(r_2-r_1+\lambda)}\cos\left[\frac{\lambda}{2\pi(r_2-r_1+\lambda)}+\frac{2\pi}{\lambda}(vt-r_2-\lambda)\right]\right\}\\
&-\frac{\pi a}{r_2r_1}\left[\frac{\lambda^2}{4\pi^2}\sin\frac{2\pi}{\lambda}(vt-r_2)+\frac{\lambda}{2\pi}(r_2-r_1)\cos\frac{2\pi}{\lambda}(vt-r_2)\right.\\
&\left.-\frac{\lambda^2}{4\pi^2}\sin\frac{2\pi}{\lambda}(vt-r_2-\lambda)-\frac{\lambda}{2\pi}(r_2-r_1+\lambda)\cos(vt-r_2-\lambda)\right];
\end{aligned}
$$

which for values of λ, that are small compared with r_2 and r_1, can be written

$$
\begin{aligned}
S=&\frac{a\lambda}{r_2}\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^2}{2\pi r_1(r_2-r_1)}\sin\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^3}{16\pi^2r_2r_1}\frac{r_2+r_1}{(r_2-r_1)^2}\cos\frac{2\pi}{\lambda}(vt-r_2)\\
&-\frac{a\lambda(r_2^2-r_1^2)}{2r_2r_1(r_2-r_1+\lambda)}\left\{\left[1-\frac{\lambda^2}{8\pi^2(r_2-r_1+\lambda)^2}\right]\cos\frac{2\pi}{\lambda}(vt-r_2)\right.\\
&\left.-\frac{\lambda}{2\pi(r_2-r_1+\lambda)}\sin\frac{2\pi}{\lambda}(vt-r_2)\right\}\\
&+\frac{a\lambda}{2r_2r_1}(r_2-r_1+\lambda)\cos\frac{2\pi}{\lambda}(vt-r_2)+\frac{a\lambda^2}{4\pi r_2r_1}\sin\frac{2\pi}{\lambda}(vt-r_2),
\end{aligned}
$$

or, if we reject all terms of higher orders of magnitude in λ than the third,

$$
S=\frac{a\lambda^2(2r_2+\lambda)}{2r_2r_1(r_2-r_1+\lambda)}\cos\frac{2\pi}{\lambda}(vt-r_2)-\frac{a\lambda^3(r_2+r_1)}{2\pi r_2r_1(r_2-r_1+\lambda)^2}\sin\frac{2\pi}{\lambda}(vt-r_2).
$$

A comparison of this expression with that (33A) for the natural electromagnetic vector shows that the given electromagnetic vector is determined by an expression that is of one higher order of magnitude in λ than that for the former (cf. also p. 156).

9. Evaluate the integral (55) for the light-vector $s_0(t)$, when the surface of integration S is a sphere with centre at the point of observation Q.

We divide the surface of the given sphere up into surface-elements ds as on p. 175; the following relations will then hold:

$$ds=2\pi\frac{r}{r_2}\rho\,d\rho,$$

$$\cos(n,\rho)=\cos(-r,\rho)=\cos\phi=\frac{r_2^2-r^2-\rho^2}{2r\rho},$$

where we are assuming the natural law of obliquity, and

$$\cos(n,r)=\cos(-r,r)=-1$$

(cf. formulae on pp. 175-176); by which the integral (55) can be written

$$\begin{aligned}s_0(t)&=\frac{a\pi}{\lambda}\int\frac{1}{r_2}\left[\frac{r_2^2-r^2-\rho^2}{2r\rho}+1\right]\cos\frac{2\pi}{\lambda}[vt-(\rho+r)]d\rho\\&=\frac{a\pi}{\lambda}\frac{r_2^2-r^2}{2r_2r}\int\frac{\cos\omega\,d\rho}{\rho}-\frac{a\pi}{\lambda}\frac{1}{2r_2r}\int\rho\cos\omega\,d\rho+\frac{a\pi}{\lambda r_2}\int\cos\omega\,d\rho,\end{aligned}$$

where
$$\omega=\frac{2\pi}{\lambda}[vt-(\rho+r)].$$

These three integrals have now been evaluated on pp. 176-177. Since we have retained terms of only the lowest order of magnitude in λ in formula (55), we can evidently retain only such terms in the evaluation of the vector sought or its integrals; the approximate values of these integrals are evidently

$$\int\frac{\cos\omega}{\rho}d\rho=-\frac{\lambda}{2\pi}\frac{\sin\omega}{\rho},\quad\int\rho\cos\omega\,d\rho=-\frac{\lambda}{2\pi}\rho\sin\omega,\text{ and }\int\cos\omega\,d\rho=-\frac{\lambda}{2\pi}\sin\omega.$$

Replace the integrals by these values in the above expression for $s_0(t)$, and we have

$$s_0(t)=-a\frac{r_2^2-r^2}{4r_2r}\frac{\sin\omega}{\rho}+a\frac{1}{4r_2r}\rho\sin\omega-a\frac{1}{2r_2}\sin\omega,$$

where the integration is to be extended from

$$\rho_1=r_2-r\text{ to }\rho_2=r_2+r.$$

Replace here ρ by these limits, and we have

$$\begin{aligned}s_0(t)=&-\frac{a}{r_2}\left[-\frac{r_2^2-r^2}{4r}\frac{1}{r_2-r}+\frac{1}{4r}(r_2-r)-\frac{1}{2}\right]\sin\frac{2\pi}{\lambda}(vt-r_2)\\&+\frac{a}{r_2}\left[-\frac{r_2^2-r^2}{4r}\frac{1}{r_2+r}+\frac{1}{4r}(r_2+r_1)-\frac{1}{2}\right]\sin\frac{2\pi}{\lambda}[vt-(r_2+2r)]\\=&\frac{a}{r_2}\sin\frac{2\pi}{\lambda}(vt-r_2),\end{aligned}$$

the actual expression for the light-vector at the distance r_2 from source.

10. Show that the function $f(\phi)$ employed in the determination of the light-vector according to Fresnel's method must be assigned the value

$$f(\phi)=\frac{1+\cos\phi}{2\lambda},$$

in order that we may obtain the correct expression for the amplitude of the light-vector.

CHAPTER VI.

DIFFRACTION.

Diffraction Phenomena.—The phenomena that appear at the boundaries of the geometrical shadow, when light rays pass through a very small aperture or by the edge of an opaque body placed in their course, are known as the phenomena of "*Diffraction*"; they arise, figuratively speaking, from the light rays deviating from their rectilinear paths. A similar class of phenomena that arise from the same cause, the deviation of the light rays from their rectilinear paths, has been examined in the preceding chapter; there we investigated briefly the behaviour of light (its intensity) directly behind small screens and small apertures in large screens—along the central axis of the geometrical shadow or image respectively. For the particular case, where the aperture or the small intercepting screen or obstacle is very small, this latter class of phenomena is, as we shall see below, to be included in the former or that of the phenomena of diffraction.

First Observations on Diffraction.—The first observations on diffraction were made by Grimaldi [*] at Bologna in 1665; he found upon placing a small opaque obstacle (wire) in the cone of light admitted into a dark room through a very small aperture that the shadow cast on a screen behind that obstacle was much larger than the geometrical shadow; he also observed that the enlarged portion of the shadow consisted of coloured bands or fringes that ran parallel to the edge of the geometrical shadow. Newton [†] was among the first to examine the complementary case or the image cast by light admitted through a very small aperture or narrow slit; the slit employed was formed by two knife blades, which admitted only a very narrow strip of light; he found that the image was then bordered exteriorly by parallel coloured fringes similar to those already observed by Grimaldi on the exterior of the shadow cast by a small obstacle (see below).

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[*] *Physico-mathesis de lumine, coloribus et iride*, Bononiae, 1665.

[†] *Opticks*, vol. iii.

Young's Explanation of Diffraction.—The first attempt to explain diffraction phenomena was made by Dr. Young;[*] he attributed the (coloured) fringes to the interference of the rays that pass very near to the edge of the obstacle and those that are reflected by the same at grazing incidence. This explanation would evidently stipulate that the given phenomena be more or less marked according to the degree of polish and sharpness of the edge. Fresnel has now shown by most exact experiments[†] that these factors, polish and sharpness of edge, have no effect whatever on the fringes produced, the fringes retaining the same position (with regard to edge) and intensity, whether the back or (sharp) edge of the knife (razor) be employed and whether that edge be highly polished or not.

Fresnel's Theory of Diffraction.—Fresnel, who had demonstrated experimentally the incorrectness of Young's explanation of diffraction phenomena, not only offered us another explanation but also confirmed the same by a series of most ingenious and exact experiments.[‡] Fresnel attributes the phenomena of diffraction to the mutual action of the elementary waves that are supposed, according to Huygens's principle, to be emitted from any wave-front, here that which passes through the edge of the intercepting obstacle; the mutual action of these waves at any external point is then calculated according to the principle of interference (cf. Chapter IV.). Diffraction phenomena are, therefore, to be conceived as due to the mutual action or interference of the elementary waves emitted from the various (elementary) sources on the wave-front in question, just as interference phenomena are due to the mutual action or interference of two systems (pencils) of waves.

Let us first examine those problems on diffraction that can be treated by the simple methods employed in the preceding chapter; these methods, which are only approximate ones, have been deduced from Huygens's principle as postulated by Fresnel and the principle of interference.

Diffraction of Light on the Edge of an Opaque Obstacle; the Exterior Bands.—Let O be the source of a system of spherical light waves, A the straight edge of an opaque obstacle AB, MN the screen of observation, and P_0 that point (line) on the screen that lies on the continuation of the line (plane) or ray (rays) OA, as indicated in the annexed figure. P_0 will then mark the upper boundary of the geometrical shadow on the screen. Let us first consider the illumination at any point Q on the screen outside the geometrical shadow. For this purpose we draw the spherical wave-front S that passes through that point of the edge A of the obstacle AB, which corresponds to the point Q. We denote the point, where the line or vector OQ intersects this wave-front by P; P is then the so-called "pole" of the given wave-front with respect to the external point Q (cf. p. 146). According to Fresnel the (light) effect at Q can now be conceived as produced by the mutual action at that point of the elementary waves emitted from the wave-front S,

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[*] "On the Theory of Light and Colours," *Phil. Trans.*, p. 12, 1802.

[†] Fresnel, *Oeuvres complètes*, tom. i., pp. 148 and 280.

[‡] "Mémoire sur la diffraction de la lumière," *Mémoires de l'Acad. franç.*, tom. v. *Poggend. Annal.*, vol. xxx. *Oeuvres complètes*, tom. i.

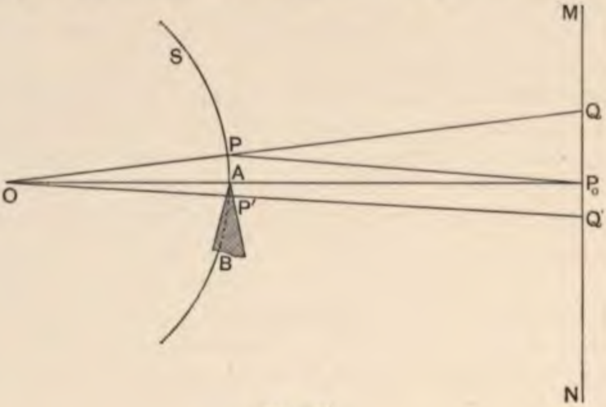

FIG. 21.

whereas for light waves the given effect may be conceived as confined to the mutual action of the elementary waves emitted alone from the immediate neighbourhood of the pole P with respect to the point Q, or, as we have seen in Chapter V., to a limited number of the so-called "half-period elements" around that pole. If the pole P be at considerable distance (compared with the wave-length) from the edge A of the obstacle, then the elementary waves that are intercepted by the obstacle will evidently have no appreciable effect at the external point Q, that is, the illumination at Q will be approximately the same as when the given obstacle is removed. On the other hand, if the point Q be very near the boundary of the geometrical shadow P_0, its pole P will be so near the edge A of the obstacle that a given portion of the elementary waves emitted from the lower half of the effective part of the given wave-front will be intercepted by the obstacle. The resultant effect at Q will then consist of two effects, that from the whole upper half of the given effective wave-front and that from that portion PA of the lower half of the same that extends from the pole P to the edge A of the obstacle. The former effect at Q will remain constant as long as the pole P is above the line (plane) OAP_0, whereas the latter will evidently vary between maxima and minima according as the number of half-period elements contained in the unscreened portion PA of the lower half of the effective wave-front be odd or even respectively; if PA contain an even number of such elements, the waves from the first and second, third and fourth, and all consecutive pairs will interfere destructively with each other, and hence produce no appreciable effect at Q; if the number of these elements be odd, $2n+1$, where n is an integer, the waves from the first n pairs will interfere destructively as in the former case, whereas those from the $(2n+1)$st or last unscreened half-period element will not interfere with those from the next element, which are intercepted by the given obstacle, and they will thus contribute materially to the total effect at Q. Or, analytically, if

$$AQ - PQ = 2n\lambda/2,$$

we shall have a minimum of intensity or a dark band or fringe at the locus of the point Q with respect to the edge A of the obstacle, and if

$$AQ - PQ = (2n+1)\,\lambda/2,$$

a maximum of intensity or bright band.

It thus follows: the shadow cast on the screen MN by the obstacle AB will not be distinctly marked, but it will be bordered exteriorly by a series of alternately bright and dark bands or fringes running parallel to the edge of the obstacle or to its geometrical shadow; and these bands will become less and less distinct, the further we recede from the boundary of the geometrical shadow.

Condition for the Appearance of Exterior Bands.—Diffraction bands can evidently appear on the boundary of a geometrical shadow only when the angular diameter of the source of light is very small; for, if this diameter is not small, each point of the source will emit a different wave-front and each such wave front a corresponding system of elementary waves, each one of which will produce a different set of bands on the screen; and these numerous sets of bands or fringes upon being superposed will interfere or destroy one another, and thus produce an uniform illumination of the screen (outside the geometrical shadow) (cf. also p. 111).

The Diffraction Phenomena within the Shadow.—Let us next examine the illumination at any external point Q' on the screen MN within the geometrical shadow, that is, below the central point (line) P_0 (cf. Fig. 21). Here only a portion of the upper half of the wave-front

S will transmit elementary waves to point Q', and the waves from that portion will be less effective than those from the lower portion of the same, which are intercepted by the obstacle AB, since their sources are more remote from the pole P' with respect to the point Q' (cf. Fig. 21) than those of the latter. To find the total effect at Q' we divide the unscreened portion of the given wave-front up into half-period elements, beginning at the edge A of the obstacle AB, and calculate the resultant effect of the waves from those elements according to the method employed in the foregoing chapter. This effect will evidently be determined by the same expression as that for the resultant effect at an external point, when its pole and the immediate neighbourhood of the same is covered by a small screen; the latter has already been determined in Chapter V. (cf. pp. 153-154); we found, namely, that the resultant effect was half that produced by the waves emitted from the first half-period element of the unscreened portion of the given wave-front (cf. formula (21, V.)). As Q' recedes from the central point P_0 downwards into the geometrical shadow, the resultant effect will, therefore, diminish *continuously* but rapidly, since at a short distance below the central point the elementary waves from the whole effective portion of the given wave-front, that around the pole P', will all be intercepted by the given obstacle. The illumination within the given geometrical shadow will, therefore, decrease rapidly but continuously, as we recede from its boundary.

Diffraction Phenomena produced by a Small Opaque Obstacle; Exterior and Interior Bands.—Let us, next, examine the diffraction effects due to a small opaque obstacle, such as a fine wire. We represent the relative position of the obstacle (wire) AB, the source O and the screen MN, as in the annexed figure. It follows now from the preceding problem that the geometrical shadow P_0P_0' on the screen MN, not only its upper boundary P_0 but also its lower one

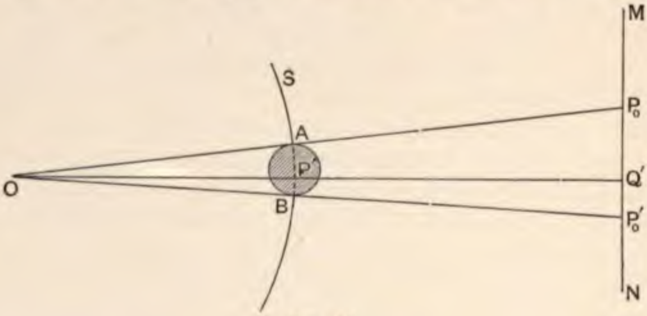

FIG. 22.

P_0', will be bordered exteriorly each by a system of alternately bright and dark bands or fringes similar to those produced by the straight edge A of the larger opaque obstacle in the problem just treated. Moreover, we have seen in the preceding problem that the effect at any point Q' within the geometrical shadow was produced alone by the mutual action of the elementary waves emitted from the unscreened portion of the given wave-front and that this could be replaced by the mutual action halved of the waves emitted from its first unscreened half-period element, that is, the first such element reckoned from the edge A of the obstacle. Hence, the effect produced here at any point Q' within the geometrical shadow P_0P_0' will evidently be determined by the mutual action halved of the waves emitted from the first unscreened half-period element of the upper half of the given wave-front and of those from the first such element of its lower half, this latter element being reckoned from the lower edge B of the obstacle. If the edges A and B of the obstacle are now equidistant from the source O and also from the screen MN, as we shall assume here, the waves from the two half-period elements in question will have the same phase upon leaving their respective sources on the given wave-front. On reaching any point Q' on the screen, they will, therefore, differ in phase only by the difference in the distances AQ' and BQ' traversed, and thus cooperate or interfere with one another accordingly. The interior of the geometrical shadow P_0P_0' will, therefore, exhibit a set of alternately bright and dark bands or fringes, and these bands will evidently be similar to the interference-bands produced by waves emitted from one and the same source and brought to interfere, after having traversed slightly different paths (cf. p. 108). That these bands are, in fact, due to the interference of the two systems of elementary waves that bend round the edges of the given obstacle, has been demonstrated by Dr. Young as follows: he intercepted the one system of light rays by an opaque screen placed first in front of the obstacle, that is, between it and the source, and then behind the same, and found that in both cases the set of interior fringes disappeared completely, whereas the exterior fringes remained unaltered on that side of the geometrical shadow, where the waves were not intercepted; from which it is evident that the interior fringes are due to the interference of the two systems of elementary waves that pass over the edges of the obstacle, just as the phenomena of interference proper are due to the mutual action of two systems of ordinary waves, whereas the exterior fringes on either boundary of the shadow are produced by the mutual action alone of the elementary waves that pass over the respective edge of the obstacle, as maintained above.

Conditions for the Appearance of Interior Bands.—The interior diffraction bands, like the exterior ones, can evidently only appear when the angular diameter of the source of light is very small or narrow. But this is not the only condition for the appearance of interior fringes; their appearance also demands that the diffracting obstacle be very narrow; for otherwise the luminous effect within the geometrical shadow due to the action of the elementary waves that pass over either edge of the obstacle will extend only to a comparatively short distance within the shadow, since the effective portion of that half of the given wave-front, which is confined, as we have seen above, to the immediate neighbourhood of the pole, will be screened off by the obstacle, except at points very near the boundaries of the shadow; the waves that pass over the one edge of the obstacle and those that pass over the other will not then overlap within the geometrical shadow, that is, there will be no interference within the same between the two systems of elementary waves. The interior of the given shadow, instead of exhibiting a set of fringes, as is the case when the diffracting obstacle is very narrow, will, therefore, be illuminated only at and near its boundaries; this illumination, which is brightest at the boundaries of the geometrical shadow, will evidently decrease *continuously* but rapidly, as we recede from the same towards the centre of the shadow (cf. p. 188).

Expressions for the Breadths of the Bands.—Lastly, let us determine the breadths of the exterior and interior diffraction fringes produced on the screen MN by the narrow wire AB of Fig. 22. It is now easy to show that the breadth of the nth exterior fringe from the boundary of the geometrical shadow is given by the expression

$$[\sqrt{2n+1}-\sqrt{2n-1}]\sqrt{\lambda\frac{b(a+b)}{a}}, \quad\text{.....................(1)}$$

when the fringe is a dark one, and by

$$[\sqrt{n}-\sqrt{n-1}]\sqrt{2\lambda\frac{b(a+b)}{a}}, \quad\text{.......................(2)}$$

when the fringe is a bright one (cf. Ex. 1), where a denotes the distance of the edge A of the obstacle AB from the source O and b its distance from the screen MN. By Ex. 3 at end of chapter the breadth of *any* interior fringe is given by the simple expression

$$\frac{\lambda b}{c}, \quad\text{.......................................(3)}$$

where b denotes the distance between the screen and the obstacle, as above, and c the distance (breadth) between its two diffracting edges.

We observe that the expression (1) or (2) for the breadth of an exterior band is a function not only of the distances a and b but also of the band n in question; that is, the exterior bands will not be of equal breadth, but they will evidently decrease in breadth, as we recede from the geometrical shadow (cf. Ex. 2). On the other hand, the expression (3) for the breadth of an interior band is a function only of the distances b and c and not of the band in question; these bands will, therefore, be of equal breadth, and in this respect they will differ from the exterior bands.

Relative Breadths of the Exterior and Interior Bands.—Let us now attempt to compare the breadths of the exterior and interior bands, the expressions (1) or (2) and (3), with each other. We have seen above that for the appearance of interior bands the breadth c of the obstacle AB must be taken small compared with easily measurable quantities, such as the distances a and b. We can express this in the form

$$c=\epsilon b, \quad\dots\dots\dots(4)$$

where ϵ shall denote a small quantity. Let now 2μ denote the number of bands, dark or bright, within the geometrical shadow. The difference (DE) in the paths traversed by the waves from $B(D)$ to $P_0(P_n)$ and from $A(C)$ to $P_0(P_n)$—the letters in the brackets refer to Fig. 12, those without brackets to Fig. 22—will then be

$$(DE)=\mu\lambda=\eta c, \quad\dots\dots\dots(5)$$

where η shall denote a small quantity. The relative magnitudes of these two small quantities, ϵ and η, are now to be sought.

From the similarity of the two right-angle triangles (CED) and (OPP_n) (cf. p. 110), we have

$$(DE):(CE)=(P_nP):(OP),$$

or, by (5) and Fig. 22,

$$\eta c : c\cos\xi=\mu\omega_i : b,$$

where ξ denotes the very small angle (ECD) (cf. Fig. 12) and ω_i the width of any interior dark or bright band. Since now ξ is very small, we can put $\cos\xi=1$ approximately and thus write the last proportion in the form

$$\eta : 1=\mu\omega_i : b,$$

or, by (4),

$$\eta : \epsilon=\mu\omega_i : c. \quad\dots\dots\dots(6)$$

It is now evident from Fig. 22 that

$$\mu\omega_i : c/2=a+b : a; \quad\dots\dots\dots(7)$$

we observe here that, although the breadth ω_i of the interior bands is *not* a function of the distance a (cf. formula (3)), the number of bands 2μ that will appear within the geometrical shadow is a function of that distance.

The proportions (6) and (7) give

$$\eta : \epsilon = a + b : 2a, \qquad \text{.............................. (8)}$$

the relation sought between the two small quantities ϵ and η.

By this and relation (5) we can write the wave-length λ as follows:

$$\lambda = \frac{\eta c}{\mu} = \frac{a+b}{2a}\,\frac{c}{\mu}\epsilon,$$

or, by (4),

$$\lambda = \frac{(a+b)b}{2a}\,\frac{\epsilon^2}{\mu}.$$

Replace λ by this value in the expressions (2) and (3) for the breadths of the exterior and interior bands respectively, and we have

$$\omega_e = (\sqrt{n} - \sqrt{n-1})\sqrt{\frac{(a+b)b}{a}\,\frac{\epsilon^2}{\mu}\,\frac{(a+b)b}{a}} = f(n)\,\frac{(a+b)b}{a}\,\frac{\epsilon}{\sqrt{\mu}},$$

and

$$\omega_i = \frac{\lambda b}{c} = \frac{(a+b)b}{2a}\,\frac{\epsilon}{\mu},$$

where ω_e and ω_i are expressed in terms of ϵ and μ in place of λ, and hence

$$\omega_e : \omega_i = f(n) : \frac{1}{2\sqrt{\mu}} \qquad \text{.............................. (9)}$$

As the number of interior bands (2μ) increases, their breadth will, therefore, diminish relatively compared with the breadths of the exterior ones, especially of those near the boundaries of the geometrical shadow (cf. the values of $f(n)$ for different n's in Ex. 2 at end of chapter). The interior bands will, therefore, in general (except for small values of μ), be finer or closer together than the interior ones, a result that is confirmed by observation.

Diffraction Phenomena produced by a Narrow Aperture; Exterior Bands.—Let us, next, examine the complementary case to the above, the effect produced on a screen MN by light that passes through a very narrow slit or aperture AB in a large screen, as indicated in the annexed figure. To determine the illumination at any point Q outside the geometrical image P_0P_0', we divide the unscreened portion of the wave-front S that passes through the two edges A and B of the slit up into half-period elements, beginning at the point A of the same. If now the slit were not narrow, the unscreened portion AB of the wave-front would be so large that it would include the whole unscreened effective portion of the same, and the resultant effect at Q would evidently be given by the mutual action halved of the elementary waves from its first (from A) unscreened half-period element (cf. p. 188); in which case the illumination outside the geometrical image would evidently decrease rapidly but continuously, as we receded from the boundary P_0 or P_0' of the image (cf. p. 188). In the given case, where the aperture is assumed to be (very) small, the unscreened portion AB of the given wave-front will include only a (small) part of the effective wave-front, that is, only a limited number n of half-period elements of the latter; the effect at Q will then depend on the number of elements in question, whether the same be odd or even, since the first and second, third and fourth and each consecutive pair of half-period elements will each emit elementary waves that will interfere mutually with one another at Q. The point Q (its locus) will, therefore, be bright or dark according as the number n of the half-period elements of the unscreened portion AB of the given wave-front be odd or even. The upper and lower portions of the screen MN outside the geometrical image P_0P_0' will, therefore, each exhibit a system of alternately bright and dark bands similar to the exterior bands on the boundary of the geometrical shadow of a large opaque obstacle or fine wire (cf. above). The distances of these bands from either boundary of the geometrical image and their breadths will evidently be determined by similar expressions to those already found for the exterior bands on the boundaries of the geometrical shadow of a small opaque obstacle (cf. formulae (1) and (2) and Ex. 1 at end of chapter).

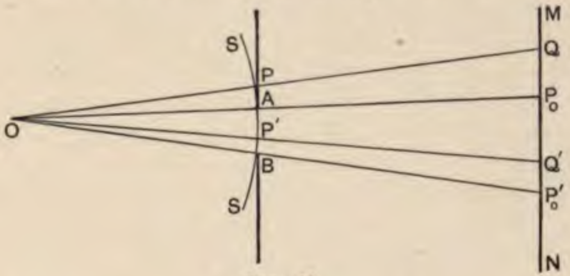

FIG. 23.

Conditions for the Appearance of Interior Bands and Method for their Determination.—If the screen MN is so remote from the aperture AB of Fig. 23 that at the boundary P_0 or P_0' of the geometrical image the unscreened portion AB of the given wave-front includes only the first half-period element, that is, if $BP_0 - AP_0 = \lambda/2$, then the first (bright) band will appear at the boundary of the geometrical image, and the diffraction phenomena exhibited will evidently consist of two sets of exterior bands, one on either boundary of the geometrical image (cf.

above). On the other hand, if the screen of observation is at such a distance from the aperture that at the point P_0 or P_0' the unscreened portion AB of the wave-front includes several half-period elements, then bands will evidently appear within the geometrical image, but the determination of their position, breadth and intensity will offer difficulties that cannot readily be solved by the simple methods employed above. This is evident from the following considerations: Take any point Q' on the screen MN within the geometrical image (cf. Fig. 23). The luminous effect at that point is now due to the mutual action of the elementary waves from the several half-period elements, into which the unscreened portion AB of the given wave-front can be divided with respect to the point Q'. At any point Q outside the geometrical image the unscreened portion of the wave-front consisted of a portion of either only the upper or the lower half of the same, and we were able to divide that unscreened portion up into half-period elements, beginning at the respective edge of the aperture, without any further reference to the pole with respect to the point Q, and to determine the effect at Q according as the number of those elements was even or odd. Here, on the other hand, the unscreened portion of the wave-front includes a portion of the upper and a portion of the lower half of the same with respect to the respective pole P'; we must, therefore, divide the given unscreened portion of the wave-front up into half-period elements with respect to that pole P' (cf. Figure 23) and on both sides of the same; this division would, in general, give not only a different number of half-period elements on either side of the pole but also a different residual of an element at either boundary of the unscreened portion, the edge A or B of the aperture. The determination of the luminous effect at any point Q' produced by the elementary waves from these half-period elements would, therefore, be quite complicated, at least, according to the simple methods employed above; it would, in fact, evidently be given by the sum (with reference to phase) of the following single effects: First, the effect due to the mutual action of the elementary waves from all the whole half-period elements on that side of the pole, which contains the smaller number m of unscreened elements, and from the first m such elements on the opposite side; the effect or illumination produced by these waves at the point Q' would now be either large, 4 times the natural effect, or very small, approximately zero, according as m is odd or even (cf. p. 156). Secondly, the effect due to the mutual action of the elementary waves from the remaining $l - m$ unscreened whole half-period elements of the given wave-front, where l denotes the total number of such elements, into which the larger unscreened portion of the wave-front with respect to the pole P' can be divided; the effect produced at Q' by the waves from these $l-m$ elements would, like the first effect, be either large or very small according as $l-m$ were odd or even. Lastly, the effect due to the mutual action of the elementary waves from the two residual half-period elements at the edges A and B of the given aperture; their effect at Q' would evidently depend on their relative size compared with the adjacent elements and with each other, and whether both $(l+1)$ and $(m+1)$ be odd or even or the one odd and the other even. The expression for the total resultant effect at Q' would, therefore, be an extremely complicated one, evidently such an one as could not readily be determined by the simple methods employed above.

For the diffraction phenomena produced by a small circular aperture in a large opaque screen, see Ex. 4 at end of chapter.

Diffraction Phenomena of the Primary and Secondary (Electromagnetic) Waves.—Let us now consider the diffraction phenomena produced by the primary electric waves, for example, the linearly polarized electromagnetic waves of the most general form, those represented by the first terms of the expressions for X, Y, Z of formulae (43, II.). It is now evident that the waves of the given system that come into consideration for diffraction phenomena or those emitted from that portion of the given wave-front, which is supposed to give rise to such phenomena, may be regarded as approximately, here linearly, polarized (cf. also p. 78), provided their source be at considerable distance from the diffracting obstacle; that is, the direction-cosines α, β, γ in the coefficients of the expressions for the component moments may be regarded here as constant. If we write the given component moments X_1, Y_1, Z_1 in the form $\quad X_1=\frac{n^2}{r}l\sin\omega,\quad Y_1=\frac{n^2}{r}m\sin\omega,\quad Z_1=\frac{n^2}{r}p\sin\omega,$

where
$$l=a_1(\beta^2+\gamma^2)-\alpha(a_2\beta+a_3\gamma),$$
$$m=a_2(\alpha^2+\gamma^2)-\beta(a_1\alpha+a_3\gamma),$$
$$p=a_3(\alpha^2+\beta^2)-\gamma(a_1\alpha+a_2\beta),$$
and
$$\omega=n[vt-(r+\delta)],$$

we may, therefore, regard these coefficients l, m, p as constant throughout that portion of the given wave-front, which comes into consideration in the determination of diffraction phenomena, that is, we may assume here, as above, that the several elementary sources on the portion in question of the given wave-front all emit similar elementary waves. The mutual action of these waves at an external point would, therefore, be subject to the same laws as those for the light waves already treated, and hence the phenomena of diffraction produced, being due to the same causes, difference in phase between the elementary waves at an external point due to difference in the paths traversed, etc., similar to those of light. The above would evidently also hold for the secondary electric waves. The diffraction phenomena exhibited by the electric waves would differ from those of light only in relative magnitude, the bands of the former, which are always proportional to the wave-length λ of the waves employed, being much broader than those of the latter. On the other hand, the bands of the primary and those of the secondary waves would be of the same dimensions, since the wave-length of the secondary wave that accompanies any primary disturbance is always that of the latter (cf. formulae (43, II.)). The bands produced by the secondary waves would be much less brilliant than those exhibited by the primary, at least, when the source of disturbance were at considerable distance from the diffracting obstacle, as assumed here, since the intensity of the secondary wave varies inversely as the fourth power of the distance from source and that of the primary inversely as its square. We could, therefore, expect to obtain bright diffraction bands from the secondary waves only either near their source or in those regions, where the secondary waves were unaccompanied by primary disturbances (cf. pp. 49-52 and 61-62). Near the source the secondary waves would now evidently be neither approximately (linearly) polarized nor of one and the same amplitude or intensity, that is, the elementary waves emitted from the different sources on that portion of the given wave-front that comes into consideration in the determination of diffraction phenomena would differ here appreciably from one another, both in intensity and direction of oscillation, so that, aside from the difficulties encountered in the actual determination of their mutual action at an external point, no great regularity in intensity or its distribution (law of dispersion of the intensity) and hence only correspondingly irregular diffraction bands could be anticipated; this would be true not only of the secondary but also of the primary waves in the neighbourhood of the source. The only regions where it might be possible to detect secondary diffraction bands of any great degree of regularity in distribution and intensity would, therefore, be those remote from the given source, and where the secondary waves were unaccompanied by primary disturbances (cf. pp. 49-52).

Diffraction of the Roentgen Rays.—In connection with the above we may call attention to the experiments of C. H. Wind* on the diffraction of the Roentgen (X) rays on a narrow wedge-shaped slit; the observation-screen exhibits systems of diffraction bands, which have been photographed and are, with the exception of certain irregularities (cf. above), remarkably similar to those obtained from actual calculation by Fresnel's method.

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* "Zum Fresnelschen Beugungsbilde eines Spaltes," *Physikalische Zeitschrift*, vol. i., p. 112 (1900), and vol. ii., p. 265 (1901).

Fresnel's Theory of Diffraction and its Shortcomings.—The above simple methods for the determination of diffraction phenomena were employed by Fresnel, and with great success, for the results obtained agreed in the cases treated with those of observation. There are, however, many problems that cannot readily be solved by the application of these methods, at least in the simple form employed above, as, for example, the determination of the distribution and intensity of the bands within the geometrical image of a small opaque obstacle (cf. pp. 194-195). Fresnel now developed these simple methods and the principles embodied therein and found new ones, by means of which he was enabled to treat and solve the more complicated problems on diffraction; these new principles and the methods deduced from them then led to a concrete representation or theory, which is known as "Fresnel's Theory of Diffraction." These methods of Fresnel, which are graphic ones, the resultant intensity being represented by means of a spiral, the so-called "Cornu spiral," the determination of which requires the evaluation of two (definite) integrals that are known as "Fresnel's Integrals," are now so exhaustively expounded and the different problems on diffraction so extensively treated in the standard text-books * on the theory of light that I shall not repeat them here, at least not in the form employed by Fresnel. Another, perhaps the chief, reason why we shall not follow Fresnel's method of treatment is the following: We have seen in the chapter on Huygens's principle that the form assumed by Fresnel for representing the elementary waves emitted by the different (elementary) sources on any wave-front was not the correct one, for the mutual action of those elementary waves at an external point gave for the simplest case, where the given waves were in no way obstructed by foreign bodies, an expression for the light vector at that point that differed materially from the actual one. On the assumption of Fresnel's form for representing the elementary waves we could not, therefore, expect to obtain the correct expression for the light vector at an external point for the cases of diffraction, where a given portion of the (light) waves is always intercepted. That the results obtained by Fresnel's theory of diffraction agree with observation is beyond all doubts; but we must realize that these results are only relative or qualitative ones; they give only qualitatively the phenomena sought, a result we might easily have anticipated, for, although an incorrect expression was chosen for the form of the elementary waves, the very principles upon which the phenomena of diffraction depend were recognized by Fresnel and embodied in his theory.

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* Cf. Preston, *The Theory of Light*, pp. 243-287.

Theory of Diffraction based on Integral Expression for Light-Vector.—Henceforth we shall base our treatment of diffraction phenomena on the formula (51) derived in the preceding chapter, which expresses the light-vector at any point as the integral of a given expression taken over any surface that encloses that point. First of all, this formula must be made to conform to the conditions prevailing in the given medium; this will necessitate the introduction of certain modifications and assumptions due to the presence of the foreign bodies or obstacles, to which the phenomena of diffraction are ascribed. Let us now consider the form of the expression under the integral sign in formula (51, V.) at any point M on the surface S enclosing the point Q, at which the light-vector s_0 shall be sought, when obstacles are placed in the course of the waves emitted by the source of disturbance O (see Fig. 19). We have now seen in Chapter V. that for the simple case, where no obstacles intercepted the waves from O, an evaluation of the given integral was possible, when we chose as surface of integration S a sphere with centre at the point Q, and we found there, in fact, the correct expression for the vector at that point (cf. Ex. 9, p. 183). If we insert obstacles, which we shall assume are opaque and reflect no waves, in the medium, it is evident that a certain portion of the waves emitted by the source O will be intercepted and hence that the resultant vector at Q will be given by a different expression from that already found for the simple case, where there was no obstruction of the waves throughout the medium. The first question that arises upon the insertion of an opaque obstacle, as a screen, in the medium is, how shall we lay the surface of integration S with respect to that screen? A knowledge of the values of the vector s and $\frac{\partial s}{\partial n}$ along the whole surface of integration chosen, where n denotes the inner normal to that surface, is now evidently necessary for an evaluation of the integral of formula (51, V.). This can be had, if we lay the surface of integration parallel and very near to that side of the obstructing screen that is turned away from the source of disturbance, for we can then assume most approximately that both s and $\frac{\partial s}{\partial n}$ vanish at every point on that surface that is sheltered from the action of the waves or is directly behind the screen (as seen from the source) and that at all other points on the same they retain the same values as those assumed by these quantities, when there is no obstruction; at points not directly behind the screen the surface of integration can evidently be laid as best suited for an evaluation of the given integral. This method of treatment of the obstructing screen and the assumptions embodied in the same are evidently similar to those employed by Fresnel (cf. pp. 144-153). The assumption that s and $\frac{\partial s}{\partial n}$ both vanish directly behind the screen but retain the same values at all other points as in the case, where there is no obstruction of the waves, will evidently be realized only, when the obstructing screen or the aperture in the large opaque screen is large compared with the wave-length of the waves employed; the results obtained on this assumption will then, in fact, be found to agree with observation (see below). On the other hand, if the obstructing screen or the aperture in the large screen is very small, the above assumption cannot well be maintained, since that portion of the given surface of integration or wave-front (according to Fresnel) that comes into consideration will then be so very small (of the dimensions of the given screen or aperture), that the vanishing of s and $\frac{\partial s}{\partial n}$ behind such a screen and the assumption of their natural values in the very next proximity or the assumption of their natural values in the small aperture and their vanishing in its very next proximity could not well be realized.* The theory in question is, therefore, only an approximate one and, like Fresnel's method employed above, it can be applied only to those problems on diffraction, where the obstructing screen or the aperture is not very small, at least compared with the wave-length.

General Problem on Diffraction: Diffraction on Small Aperture in Large Opaque Screen.—Let us now consider the above theory of diffraction, that based on formula (51, V.) and on the validity of the above assumption for the values of s and $\frac{\partial s}{\partial n}$ directly behind obstructing bodies and at other points on the surface S (cf. Fig. 19); that is, let us determine the light vector s_0 at any external point Q, expressed as integral of a given expression over any surface S enclosing that point, for the different problems on diffraction. Let us, first, examine the case, where a large opaque plane screen with a small aperture is placed between the source of disturbance O and the point of observation Q, which latter shall lie in the region, to which the diffraction phenomena sought are confined; the screen shall be so large that it intercepts all the waves from the given source except those that pass through the aperture, whereas the dimensions of the aperture shall be very small compared with the distances ρ of the source from the same on its one side and r of the point of observation Q on its other, as indicated in the annexed figure. Let now the plane of the large obstructing screen be chosen as xy-plane of a system of rectangular coordinates x, y, z, whose origin shall be taken so near the given aperture that the distance of any point M in the same will be small compared with the distances ρ_1 and ρ_0 of the points O and Q from that origin. We denote the coordinates of the source O by x_1, y_1, z_1, of the point of observation Q by x_0, y_0, z_0, and of any point M in the aperture by x, $y (z=0)$. The distances ρ_1, ρ_0, ρ and r will then be given by the expressions

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* This could also be stated as follows: When the screen or aperture is very small, its edges and the behaviour of s and $\frac{\partial s}{\partial n}$ at and near the same will alone come into consideration, whereas, when the screen or aperture is large, its edges will constitute so small a part of the screened or open portion respectively of the surface of integration or wave-front, that the behaviour of these quantities at and near the edges will not come into consideration. For this reason the above assumption on the behaviour of s and $\frac{\partial s}{\partial n}$ can evidently be maintained only in the latter case.

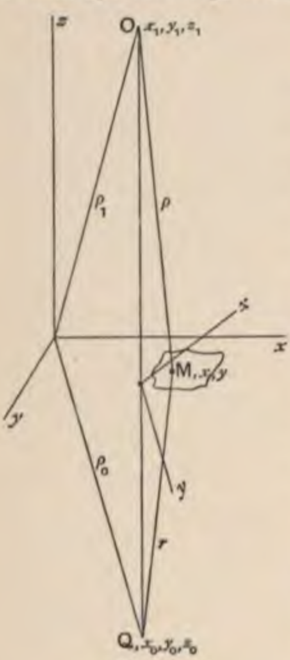

FIG. 24.

$$\left.\begin{aligned} \rho_1^2 &= x_1^2+y_1^2+z_1^2 \\ \rho_0^2 &= x_0^2+y_0^2+z_0^2 \\ \rho^2 &= (x_1-x)^2+(y_1-y)^2+z_1^2 \\ \text{and} \qquad r^2 &= (x_0-x)^2+(y_0-y)^2+z_0^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

By the first two of these relations we can write the last two in the form

$$\rho^2 = \rho_1^2 + x^2 + y^2 - 2(xx_1 + yy_1)$$

and

$$r^2 = \rho_0^2 + x^2 + y^2 - 2(xx_0 + yy_0),$$

hence and

$$\left.\begin{aligned}\rho&=\rho_1\sqrt{1+\frac{x^2+y^2-2(xx_1+yy_1)}{\rho_1{}^2}}\\ r&=\rho_0\sqrt{1+\frac{x^2+y^2-2(xx_0+yy_0)}{\rho_0{}^2}}\end{aligned}\right\}\text{..................(11)}$$

Since now, by assumption, x and y are small compared with ρ_1 and ρ_0, the second terms of the expressions under the square-root signs of these expressions for ρ and r will be small compared with their first ones or unity; we obtain, therefore, approximate values for ρ and r, on developing these expressions by the binomial theorem according to the ascending powers of their second terms and on rejecting the terms of the higher orders of magnitude in x/ρ_1, y/ρ_1 and x/ρ_0, y/ρ_0 respectively; if we retain here the terms of only the null, first and second orders of magnitude in these quantities, we evidently have

$$\rho=\rho_1\left[1+\frac{1}{2}\,\frac{x^2+y^2-2(xx_1+yy_1)}{\rho_1{}^2}-\frac{1}{8}\,\frac{4(xx_1+yy_1)^2}{\rho_1{}^4}\right]$$

$$\left.\begin{aligned}&=\rho_1\left[1-\frac{xx_1+yy_1}{\rho_1{}^2}+\frac{x^2+y^2}{2\rho_1{}^2}-\frac{(xx_1+yy_1)^2}{2\rho_1{}^4}\right]\\ &\text{and similarly,}\\ r&=\rho_0\left[1-\frac{xx_0+yy_0}{\rho_0{}^2}+\frac{x^2+y^2}{2\rho_0{}^2}-\frac{(xx_0+yy_0)^2}{2\rho_0{}^4}\right]\end{aligned}\right\}\text{..................(12)}$$

If we denote the direction-cosines of the vectors ρ_1 and ρ_0 by α_1, β_1 and α_0, β_0 respectively, we can write these expressions for ρ and r in the form

$$\left.\begin{aligned}\rho&=\rho_1-(\alpha_1x+\beta_1y)+\frac{x^2+y^2}{2\rho_1}-\frac{(\alpha_1x+\beta_1y)^2}{2\rho_1}\\ \text{and}\quad r&=\rho_0-(\alpha_0x+\beta_0y)+\frac{x^2+y^2}{2\rho_0}-\frac{(\alpha_0x+\beta_0y)^2}{2\rho_0}\end{aligned}\right\}\text{..........(12A)}$$

Let us now employ formula (51, V.) for an examination of the given problem. If we assume that the light-vector s at any point on the surface of integration due to the action of the waves emitted by the source O, which shall lie outside that surface, be given by the expression

$$s=\frac{a}{\rho}\sin\frac{2\pi}{\lambda}(vt-\rho)$$

(cf. formula (52, V.)), where ρ denotes the distance of that point from the source, then the light-vector s_0 at any point Q within that surface will evidently be given, for light-waves, by the expression

$$s_0=\frac{a}{2\lambda}\int\frac{1}{\rho r}\left[\cos(n,\,\rho)-\cos(n,\,r)\right]\cos\frac{2\pi}{\lambda}\left[vt-(\rho+r)\right]ds\text{.......(13)}$$

(cf. formula (55, V.)), where, however, here ds shall denote any surface-element of the *unscreened* portion of the surface of integration chosen (cf. above); we choose here as surface of integration best suited for an evaluation of the given integral the xy-plane. For electric waves or waves of long wave length, the vector in question would be given by the more approximate formula (54, V.).

Since now the aperture in the large obstructing screen is, by assumption, very small compared with the distances ρ and r, the angles (n, ρ) and (n, r) in the expression (13) may be regarded as constant by the given integration, which is extended only over that aperture; for the same reasons ρ and r may also be regarded as constant by the given integration, except where they are divided by very small quantities, as the wave-length λ of the light-waves. We can, therefore, write the above integral (13) for s_0 here in the form

$$s_0 = \frac{a}{2\lambda}\,\frac{1}{\rho r}[\cos(n, \rho) - \cos(n, r)]\int \cos\frac{2\pi}{\lambda}[vt - (\rho + r)]\,ds. \quad \ldots\ldots(13\text{A})$$

By formulae (12A), which express ρ and r (most approximately) in terms of the coordinates x, y of any point M in the aperture, the expression to be integrated here can now be written

$$\begin{aligned}
&\cos\frac{2\pi}{\lambda}[vt - (\rho + r)]\\
&= \cos\frac{2\pi}{\lambda}\left\{vt - \left[\rho_1 + \rho_0 - (\alpha_1 + \alpha_0)x - (\beta_1 + \beta_0)y + \frac{x^2 + y^2}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_0}\right)\right.\right.\\
&\qquad\qquad\qquad\qquad\left.\left. - \frac{(\alpha_1 x + \beta_1 y)^2}{2\rho_1} - \frac{(\alpha_0 x + \beta_0 y)^2}{2\rho_0}\right]\right\}\\
&= \cos\frac{2\pi}{\lambda}[vt - (\rho_1 + \rho_0)]\cos[f(x, y)] - \sin\frac{2\pi}{\lambda}[vt - (\rho_1 + \rho_0)]\sin[f(x, y)],
\end{aligned}$$

where

$$\left.\begin{aligned}
f(x, y) = \frac{2\pi}{\lambda}\Big[(\alpha_1 + \alpha_0)x + (\beta_1 + \beta_0)y - \frac{x^2 + y^2}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_0}\right)\\
+ \frac{(\alpha_1 x + \beta_1 y)^2}{2\rho_1} + \frac{(\alpha_0 x + \beta_0 y)^2}{2\rho_0}\Big]
\end{aligned}\right\} \ldots\ldots(14)$$

We can, therefore, write formula (13A) in the form

$$\left.\begin{aligned}
s_0 = \frac{a}{2\lambda}\,\frac{1}{\rho r}[\cos(n, \rho) - \cos(n, r)]\Big\{\cos\frac{2\pi}{\lambda}[vt - (\rho_1 + \rho_0)]\int\cos[f(x, y)]\,ds\\
- \sin\frac{2\pi}{\lambda}[vt - (\rho_1 + \rho_0)]\int\sin[f(x, y)]\,ds\Big\}
\end{aligned}\right\}. \quad (15)$$

Light-Vector Resultant of two Systems of Waves.—By formula (15) we could conceive the light-vector s_0 as due to the mutual action of two systems of (elementary) waves emitted from the aperture s, whose difference in phase is $\pi/2$ and whose amplitudes are proportional to the integrals

$$\left.\begin{aligned} C&=\int\cos[f(x,\ y)]ds \\ \text{and} \qquad S&=\int\sin[f(x,\ y)]ds \end{aligned}\right\} \dots\dots\dots\dots(16)$$

To obtain a simpler form for $f(x,\ y)$ than the above (14), we lay the origin of our system of coordinates $x,\ y,\ z$ at that point of the xy-plane, where the same is intersected by the straight line joining the source O and the point of observation Q (cf. Fig. 24); the vectors ρ_1 and ρ_0 will then lie on one and the same straight line, but they will be oppositely directed; hence

$$a_1=-a_0 \quad \text{and} \quad \beta_1=-\beta_0$$

and formula (14) may then be written in the simpler form

$$f(x,\ y)=-\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)[x^2+y^2-(a_1x+\beta_1y)^2].$$

Lastly, we may choose the projection of the line OQ on the xy-plane as x-axis of a new system of rectangular coordinates (cf. Fig. 24); β_1 will then vanish, and hence $f(x,\ y)$ assume the form

$$\left.\begin{aligned} f(x,\ y)&=-\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2+y^2-a_1{}^2x^2) \\ &=-\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2) \end{aligned}\right\},\dots\dots\ \dots\dots(17)$$

where ϕ denotes the angle which the vector ρ_1 makes with the z-axis.

On referring formula (15) to this new uniquely determined system of coordinates, we can, therefore, write the same in the explicit form

$$\left.\begin{aligned} s_0=A\Big\{&\sin\omega\int\sin\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]ds \\ &+\cos\omega\int\cos\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]ds\Big\} \end{aligned}\right\},\dots\dots(18)$$

where $A=\dfrac{a}{2\lambda}\dfrac{1}{\rho r}[\cos(n,\ \rho)-\cos(n,\ r)]$ and $\omega=\dfrac{2\pi}{\lambda}[vt-(\rho_1+\rho_0)]$...(18A)

Diffraction on Straight Edge of Large Opaque Screen.—Let us, first, employ formula (18) for an examination of the diffraction on the straight edge of a large opaque screen; we shall see below that this problem may be treated as a particular case of the above general one, where namely the (small) aperture in the large obstructing screen is of quite arbitrary contour. We choose the straight obstructing edge of the screen parallel to the y-axis; the screen itself shall extend in the direction of the positive x-axis, from $x=x'$ to $x=+\infty$, and its diffracting edge (parallel to the y-axis) from $y=+\infty$ to $y=-\infty$, as indicated in the annexed figure. Let us now examine the expression (18) for the light-vector s_0 at any point Q outside the geometrical shadow and in the plane that passes through the given source O and intersects the large opaque screen in a line, which we shall choose as x-axis, that makes a right angle with its diffracting edge (cf. Fig. 25). This system of coordinates is evidently that uniquely determined one chosen above, the system to which formula (18) has been referred.

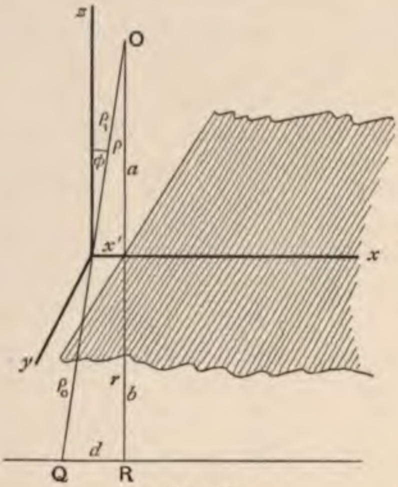

FIG. 25.

We have assumed above in the general development that the aperture in the large obstructing screen or the unscreened portion of the given surface of integration be small and that any point of the same be at a distance from the origin of our system of coordinates that is small compared with the distances ρ_1 and ρ_0 of the source O and the point of observation Q respectively from that origin. Here we can now image the unscreened portion of the given surface of integration or wave-front (the xy-plane) as divided into two regions, the one extending from the diffracting edge of the screen to only a short distance within its unscreened portion (in the direction of the negative x-axis) and the other from that line of division to $x=-\infty$; the former region shall include the effective portion of the given wave-front or that, which emits the waves, whose action at the point of observation determines most approximately the light-vector sought, and the latter its ineffective portion or that which contributes only inappreciably to that vector. We have now seen on p. 153 that the effective portion of any wave-front was confined alone to the region in the next proximity of the pole (with respect to the point Q); the dimensions of this region, although large compared with the wave-length of the waves employed, will always be small compared with the distances of the source and the point of observation from the same, and especially here, where these points have been assumed to be at considerable distance from the screen. Since now the effective portion of the given wave-front is small compared with the distances ρ_1 and ρ_0 and hence any point of the same at a distance from the given origin that is small compared with those distances, the above development will evidently hold here, that is, the formulae already derived may be employed for an examination of the given problem, and this problem may thus be regarded as a particular case of the above general one. At the same time, since the integration over the ineffective portion of the given wave-front or surface of integration will contribute approximately nothing to the value of the light-vector sought, the integration itself may be extended not only over the effective but also over the ineffective portion of the same, that is, here from $x=x'$ to $x=-\infty$.

We may, therefore, write the above expression (18) for s_0 here

$$\left.\begin{aligned} s_0=A\Bigg\{\sin\omega\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\sin\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dxdy \\ +\cos\omega\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\cos\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dxdy\Bigg\}\end{aligned}\right\}\ldots\ldots\ldots(19)$$

To evaluate the integrals of this expression for s_0, we, first, replace the variables x and y by the two new ones, v and u, where

$$\frac{1}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)x^2\cos^2\phi=\frac{v^2}{2}\text{ and }\frac{1}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)y^2=\frac{u^2}{2},$$

hence
$$dx=\frac{vdv}{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)x\cos^2\phi}=\frac{dv}{\cos\phi\sqrt{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}}$$

and
$$dy=\frac{udu}{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)y}=\frac{du}{\sqrt{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}},$$

and we have
$$\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\sin\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dxdy$$
$$=\frac{1}{\cos\phi\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\int_{-\infty}^{v'}\int_{-\infty}^{\infty}\sin\frac{\pi}{2}(v^2+u^2)dvdu,$$

and
$$\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\cos\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dxdy$$
$$=\frac{1}{\cos\phi\,\dfrac{2}{\lambda}\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho_0}\right)}\int_{-\infty}^{v'}\int_{-\infty}^{\infty}\cos\frac{\pi}{2}(v^2+u^2)dvdu,$$

where the upper limits v' are to be replaced by the expression

$$v'=x'\cos\phi\sqrt{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}. \quad\text{..............................(20)}$$

On expanding here the sine and the cosine of the angle $\frac{\pi}{2}(v^2+u^2)$ as functions of the sines and cosines of the single angles $\frac{\pi v^2}{2}$ and $\frac{\pi u^2}{2}$, we can write these integrals

$$\left.\begin{aligned}
&\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\sin\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dxdy\\
&\quad=\frac{1}{\cos\phi\,\dfrac{2}{\lambda}\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho_0}\right)}\left\{\int_{-\infty}^{\infty}\cos\frac{\pi u^2}{2}du\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right.\\
&\qquad\left.+\int_{-\infty}^{\infty}\sin\frac{\pi u^2}{2}du\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right\}\\
\text{and}\quad&\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\cos\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dxdy\\
&\quad=\frac{1}{\cos\phi\,\dfrac{2}{\lambda}\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho_0}\right)}\left\{\int_{-\infty}^{\infty}\cos\frac{\pi u^2}{2}du\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right.\\
&\qquad\left.-\int_{-\infty}^{\infty}\sin\frac{\pi u^2}{2}du\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right\}
\end{aligned}\right\}\text{......(21)}$$

Fresnel's Integrals and Cornu's Spiral.—The integrals that appear in formulae (21),

$$\left.\begin{aligned}
&\int\cos\frac{\pi u^2}{2}du=\int\cos\frac{\pi v^2}{2}dv=\xi\\
\text{and}\quad&\int\sin\frac{\pi u^2}{2}du=\int\sin\frac{\pi v^2}{2}dv=\eta
\end{aligned}\right\}\text{..................(22)}$$

—we denote them by ξ and η—are known as "Fresnel's Integrals." Since now these integrals appear in all problems on diffraction, a knowledge of the values they assume according to the limits of integration in question will be indispensable to a further treatment of the problems on diffraction; before we proceed further with the examination of the given problem, let us, therefore, consider the different properties of and the values assumed by these integrals according to the different limits of integration chosen. Let us, first, examine the two fundamental integrals

$$\xi=\int_0^v \cos\frac{\pi v^2}{2}\,dv \text{ and } \eta=\int_0^v \sin\frac{\pi v^2}{2}\,dv. \quad\ldots\ldots\ldots\ldots\ldots(22\text{A})$$

The values of these integrals will evidently vary according to the value of their upper limits v. Let us now choose these values, ξ and η, as coordinates of a system of rectangular coordinates and seek the curve described by the point ξ, η for different values of v. This curve will pass through the origin of these coordinates, since for $v=0$ both ξ and η evidently vanish. If we replace the upper limits v by $-v$, the values assumed by these integrals will evidently differ only in sign from those of the above (22A), that is, both ξ and η will change signs only; the curve sought will, therefore, be symmetrical with respect to the origin $\xi=\eta=0$, since for any and every pair of values ξ, η corresponding to any positive value of v there will always be a similar pair $-\xi$, $-\eta$ that corresponds to a negative v.

To determine the length of any element ds of the given curve, we observe that the projections of any such element on the ξ- and η-axes will be

$$d\xi=d\int_0^v \cos\frac{\pi v^2}{2}\,dv=\cos\frac{\pi v^2}{2}\,dv$$

and, similarly, $d\eta=\sin\dfrac{\pi v^2}{2}\,dv$;

hence $ds=\sqrt{d\xi^2+d\eta^2}=dv$

or integrated $s=v$ $\Big\}$, ……………………………(23)

provided the curve be measured from the origin, the lower limits zero.

The angle, which the tangent to the curve ξ, η at any point ξ, η (v) makes with the ξ-axis, will evidently be given by the expression

$$\tan\tau=\frac{d\eta}{d\xi}=\tan\frac{\pi v^2}{2},$$

hence $\tau=\dfrac{\pi v^2}{2}$. ……………………………………(24)

At the origin or for $s=v=0$ (cf. formula (23)) $\tau=0$, that is, the curve will leave the origin on either side running tangential to the ξ-axis; for $s=v=1$, $\tau=\pi/2$, that is, the curve will run here parallel to the η-axis; it will next run parallel to the ξ-axis for $\tau=\pi$, that is, for $v^2=2=s^2$, then again parallel to the η-axis for $\tau=\dfrac{3\pi}{2}$ or $v^2=3=s^2$, etc. (cf. Fig. 26).

The radius of curvature ρ of the curve ξ, η at any point (ξ, η) will evidently be given by

$$\rho=\frac{ds}{d\tau}=\frac{dv}{\pi v dv}=\frac{1}{\pi v}=\frac{1}{\pi s} \text{ ....}(25)$$

(cf. formulae (23) and (24)); that is, the radius of curvature will vary inversely as the length of the curve s from the origin. The curve described by the point ξ, η on either side of the origin will, therefore, be one whose radius of curvature decreases continously, as we recede from the origin along the same; such a curve is now a spiral or double spiral; that in question is known as "Cornu's spiral." Each spiral approaches now a given asymptotic point A or A' (cf. Fig. 26), closing in more and more on that point as the length of the curve s from the origin increases, until it finally for $s=v=\infty$ reaches that point.

Determination of Coordinates of the Asymptotic Points of Cornu's Spiral.—Let us, next, determine the coordinates of the asymptotic point A of Cornu's spiral (cf. Fig. 26); they are evidently given by the values of the integrals

$$\xi_A=\int_0^\infty \cos\frac{\pi v^2}{2}dv \text{ and } \eta_A=\int_0^\infty \sin\frac{\pi v^2}{2}dv. \text{ . ...}(26)$$

To evaluate these integrals, we make use of the integrals

$$\int_0^\infty e^{-x^2}dx=M, \text{ ....}(27)$$

$$\int_0^\infty e^{-y^2}dy=M,$$

and their product

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy=M^2. \text{ ....}(28)$$

We, first, seek the value of this double integral on the assumption that x and y are the rectangular coordinates of any point P (in the xy-plane); $dxdy$ will then denote any rectangular surface-element of the xy-plane. To evaluate this integral, we replace the surface-element $dxdy$ by the surface-element do bounded on the one hand by any two vectors r, that subtend the infinitesimal angle $d\phi$ at the origin, and on the other hand by the segments of arc, intercepted by those vectors, of any two circles of radii, that differ by the infinitesimal quantity dr. The area of any such element do will evidently be

$$do=rdrd\phi.$$

On replacing the rectangular coordinates x, y by these new ones r and ϕ, we can write the above double integral in the form

$$\int_0^{\frac{\pi}{2}}d\phi\int_0^\infty e^{-r^2}rdr=M^2.$$

This integral can now easily be evaluated; we find namely, since

$$\int_0^\infty e^{-r^2} r\,dr = \left| -\frac{1}{2}e^{-r^2} \right|_0^\infty = \frac{1}{2}$$

and
$$\int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{2},$$

that
$$\int_0^{\frac{\pi}{2}} d\phi \int_0^\infty e^{-r^2} r\,dr = \frac{\pi}{4} = M^2,$$

hence
$$M = \frac{1}{2}\sqrt{\pi}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

We, next, replace the variable x in integral (27) by a new variable v, where the following relation shall hold between these two variables:

$$x = \sqrt{-\frac{i\pi^2}{2}}\, v,$$

hence
$$dx = \sqrt{-\frac{i\pi^2}{2}}\, dv,$$

where i shall denote the imaginary unit $\sqrt{-1}$; and we have

$$\int_0^\infty e^{-x^2} dx = \sqrt{-\frac{i\pi}{2}} \int_0^\infty e^{i\frac{\pi v^2}{2}} dv = M = \frac{1}{2}\sqrt{\pi}$$

(cf. formula (29)), which gives

$$\int_0^\infty e^{i\frac{\pi v^2}{2}} dv = \frac{1}{\sqrt{2}\sqrt{-i}}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

We can now write the value of this integral in another form, one, in which the real and the imaginary parts appear separate. To accomplish this, we observe that

$$\frac{1}{\sqrt{-i}} = \sqrt{i}$$

and
$$\sqrt{i} = \frac{2i}{2\sqrt{i}} = \frac{1 + 2i + i^2}{2\sqrt{i}},$$

which gives
$$2i = (1 + i)^2,$$

and hence
$$\sqrt{i} = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{-i}}.$$

Replace $\frac{1}{\sqrt{-i}}$ in formula (30) by this value $\frac{1+i}{\sqrt{2}}$, and we have

$$\int_0^\infty e^{i\frac{\pi v^2}{2}} dv = \frac{1+i}{2}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30\text{A})$$

Since now
$$e^{i\frac{\pi v^2}{2}} = \cos\frac{\pi v^2}{2} + i\sin\frac{\pi v^2}{2},$$

we can write this integral in the form

$$\int_0^\infty \left(\cos\frac{\pi v^2}{2} + i \sin\frac{\pi v^2}{2}\right) dv = \frac{1+i}{2}.$$

Lastly, since both the real and the imaginary parts on the two sides of any equation must always be equal, it follows from this integral equation that

$$\left.\begin{aligned} &\int_0^\infty \cos\frac{\pi v^2}{2}\,dv = \frac{1}{2} \\ \text{and}\qquad &\int_0^\infty \sin\frac{\pi v^2}{2}\,dv = \frac{1}{2} \end{aligned}\right\}, \qquad\qquad (31)$$

the integrals, whose values were sought.

The coordinates of the asymptotic point A of the spiral described by the point

$$\xi = \int_0^v \cos\frac{\pi v^2}{2}\,dv, \quad \eta = \int_0^v \sin\frac{\pi v^2}{2}\,dv,$$

will, therefore, be

$$\xi_A = \eta_A = \frac{1}{2} \qquad\qquad (31\text{A})$$

(cf. formulae (26)); this point is thus situated on the line bisecting the right angle between the coordinate-axes ξ and η.

Construction of Cornu's Spiral.—To plot Cornu's spiral, we start at the origin and lay off the distance $s = 0{\cdot}1$ on the ξ-axis (cf. Fig. 26). From this point we draw a straight line intersecting the η-axis at the angle

$$\tau = \frac{\pi s^2}{2} = \frac{\pi}{2}\,0{\cdot}01 = \pi\,0{\cdot}005 \quad (s = 0{\cdot}1)$$

(cf. formula (24)), and describe with this point of intersection as centre a circle of radius

$$\rho = \frac{1}{\pi s} = \frac{10}{\pi} \quad (s = 0{\cdot}1)$$

(cf. formula (25)); this circle will pass (most approximately) through the origin of our coordinates. On this circle we lay off from the origin the arc of length $s = 0{\cdot}1$; at the end of this arc $s = 0{\cdot}1$ the given circle will evidently have the radius of curvature of the spiral sought at that point of the same, which is at the distance $s = 0{\cdot}1$ from the origin measured along the spiral. On joining this point with the origin (by a curve whose radius of cur-

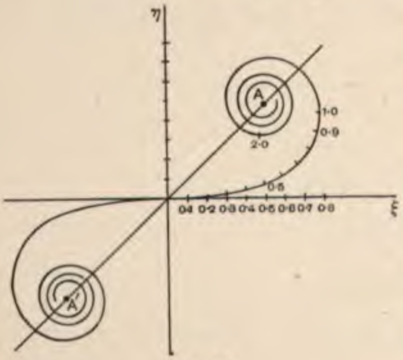

Fig. 26.

vature increases from that of the spiral at the point $s=0{\cdot}1$ to $\rho=\infty$ at the origin), we obtain the first portion of the given spiral.

To obtain the next portion of the spiral, we continue the curve from the point $s=0{\cdot}1$ of the same, by describing a circle of radius $\rho=\frac{5}{\pi}$ $(s=0{\cdot}2)$, whose centre is to be determined similarly to that of the first circle, lay off from the end of the first portion of the spiral already plotted the arc of length $s=0{\cdot}1$ on the same and join the end of this arc with that of the first one by a curve of the radius of curvature in question. Similarly, we can construct the successive portions of the given spiral.

Evaluation of Fresnel's Integrals.—The above method of constructing Cornu's spiral, like the other methods based on the approximate evaluations of the integrals themselves (cf. Exs. 6-9), is evidently only an approximate one. The values of Fresnel's integrals, as determined by Gilbert,* are given in the following table:

v	$\xi=\int_0^v \cos\frac{\pi v^2}{2}\,dv$	$\eta=\int_0^v \sin\frac{\pi v^2}{2}\,dv$	v	$\xi=\int_0^v \cos\frac{\pi v^2}{2}\,dv$	$\eta=\int_0^v \sin\frac{\pi v^2}{2}\,dv$
0·0	0·0000	0·0000	2·6	0·3389	0·5500
0·1	0·0999	0·0005	2·7	0·3926	0·4529
0·2	0·1999	0·0042	2·8	0·4675	0·3915
0·3	0·2994	0·0141	2·9	0·5624	0·4102
0·4	0·3975	0·0334	3·0	0·6057	0·4963
0·5	0·4923	0·0647	3·1	0·5616	0·5818
0·6	0·5811	0·1105	3·2	0·4663	0·5933
0·7	0·6597	0·1721	3·3	0·4057	0·5193
0·8	0·7230	0·2493	3·4	0·4385	0·4297
0·9	0·7648	0·3398	3·5	0·5326	0·4153
1·0	0·7799	0·4383	3·6	0·5880	0·4923
1·1	0·7638	0·5365	3·7	0·5419	0·5750
1·2	0·7154	0·6234	3·8	0·4481	0·5656
1·3	0·6386	0·6863	3·9	0·4223	0·4752
1·4	0·5431	0·7135	4·0	0·4984	0·4205
1·5	0·4453	0·6975	4·1	0·5737	0·4758
1·6	0·3655	0·6383	4·2	0·5417	0·5632
1·7	0·3238	0·5492	4·3	0·4494	0·5540
1·8	0·3337†	0·4509	4·4	0·4383	0·4623
1·9	0·3945	0·3734	4·5	0·5258	0·4342
2·0	0·4883	0·3434	4·6	0·5672	0·5162
2·1	0·5814	0·3743	4·7	0·4914	0·5669
2·2	0·6362	0·4556	4·8	0·4338	0·4968
2·3	0·6268	0·5525	4·9	0·5002	0·4351
2·4	0·5550	0·6197	5·0	0·5636	0·4992
2·5	0·4574	0·6192	∞	0·5000	0·5000

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* Gilbert, *Mém. couronnés de l'Acad. de Bruxelles*, tom. xxxi., p. 1, 1863.

† The value 0·3363 found by Gilbert for the integral ξ for $v=1{\cdot}8$ varies appreciably from those determined subsequently by other methods; we have, therefore, replaced it in the above table by the more exact value 0·3337 found by C. W. Wind (cf. *Phys. Zeitschrift*, No. 18, p. 265).

This table shows that the given integrals pass through a series of maxima and minima, which become less and less marked as we approach the asymptote point A. Lommel[*] has evaluated Fresnel's integrals to the sixth decimal for the variable (upper limit) $z = \frac{\pi v^2}{2}$ for $z = 0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}3 \ldots 0{\cdot}9,\ 1{\cdot}0,\ 1{\cdot}5,\ 2{\cdot}0,\ 2{\cdot}5 \ldots 49{\cdot}5,\ 50$, which correspond to 110 values of v between 0·064 and 5·642.

The Given Problem.—Let us now return to the examination of the above problem on diffraction, which we were obliged to interrupt on account of the appearance of Fresnel's integrals (cf. formulae (21)), which we have just investigated.

The integrals of formulae (21), whose limits are $-\infty$ and $+\infty$, can evidently be evaluated at once; we have namely

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \cos\frac{\pi u^2}{2}\,du = 2\int_0^{\infty} \cos\frac{\pi u^2}{2}\,du = 1 \\ &\text{(cf. formulae (31)), and, similarly,} \\ &\int_{-\infty}^{\infty} \sin\frac{\pi u^2}{2}\,du = 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(32)$$

We can, therefore, write formulae (21)

$$\begin{aligned} &\int_{-\infty}^{x'}\int_{-\infty}^{\infty} \sin\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi + y^2)\right]dx\,dy \\ &\qquad = \frac{1}{\cos\phi\,\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\left\{\int_{-\infty}^{v'} \sin\frac{\pi v^2}{2}dv + \int_{-\infty}^{v'} \cos\frac{\pi v^2}{2}dv\right\} \\ &\int_{-\infty}^{x'}\int_{-\infty}^{\infty} \cos\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi + y^2)\right]dx\,dy \\ &\qquad = \frac{1}{\cos\phi\,\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\left\{\int_{-\infty} \cos\frac{\pi v^2}{2}dv - \int_{-\infty}^{v'} \sin\frac{\pi v^2}{2}dv\right\}, \end{aligned}$$

and hence formula (19) for s_0

$$\left.\begin{aligned} s_0 = \frac{A}{\cos\phi\,\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}&\left\{\sin\omega\left[\int_{-\infty}^{v'} \sin\frac{\pi v^2}{2}dv + \int_{-\infty}^{v'} \cos\frac{\pi v^2}{2}dv\right]\right. \\ &\left. + \cos\omega\left[\int_{-\infty}^{v'} \cos\frac{\pi v^2}{2}dv - \int_{-\infty}^{v'} \sin\frac{\pi v^2}{2}dv\right]\right\} \end{aligned}\right\}. \quad \ldots.(33)$$

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[*] v. Lommel: *Abhandlungen der math.-phys. Classe der kgl. bayr. Akademie der Wissenschaften*, Band 15, Tabelle iii., p. 648.

This light-vector s_0, like the light-vector s_0 of formula (15), can now be conceived as due to the mutual action of two systems of (elementary) waves, whose difference in phase is $\pi/2$ and whose amplitudes are

$$\frac{A}{\cos\phi\,\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\left\{\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}\,dv+\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}\,dv\right\}$$

and

$$\frac{A}{\cos\phi\,\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\left\{\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}\,dv-\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}\,dv\right\}.$$

The Resultant Intensity.—According to the principle of interference the resultant intensity I produced by the mutual action of two waves, whose difference in phase is $\pi/2$, is now proportional to the sum of the amplitudes squared of the given single waves (cf. formula (4, IV)). The resultant intensity I of the two systems of waves, to whose mutual action the light-vector s_0 of formula (33) is conceived as due, will, therefore, be proportional to the expression

$$\begin{aligned}&\frac{A^2\lambda^2}{4\cos^2\phi\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)^2}\\ &\qquad\times\left\{\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}\,dv+\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}\,dv\right]^2+\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}\,dv-\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}\,dv\right]^2\right\}\\ &\qquad=\frac{A^2\lambda^2}{2\cos^2\phi\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\left\{\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}\,dv\right]^2+\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}\,dv\right]^2\right\},\end{aligned}$$

or, by formula (18A),

$$\left.\begin{aligned}&=\frac{a^2[\cos(n,\rho)-\cos(n,r)]^2}{8\rho^2r^2\cos^2\phi}\,\frac{{\rho_1}^2{\rho_0}^2}{(\rho_1+\rho_0)^2}\\ &\qquad\times\left\{\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}\,dv\right]^2+\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}\,dv\right]^2\right\}\text{ prop. to } I\end{aligned}\right\}.\quad\ldots\ldots(34)$$

Expression for the Resultant Intensity.—Let us now examine the expression (34) for the intensity. We have now seen on p. 205 that only those elements of the surface of integration or half-period elements of the given wave-front come into consideration that are in the immediate neighbourhood of the pole on that surface or wave-front with respect to the point of observation Q (see Fig. 25), which pole has been chosen above as origin of our coordinates (cf. p. 203 and Fig. 25). We can, therefore, replace most approximately ρ by ρ_1 and r by ρ_0 in the *coefficient* of the above expression

(34) for I (cf. also p. 155). This coefficient will then assume the form

$$\frac{a^2[\cos(n, \rho_1) - \cos(n, \rho_0)]^2}{8\cos^2\phi(\rho_1 + \rho_0)^2}.$$

Moreover, since the origin of our coordinates has been laid on the line joining the source O and the point of observation Q (cf. Fig. 24), we evidently have

$$\cos(n, \rho_1) = -\cos(n, \rho_0) = \cos\phi,$$

and the given coefficient will then reduce to the simple form

$$\frac{a^2}{2(\rho_1 + \rho_0)^2}.$$

The expression (34) can thus be written (most approximately)

$$I \text{ prop. to } \frac{a^2}{2(\rho_1 + \rho_0)^2}\left\{\left[\int_{-\infty}^{w} \sin\frac{\pi v^2}{2} dv\right]^2 + \left[\int_{-\infty}^{w} \cos\frac{\pi v^2}{2} dv\right]^2\right\}. \quad \text{......(35)}$$

For the determination of I for waves of long wave-length see Ex. 5 at end of chapter.

Geometrical Interpretation of Expression for Resultant Intensity.—We can now interpret geometrically the integral expression in the largest brackets of formula (35) for I by means of Cornu's spiral. For this purpose we denote the coordinates of any two points of the (Cornu's) spiral of Fig. 26 by ξ, η and ξ', η', where

$$\left.\begin{array}{ll} & \xi = \int_0^v \cos\frac{\pi v^2}{2} dv, \quad \eta = \int_0^v \sin\frac{\pi v^2}{2} dv \\ \text{and} & \xi' = \int_0^{v'} \cos\frac{\pi v^2}{2} dv, \quad \eta' = \int_0^{v'} \sin\frac{\pi v^2}{2} dv \end{array}\right\}. \quad \text{..................(36)}$$

The direct distance Δ between any two such points is evidently determined by the expression

$$\left.\begin{aligned} \Delta^2 &= (\xi' - \xi)^2 + (\eta' - \eta)^2 \\ &= \left[\int_0^{v'} \cos\frac{\pi v^2}{2} dv - \int_0^{v} \cos\frac{\pi v^2}{2} dv\right]^2 + \left[\int_0^{v'} \sin\frac{\pi v^2}{2} dv - \int_0^{v} \sin\frac{\pi v^2}{2} dv\right]^2 \\ &= \left[\int_v^{v'} \cos\frac{\pi v^2}{2} dv\right]^2 + \left[\int_v^{v'} \sin\frac{\pi v^2}{2} dv\right]^2 \end{aligned}\right\}, \quad (37)$$

which we can interpret as follows: The sum of the squares of the two (Fresnel's) integrals

$$\int_v^{v'} \cos\frac{\pi v^2}{2} dv \quad \text{and} \quad \int_v^{v'} \sin\frac{\pi v^2}{2} dv$$

can be represented by the square of the direct distance between the two points of Cornu's spiral (cf. Fig. 26), whose coordinates ξ, η and ξ', η' are determined by the values of the integrals (36).

For $v = -\infty$ formula (37) will assume the form

$$\Delta^2 = (\xi' - \xi_{A'})^2 + (\eta' - \eta_{A'})^2 = \left[\int_{-\infty}^{v'} \cos\frac{\pi v^2}{2}\,dv\right]^2 + \left[\int_{-\infty}^{v'} \sin\frac{\pi v^2}{2}\,dv\right]^2,$$

where $\xi_{A'}$, $\eta_{A'}$ denote the coordinates of the asymptotic point A' of the spiral in the quadrant $-\xi$, $-\eta$; this integral expression is now the one in the largest brackets of formula (35). We thus have

$$\left.\begin{aligned} I \text{ prop. to } & \frac{a^2}{2(\rho_1+\rho_0)^2}[(\xi' - \xi_{A'})^2 + (\eta' - \eta_{A'})^2] \\ & = \frac{a^2}{2(\rho_1+\rho_0)^2}[(\xi' + 1/2)^2 + (\eta' + 1/2)^2] \end{aligned}\right\} \dots\dots\dots\dots(38)$$

(cf. formula (31)), that is, the intensity at the given point will be proportional to the square of the direct distance of the point ξ', η' from the asymptotic point $\xi_{A'} = \eta_{A'} = -1/2$.

Exterior Diffraction Bands.—It is evident from the form of the given (Cornu's) spiral (cf. Fig. 26) that the distance of any point ξ', η' of the same from the asymptotic point $\xi_{A'}$, $\eta_{A'}$ will pass through a series of maxima and minima, as we recede from the origin, $\xi = \eta = 0$ $(v' = 0)$, along the spiral towards the asymptotic point $\xi_A = \eta_A = 1/2$ $(v' = \infty)$, that is, as the distance x' of the obstructing edge of the given screen is increased from $x' = 0$ to $x' = \infty$; positive values of x' correspond now to points of observation Q outside the geometrical shadow (cf. Fig. 25). On the other hand, for negative values of v' or x', which will correspond to points Q inside the geometrical shadow, the distance of any point ξ', η' of the spiral will evidently diminish *continuously*, without passing through maxima or minima, as we recede from the origin along the spiral towards the asymptotic point $\xi_{A'} = \eta_{A'} = -1/2$ $(v' = -\infty)$. These results, which are confirmed by observation, are similar to those obtained by the former less exact method (cf. pp. 187, 188).

Approximate Determination of Distribution of Bands.—The exact determination of the position of the maxima and minima of intensity (outside the geometrical shadow) would require the determination of the points of the spiral in the quadrant ξ, η that are furthest and nearest respectively to the asymptotic point $\xi_{A'} = \eta_{A'} = -1/2$; this problem has been solved in all its details by Lommel.* These points may now be determined approximately; approximately they will evidently be the points of intersection of the given spiral and the straight line bisecting the quadrants ξ, η and $-\xi$, $-\eta$ of Fig. 26 and passing through the asymptotic points of both spirals. Since this line cuts the given spiral approximately orthogonally (cf. Fig. 26), the angle τ, which the tangent to the spiral at any such point makes with the ξ-axis, will evidently be approximately

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* "Die Beugungserscheinungen einer kreisrunden Oeffnung und eines kreisrunden Schirmchens etc.," and "Die Beugungserscheinungen gradlinig begrenzter Schirme," in the *Abhandlungen der math.-phys. Classe der kgl. bayr. Akademie der Wissenschaften*, pp. 233-329 and pp. 531-664 respectively.

$$\tau=(3/4+2h)\pi$$

for maxima of distance Δ or intensity I and

$$\tau=(7/4+2h)\pi$$

for minima, where $h=0, 1, 2, \dots$.

By formula (24), $$\tau=\frac{\pi v'^2}{2};$$

these maxima and minima will, therefore, correspond to the following values of v':

$$\left.\begin{aligned} v'_{\text{max.}}&=\sqrt{\frac{2\tau}{\pi}}=\sqrt{\frac{3}{2}+4h} \\ \text{and}\qquad v'_{\text{min.}}&=\sqrt{\frac{7}{2}+4h}. \end{aligned}\right\}\qquad\text{(39)}$$

These different values of v' correspond now to different positions of the point of observation Q; let us, next, determine v' in terms of the position (with regard to source and screen) of any such point. For this purpose we assume that the vector from the source O to the obstructing edge of the given screen coincide with the normal to the screen at that edge; this assumption is evidently consistent with the construction represented in Fig. 25. Let us denote this shortest distance from source to screen by a (cf. Fig. 25). We now continue the vector from O through the obstructing edge of the screen downwards till it meets the line (screen of observation) drawn through the point Q parallel to the x-axis at the point R (cf. Fig. 25); this point R marks the edge of the geometrical shadow on the screen of observation. Let us denote the distance of the point R from the obstructing edge of the given screen by b and the distance of the point of observation Q from R by d. It follows then from Fig. 25 that

$$a:a+b=x':d, \qquad\text{(40)}$$

and $$a:\rho_1=\cos\phi.$$

By formula (20), we have now

$$v'=x'\cos\phi\sqrt{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}.$$

Replace here x' by its value from formula (40), and we have

$$v' = \frac{ad}{a+b}\cos\phi\sqrt{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}.$$

For points Q near the edge of the geometrical shadow R—only such points come into consideration here (cf. p. 205)—we may now replace ρ_1 by a and ρ_0 by b most approximately and hence put

$$\cos\phi = a/\rho_1 = 1.$$

We can then write v' most approximately

$$v' = \frac{ad}{a+b}\sqrt{\frac{2}{\lambda}\left(\frac{1}{a}+\frac{1}{b}\right)} = d\sqrt{\frac{2a}{\lambda b(a+b)}}, \qquad\qquad (41)$$

that is, v' will be directly proportional to the distance d of the point of observation Q from the edge of the geometrical shadow R.

By formulae (39) maxima and minima of intensity will appear for the following particular values of v':

$$v'_{\text{max.}} = \sqrt{\frac{3}{2}+4h} \quad\text{and}\quad v'_{\text{min.}} = \sqrt{\frac{7}{2}+4h}.$$

Replace here v' by its value (41) in terms of the distance d, and we have

$$v'_{\text{max.}} = d_{\text{max.}}\sqrt{\frac{2a}{\lambda b(a+b)}} = \sqrt{\frac{3}{2}+4h},$$

hence

$$d_{\text{max.}} = \sqrt{\frac{3}{2}+4h}\,\sqrt{\frac{\lambda b(a+b)}{2a}},$$

and

$$v'_{\text{min.}} = d_{\text{min.}}\sqrt{\frac{2a}{\lambda b(a+b)}} = \sqrt{\frac{7}{2}+4h},$$

hence

$$d_{\text{min.}} = \sqrt{\frac{7}{2}+4h}\,\sqrt{\frac{\lambda b(a+b)}{2a}}.$$

On evaluating here the expressions $\sqrt{3/2+4h}$ and $\sqrt{7/2+4h}$ for the different values of h, $h=0, 1, 2, \ldots$, we find the following values for the d's:

$$d_{0\,\text{max.}} = p\,1{\cdot}225,\quad d_{1\,\text{max.}} = p\,2{\cdot}345,\quad d_{2\,\text{max.}} = p\,3{\cdot}082,\quad d_{3\,\text{max.}} = p\,3{\cdot}674,$$

and $d_{0\,\text{min.}} = p\,1{\cdot}871,\quad d_{1\,\text{min.}} = p\,2{\cdot}739,\quad d_{2\,\text{min.}} = p\,3{\cdot}391,\quad d_{3\,\text{min.}} = p\,3{\cdot}937,$

where

$$p = \sqrt{\frac{\lambda b(a+b)}{2a}}. \qquad\qquad (42)$$

These approximate values for the distances of the first maxima and minima of intensity from the boundary of the geometrical shadow differ only inappreciably from the exacter ones, found by Lommel;*

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* "Die Beugungserscheinungen geradlinig begrenzter Schirme"; *Abhandlungen der math.-phys. Classe der kgl. bayr. Akademie der Wissenschaften*, Band XV., Tabelle XXIa, p. 662.

the latter values for these distances, expressed in terms of the quantity x, where $x=\frac{\pi d^2}{p^2}$, are

4·654, 17·268, 29·840, 42·409,
and 11·015, 23·569, 36·132, 48·698 respectively,

where we have retained only the first three decimals; these values correspond to the following values of d:

$p1·217$, $p2·344$, $p3·082$, $p3·674$,
and $p1·873$, $p2·739$, $p3·391$, $p3·937$ respectively.

Intensity of Maxima and Minima.—The intensities of the above maxima and minima will now, by formula (38), be proportional approximately to the squares of the direct distances of the points of intersection of the line $A'A$ and the upper (in the quadrant ξ, η) (Cornu's) spiral from the asymptotic point A' of the lower spiral. If we choose the natural intensity as unity, we find,* on measuring these distances, the following values for the maximum and minimum intensities in question:

$I_{0\,\text{max.}}=1·34$, $I_{1\,\text{max.}}=1·20$, $I_{2\,\text{max.}}=1·16$,
and $I_{0\,\text{min.}}=0·78$, $I_{1\,\text{min.}}=0·84$, $I_{2\,\text{min.}}=0·87$.

These values also differ only inappreciably from those found by the exacter method. Lommel† has found the following values for these intensities:

1·370, 1·199, 1·151,
and 0·778, 0·843, 0·872,

where we have retained only the first three decimals.

Diffraction on Narrow Slit in Large Opaque Screen.—Let us, next, examine the behaviour of the intensity after the passage of light-waves through a narrow slit in a large opaque screen. We can now regard the preceding problem as a particular case of the given one, the slit being conceived in the former as so broad that there is a diffraction of the waves on its one edge only. The given problem will, therefore, differ from the preceding one only therein that the lower limits of integration, instead of being $-\infty$ as in formula (35), will be x_2' (v_2'), the distance of the other edge of the slit from the origin. On replacing the lower limits $-\infty$ by x_2' or v_2' in the formulae deduced above, we obtain, therefore, the formulae sought for the given problem. Formula (38) for the resultant intensity then becomes

$$I \text{ prop. to } \frac{a^2}{2(\rho_1+\rho_0)^2}[(\xi_1'-\xi_2')^2+(\eta_1'-\eta_2')^2], \quad \ldots\ldots\ldots\ldots(43)$$

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* Cf. Drude: *Lehrbuch der Optik*, p. 183.

† Cf. paper cited in footnote, p. 217.

where ξ_1', η_1' and ξ_2', η_2' denote the coordinates of the two points of Cornu's spiral that are determined by the values of the integrals

$$\xi_1' = \int_0^{v_1'} \cos\frac{\pi v^2}{2}\,dv, \quad \eta_1' = \int_0^{v_1'} \sin\frac{\pi v^2}{2}\,dv$$

and

$$\xi_2' = \int_0^{v_2'} \cos\frac{\pi v^2}{2}\,dv, \quad \eta_2' = \int_0^{v_2'} \sin\frac{\pi v^2}{2}\,dv$$

(cf. formulae (36)), where

$$v_1' = x_1'\cos\phi\sqrt{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)} \text{ and } v_2' = x_2'\cos\phi\sqrt{\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)} \quad \ldots.(44)$$

(cf. formula (20)), x_1' and x_2' denoting the distances of the two edges of the slit from the given origin, as indicated in Fig. 27. The intensity at Q will thus vary directly as the square of the direct distance between the two points ξ_1', η_1' and ξ_2', η_2'.

Let now the source O be so chosen that the centre of the given slit and not the (one) diffracting edge of the screen, as in the preceding problem, be nearest to the same, as indicated in the annexed figure. Since the origin of our coordinates has been chosen on the line joining the source O and the point of observation Q (cf. p. 203), x_1' and x_2' will evidently be opposite in sign for any point Q within the geometrical image, and of the same sign, either both positive or both negative, for any point within the geometrical shadow. If we denote the distance of the source O from the centre of the slit by a, the distances of that centre projected on the line (screen of observation) drawn through the point of observation Q parallel to the x-axis from the centre of the slit by b and from the point of observation Q by d and the width of the

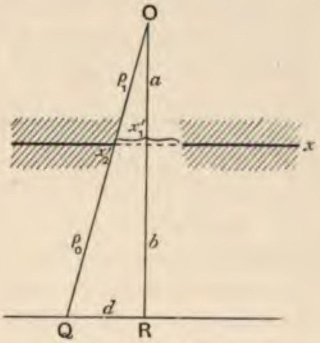

FIG. 27.

slit itself by δ (cf. Fig. 27), the following relations will evidently hold between these distances and the distances x_1' and x_2':

$$x_1' - x_2' = \delta$$

and

$$x_1' - \delta/2 : d = a : a + b = \rho_1 : \rho_1 + \rho_0.$$

As in the preceding problem, we may now replace ρ_1 by a and ρ_0 by b, and hence put $\cos\phi = 1$. The values (44) of v_1' and v_2', which correspond to any given point Q, may, therefore, be written most approximately

$$v_1' = x_1'\sqrt{\frac{2}{\lambda}\left(\frac{1}{a}+\frac{1}{b}\right)} \text{ and } v_2' = x_2'\sqrt{\frac{2}{\lambda}\left(\frac{1}{a}+\frac{1}{b}\right)}. \quad \text{..........(44A)}$$

By the above relations we can now express x_1' and x_2', the distances of the two edges of the slit from the origin, whose position on the x-axis varies according to that of the point of observation Q chosen, in terms of the distances a, b, d and δ; we evidently find

$$x_1' = \frac{ad}{a+b} + \frac{\delta}{2} \text{ and } x_2' = \frac{ad}{a+d} - \frac{\delta}{2}.$$

Replace x_1' and x_2' by these values in the expressions (44A) for v_1' and v_2', and we have

$$\left.\begin{aligned} v_1' &= \left[\frac{ad}{a+b}+\frac{\delta}{2}\right]\sqrt{\frac{2}{\lambda}\left(\frac{1}{a}+\frac{1}{b}\right)} \\ \text{and} \qquad v_2' &= \left[\frac{ad}{a+b}-\frac{\delta}{2}\right]\sqrt{\frac{2}{\lambda}\left(\frac{1}{a}+\frac{1}{b}\right)} \end{aligned}\right\}; \quad \text{................(44B)}$$

which give the two following simple relations between the v's:

$$v_1' - v_2' = \delta\sqrt{\frac{2}{\lambda}\left(\frac{1}{a}+\frac{1}{b}\right)} \text{ and } \frac{v_1'+v_2'}{2} = d\sqrt{\frac{2a}{\lambda b(a+b)}} = \frac{d}{p}, \quad \text{.....(45)}$$

where p is given by formula (42).

Interpretation of the Relations between the v''s and δ; Diffusion of the Waves.—We can now interpret the two relations (45) as follows: For given λ, a and b the difference in the distances v_1' and v_2' (s_1' and s_2') along the given spiral will be directly proportional to the width δ of the slit and their sum, $v_1' + v_2'$, along the spiral directly proportional to the distance d of the point of observation Q from the centre of the geometrical shadow. For given δ the difference in these distances, $v_1' - v_2'$, along the spiral will, therefore, remain constant for all values of d; the intensity at any point Q, which is proportional to the square of the direct distance between the two points ξ_1', η_1' and ξ_2', η_2' of the spiral (cf. formula (43)), will, therefore, depend alone on the curvature of the given constant segment $v_1' - v_2'$ of the spiral. If now the slit is very narrow, the segment $v_1' - v_2'$ will be very short and the intensity will evidently retain approximately one and the same value, as we recede from the centre of the geometrical image, where $d=0$, to a considerable distance within the geometrical shadow (cf. Fig. 26). The screen of observation will, therefore, be illuminated approximately uniformly, the illumination diminishing very gradually, as we pass from the centre of the geometrical image into the geometrical shadow, that is, no marked boundary will be discernible between shadow and image; the waves are then said to be diffused.

Exterior Diffraction Bands.—If the width δ of the slit is increased to such dimensions that the constant segment $v_1' - v_2'$ of the spiral, which is always proportional to δ (cf. formulae (45)), embraces a comparatively long portion of the same, then the intensity will evidently retain approximately one and the same value within the geometrical image, but diminish gradually as we pass into the geometrical shadow (cf. Figs. 26-28); as we pass from geometrical image into shadow, x_1' and x_2' (v_1' and v_2'), which are always opposite in sign in the former, will evidently assume one and the same sign in the latter (cf. p. 219). As we recede further into the geometrical shadow, the intensity will evidently diminish more rapidly and then less rapidly, until we finally reach a point Q at the distance d from the centre of the geometrical image, where the intensity becomes a minimum; the position of this first minimum will evidently be determined approximately thereby, that the tangents to the end points of the given segment $v_1' - v_2'$ of the spiral run parallel to each other and in the same direction, as indicated in Fig. 28 below. From this minimum the intensity will evidently increase first gradually, then rapidly, and finally gradually again, until it reaches a maximum, whose position will evidently be determined approximately thereby, that the tangents to the end points of the given segment $v_1' - v_2'$ run parallel to each other, but in opposite directions (cf. Fig. 28 below). The screen of observation will, therefore, be illuminated approximately uniformly from the centre of the geometrical image to a certain distance within the geometrical shadow, the illumination diminishing gradually from the central line towards the shadow, then the illumination will diminish more rapidly, until it finally reaches a minimum, and from this point on the screen will exhibit a series of bright and dark (coloured) bands (cf. pp. 193 and 224 below).

Approximate Determination of Distribution of Bands.—Let us now determine the distances d of the above minima and maxima of intensity from the centre of the geometrical image. As we have observed above, the positions of these minima and maxima are determined approximately thereby, that the tangents to the two end points of the given segment $v_1' - v_2'$ run parallel to each other, for the minima in the same direction and for the maxima in opposite directions; this will now evidently be the case, when the angles which the tangents to the two end points of the given segment make with the ξ-axis, differ by 2π and multiples of the same for the minima, and by 3π, 5π, 7π, etc., for the maxima; the positions of the first minimum and maximum determined in this manner are represented roughly in the annexed figure. The angle τ, which the tangent to any point ξ, η of the (Cornu's) spiral makes with the ξ-axis, is now, by formula (24),

$$\tau = \frac{\pi t^2}{2}.$$

That the two end points of the given segment $v_1' - v_2'$ of the spiral be so

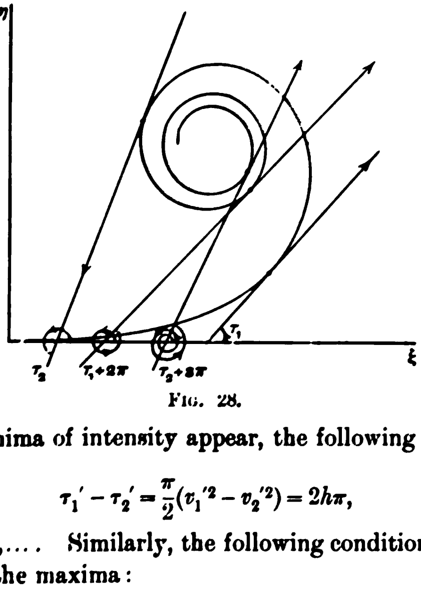

FIG. 28.

situated that minima of intensity appear, the following condition must, therefore, hold:

$$\tau_1' - \tau_2' = \frac{\pi}{2}(v_1'^2 - v_2'^2) = 2h\pi,$$

where $h = 1, 2, 3, \ldots$. Similarly, the following condition will determine the positions of the maxima:

$$\tau_1' - \tau_2' = \frac{\pi}{2}(v_1'^2 - v_2'^2) = (1 + 2h)\pi.$$

We can now write these two conditions in the form

$$(v_1' - v_2')(v_1' + v_2') = 4h$$

and

$$(v_1' - v_2')(v_1' + v_2') = 2(1 + 2h),$$

and hence, on replacing $v_1' - v_2'$ and $v_1' + v_2'$ by their values (45),

$$\frac{\delta d\sqrt{\frac{2}{\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)}}{p} = 2h$$

and

$$\frac{\delta d\sqrt{\frac{2}{\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)}}{p} = 1 + 2h.$$

Lastly, replace here p by its value (42), and we find

$$\left.\begin{aligned} \delta d &= \lambda b h \\ \text{and} \qquad \delta d &= \lambda b \frac{1+2h}{2} \end{aligned}\right\}, \quad \text{.............................(46)}$$

as conditions for the appearance of minima and maxima respectively.

On replacing h by 1, 2, 3... in these formulae (46), we find the following values for the d's:

$$\left.\begin{aligned} &d_{1\,\text{min.}} = \Delta, \quad d_{2\,\text{min.}} = 2\Delta, \quad d_{3\,\text{min.}} = 3\Delta, \text{ etc.} \\ \text{and} \qquad &d_{1\,\text{max.}} = \frac{3}{2}\Delta, \ d_{2\,\text{max.}} = \frac{5}{2}\Delta, \ d_{3\,\text{max.}} = \frac{7}{2}\Delta, \text{ etc.} \end{aligned}\right\}, \quad \text{...........(47)}$$

where

$$\Delta = \frac{\lambda b}{\delta};$$

that is, the distances of the maxima and minima of intensity from the centre of the geometrical image will be independent of the distance a of the source of the disturbance from the slit, but directly proportional to the distance b of the screen of observation behind the same and to the wave-length λ of the waves employed and indirectly proportional to the width δ of the slit, whereas the maxima and minima themselves or the bright and dark (coloured) bands will be equidistant. The exacter method for the determination of the positions of these maxima and minima shows, however, that this last result is only approximately correct. Lommel* has namely found the following values for these distances d of *maximum* intensity in terms of the distance z, where $z = \pi d$:

$$\begin{aligned} &z_1 = \ \ 4{\cdot}493\,\Delta, \quad z_2 = \ \ 7{\cdot}726\,\Delta, \quad z_3 = 10{\cdot}904\,\Delta, \quad z_4 = 14{\cdot}066\,\Delta, \\ &z_5 = 17{\cdot}221\,\Delta, \quad z_6 = 20{\cdot}371\,\Delta, \quad z_7 = 23{\cdot}520\,\Delta, \quad z_8 = 26{\cdot}666\,\Delta, \end{aligned}$$

where we have retained only the first three decimals; these values evidently correspond to the following values of d:

$$\left.\begin{aligned} &d_1 = 1{\cdot}430\,\Delta, \quad d_2 = 2{\cdot}459\,\Delta, \quad d_3 = 3{\cdot}472\,\Delta, \quad d_4 = 4{\cdot}478\,\Delta, \\ &d_5 = 5{\cdot}482\,\Delta, \quad d_6 = 6{\cdot}485\,\Delta, \quad d_7 = 7{\cdot}488\,\Delta, \quad d_8 = 8{\cdot}490\,\Delta \end{aligned}\right\}. \quad \text{(47A)}$$

A comparison of the above approximate values (47) for the distances d with these shows that the former differ only inappreciably from the latter.

Intensity of Maxima.—To find the intensity of any maximum or minimum, we replace d by its respective value from (47) or (47A) in formulae (44B), determining thereby the corresponding values of v_1' and v_2', evaluate the integrals ξ_1', η_1' and ξ_2', η_2' on p. 219 for those values of v as limits, and then determine by formula (43) and actual measurement of the direct distance between the two points ξ_1', η_1' and ξ_2', η_2' of the spiral the intensity sought. The values of these maxima of intensity, as determined by Lommel,* are

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* Cf. paper cited in foot-note, p. 217: Tabelle IVa, p. 651.

$$\left.\begin{array}{llll} \text{at } d_0=0, & I_0=1, & \text{at } d_1=1\cdot430\Delta, & I_1=0\cdot0472, \\ \text{at } d_2=2\cdot459\Delta, & I_2=0\cdot0165, & \text{at } d_3=3\cdot472\Delta, & I_3=0\cdot0083, \\ \text{at } d_4=4\cdot478\Delta, & I_4=0\cdot0050, \ldots, & & \\ \text{and at } d_{16}=16\cdot497\Delta, & I_{16}=0\cdot0003(7) & & \end{array}\right\}, \quad (48)$$

where we have retained only the first four decimals and the natural intensity has been taken as unity. It is evident from these values that only a limited number of bands will be observable, for these maxima decrease very rapidly in intensity, as we recede from the central axis $d=0$ of the geometrical image (cf. p. 193).

Behaviour of Intensity along Central Axis of Image; Determination of Positions of Maxima and Minima on that Axis.—It is evident from formulae (44B) that, for given a and δ, v_1' and v_2' will vary along the central axis $d=0$ of the geometrical image according to the distance b of the point of observation Q on that axis from the centre of the slit; as this distance b decreases, v_1' or v_2' will increase in absolute value, and the direct distance between the two corresponding symmetrically situated points ξ_1', η_1' and ξ_2', η_2' $(-\xi_1', -\eta_1')$ of the spiral will evidently pass through a series of maxima and minima and hence the respective intensity also. Since now for $d=0$, $v_1'=-v_2'$, so that the straight line joining any two points ξ_1', η_1' and $-\xi_1', -\eta_1'$ of the spiral will always pass through the origin of the same, the points of maximum and minimum distance from the origin will be determined approximately by the points of intersection of the spiral and the straight line AA' joining the two asymptotic points and passing through the origin of the spiral. These points of intersection have now been determined on p. 216; we found namely

$$v_1'=\sqrt{\frac{3}{2}+4h} \text{ for the maxima}$$

and

$$v_1'=\sqrt{\frac{7}{2}+4h} \text{ for the minima,}$$

where $h=0, 1, 2 \ldots$.

Replace here v_1' by its value for $d=0$ from formulae (44B), and we have

$$\left.\begin{array}{l} v_1'{}_{\text{max.}}=\dfrac{\delta}{2}\sqrt{\dfrac{2}{\lambda}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}=\sqrt{\dfrac{3}{2}+4h} \\ \text{and} \quad v_1'{}_{\text{min.}}=\dfrac{\delta}{2}\sqrt{\dfrac{2}{\lambda}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}=\sqrt{\dfrac{7}{2}+4h} \end{array}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(49)$$

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*Cf. paper cited in foot-note, p. 217: p. 606.

which evidently give the following expressions for the b's:

$$\left.\begin{aligned} & b_{\text{max.}} = \frac{a\delta^2}{-\delta^2+\lambda a(3+8h)} \\ \text{and}\qquad & b_{\text{min.}} = \frac{a\delta^2}{-\delta^2+\lambda a(7+8h)} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots(50)$$

(cf also formula (b) Ex. 4).

Lommel[*] has now determined by his exacter method the distances of these maxima and minima from the slit in terms of the quantity y, which is related to the distance b by the formula

$$b = \frac{a\delta^2}{-\delta^2+\lambda a\dfrac{2y}{\pi}}.\dagger$$

On replacing the y's by the values[‡] determined by Lommel in this formula, we find

$$\left.\begin{aligned} & b_{0\,\text{max.}} = \frac{a\delta^2}{-\delta^2+\lambda a(0)} = -a, \quad b_{0\,\text{min.}} = \frac{a\delta^2}{-\delta^2+\lambda a(7{\cdot}308)} \\ & b_{1\,\text{max.}} = \frac{a\delta^2}{-\delta^2+\lambda a(10{\cdot}420)}, \quad b_{1\,\text{min.}} = \frac{a\delta^2}{-\delta^2+\lambda a(15{\cdot}240)} \\ & b_{2\,\text{max.}} = \frac{a\delta^2}{-\delta^2+\lambda a(18{\cdot}606)}, \quad b_{2\,\text{min.}} = \frac{a\delta^2}{-\delta^2+\lambda a(23{\cdot}204)} \end{aligned}\right\}. \dots(50\text{A})$$

On comparing these values for the b's with the approximate ones obtained by putting $h=0, 1, 2, \dots$ in formulae (50), we observe that the latter, with the exception of the first maximum,[§] differ only inappreciably from the former.

Intensity of Maxima and Minima.—The intensities of the above maxima and minima could evidently be determined in a similar manner to the preceding ones (48), namely, by formulae (50) or (50A), (49) and (43) ($\xi_2' = -\xi_1'$ and $\eta_2' = -\eta_1'$) and by actual measurement of the direct distance between the two points ξ_1', η_1' and $-\xi_1'$, $-\eta_1'$ of the spiral thus determined. Lommel[||] has found the following values for these maxima and minima:

$$\begin{aligned} &\text{at } b_{0\,\text{max.}}(=-a),\ I_{0\,\text{max.}} = 1, && \text{at } b_{0\,\text{min.}},\ I_{0\,\text{min.}} = 0{\cdot}0816, \\ &\text{at } b_{1\,\text{max.}},\ I_{1\,\text{max.}} = 0{\cdot}1323, && \text{at } b_{1\,\text{min.}},\ I_{1\,\text{min.}} = 0{\cdot}0463, \\ &\text{at } b_{2\,\text{max.}},\ I_{2\,\text{max.}} = 0{\cdot}0698, && \text{at } b_{2\,\text{min.}},\ I_{2\,\text{min.}} = 0{\cdot}0326, \end{aligned}$$

where we have retained only the first four decimals.

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[*] Cf paper cited in foot-note, p. 217, Tabelle Va, p. 652.

[†] Cf. ditto, p. 606.

[‡] $y_{0\,\text{max.}} = 0$, $y_{0\,\text{min.}} = 11{\cdot}479$, $y_{1\,\text{max.}} = 16{\cdot}371$, $y_{1\,\text{min.}} = 23{\cdot}939$,
$y_{2\,\text{max.}} = 29{\cdot}223$, $y_{2\,\text{min.}} = 36{\cdot}451$, $y_{3\,\text{max.}} = 41{\cdot}916$, $y_{3\,\text{min.}} = 48{\cdot}982$,
where we have retained only the first three decimals.

[§] Lommel's first maximum appears at the distance $b=-a$ from slit, that is, at the source O itself, and evidently does not correspond to the first maximum of formulae (50); the latter seems to have been overlooked by Lommel.

[||] Cf. paper cited in foot-note, p. 217, Tabelle Va, p. 652.

Diffraction on a Narrow Screen.—Lastly, let us examine the diffraction of light on its passage by a narrow screen or fine wire. As above we shall consider the behaviour of the intensity on a large screen placed parallel to the two edges of the obstructing screen or wire and at the distance b behind the latter. We denote the breadth of the obstructing screen by δ and the distances of its two edges from the origin of our coordinates by x_1' and x_2', which origin we shall lay as above on the line joining the source O and the point of observation Q. By formula (35) the intensity I at any point Q on the screen of observation will then evidently be given by the expression

$$I \text{ prop. to } \frac{a^2}{2(\rho_1+\rho_0)^2}\left\{\left[\int_{-\infty}^{v_1'} \sin\frac{\pi v^2}{2}dv + \int_{v_2'}^{\infty} \sin\frac{\pi v^2}{2}dv\right]^2 + \left[\int_{-\infty}^{v_1'} \cos\frac{\pi v^2}{2}dv + \int_{v_2'}^{\infty} \cos\frac{\pi v^2}{2}dv\right]^2\right\},$$

where v_1' and v_2' correspond to the values x_1' and x_2' respectively of x (cf. formula (20)).

By Cornu's spiral we can now interpret the (Fresnel's) integrals in the above expression for the intensity as follows: the first integral represents the projection on the η-axis of the distance $A'E_1'$, where E_1' denotes that point of the spiral, whose η-coordinate is determined by the value of the integral

$$\int_0^{v_1'} \sin\frac{\pi v^2}{2}dv,$$

the second integral the projection on the η-axis of the distance $E_2'A$, where E_2' denotes the point of the spiral, whose coordinate η is determined by the value of the integral

$$\int_0^{v_2'} \sin\frac{\pi v^2}{2}dv,$$

and, similarly, the third and fourth or last integrals the projections on the ξ-axis of those same distances $A'E_1'$ and $E_2'A$ respectively, as indicated in Fig. 29 below. If we denote these four projections by η_1', η_2', ξ_1', ξ_2' respectively, we can then write the above expression for the resultant intensity in the form

$$I \text{ prop. to } \frac{a^2}{2(\rho_1+\rho_0)^2}[(\eta_1'+\eta_2')^2+(\xi_1'+\xi_2')^2]. \quad\ldots\ldots\ldots\ldots(51)$$

If we now lay off the distance $E_2'A$ at the point E_1' of the spiral and parallel to that direction (cf. Fig. 29), we can interpret the expression in the large brackets of this formula (51) for the intensity as the square of the distance $A'A$ thus constructed. The intensity in question will, therefore, be proportional to the square of the distance $A'A$ of the given construction (cf. Fig. 29).

Behaviour of Intensity along Central Axis of Shadow.—It follows now from the above formulae and given geometrical construction that the region directly behind the small screen or wire will always be illuminated, for there, $d=0$, v_1' and v_2' will be equal but opposite in sign, so that the points E_1' and E_2' will be situated symmetrically with respect to the origin and hence the distances $A'E_1'$ and $E_2'A$ equal and similarly directed (cf. Fig. 29). Along the central axis of the geometrical shadow the intensity will, therefore, increase gradually but continuously, without passing through maxima and minima, as we recede from the obstructing screen.*

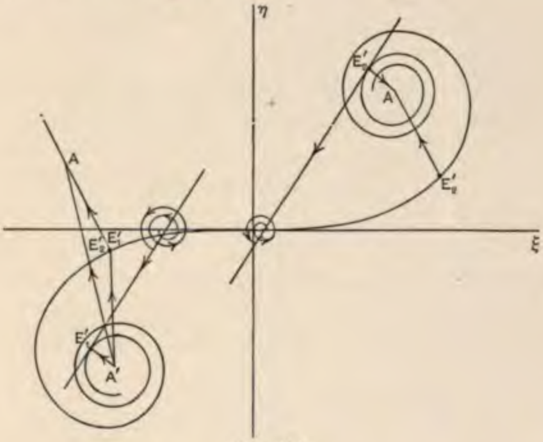

Fig. 29.

Interior Diffraction Bands.—For large values of v_1' and v_2', that is, for a comparatively broad obstructing screen or directly behind a (very) narrow screen, the two points E_1' and E_2' of the spiral will be situated near their respective asymptotic points A' and A; the distances $A'E_1'$ and $E_2'A$ will, therefore, then differ only inappreciably from each other in length, as the one point, E_1' or E_2', recedes from or approaches the origin along the spiral, whereby the other point will approach or recede from the origin respectively (cf. formulae (45)). It thus follows that marked maxima and minima of intensity will appear, when the two lines $A'E_1'$ and $E_2'A$ run parallel to each other, maxima when they are similarly directed, and minima when oppositely directed. Since now in the given case, where v_1' and v_2' are assumed to be large, the lines $A'E_1'$ and $E_2'A$ will cut the (Cornu's) spiral approximately orthogonally, the tangents to any two such points E_2' and E_1' will in both cases (maxima and minima) run approximately parallel to each other. It is now evident from the above figure, where the given construction is roughly indicated for a minimum, that maxima of intensity will appear, when the angles which these tangents make with their respective (positive and negative) ξ-axes are equal or differ from each other by 2π or multiples of the same, and minima, when they differ by odd multiples of π; that is, by formula (24), maxima will appear when

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* Cf. paper by Lommel cited in foot-note on p. 217, Tab. XVI., p. 658, for variation of intensity in question.

$$\tau_1' - \tau_2' = \frac{\pi}{2}(v_1'^2 - v_2'^2) = \pm 2\pi h,$$

and minima when

$$\tau_1' - \tau_2' = \frac{\pi}{2}(v_1'^2 - v_2'^2) = \pm (2h+1)\pi,$$

where $$h = 0,\ 1,\ 2,\ \ldots.$$

By formulae (45), which evidently hold for the given problem, these conditions for the maxima and minima can now be written

$$\left.\begin{aligned} &2\delta d_{\text{max.}} = \pm 2h\lambda b \\ \text{and}\quad &2\delta d_{\text{min.}} = \pm(2h+1)\lambda b \end{aligned}\right\}; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52)$$

which give the following values for the d's:

$$\left.\begin{aligned} &d_{0\,\text{min.}} = \pm\frac{1}{2}\frac{\lambda b}{\delta}, \quad d_{1\,\text{min.}} = \pm\frac{3}{2}\frac{\lambda b}{\delta}, \quad d_{2\,\text{min.}} = \pm\frac{5}{2}\frac{\lambda b}{\delta}, \text{ etc.,} \\ \text{and}\quad &d_{0\,\text{max.}} = \pm\frac{\lambda b}{\delta}, \quad d_{1\,\text{max.}} = \pm 2\frac{\lambda b}{\delta}, \quad d_{2\,\text{max.}} = \pm 3\frac{\lambda b}{\delta}, \text{ etc.} \end{aligned}\right\}\ldots(52\text{A})$$

It follows from these formulae that the distances of the maxima and minima from the centre of the geometrical shadow will be independent of the distance a of the source from the obstructing screen, but directly proportional to the distance b of the screen of observation behind the latter and to the wave-length λ of the waves employed, whereas the maxima and minima themselves or the bright and dark (coloured) bands (within the geometrical shadow) will be equidistant. These bands are thus similar to the diffraction bands produced by a narrow slit in a large opaque screen (cf. pp. 221-223). It is evident that the intensity of the given maxima and minima will increase as h increases, that is, as we recede from the central axis of the geometrical shadow towards its boundaries (cf. Fig. 29 and below); but the above conditions for the appearance of the bands will evidently hold only well within the geometrical shadow, that is, for values of d that lie well within the interval

$$-\frac{a+b}{a}\frac{\delta}{2} < d < \frac{a+b}{a}\frac{\delta}{2},$$

which limiting values for d evidently correspond to the conditions

$v_1' > 0$ and $v_2' < 0$ (cf. formulae (44B)). We could expect, therefore, to find only a limited number of diffraction bands of the given type within the geometrical shadow, and these only in the next proximity of an obstructing screen of such breadth that v_1' and v_2' assume there large values (cf. below). This and the above results, which are confirmed by experiment, also agree comparatively well with the exacter calculations of Lommel;* he has found the following values † for the distances of the maxima and minima from the central axis of the geometrical shadow in terms of the quantity

$$z = \frac{2\pi}{\lambda b} \frac{\delta}{2} d, \ddagger$$

and the corresponding values for the intensities:

(1) At the distance $y = \frac{2\pi}{\lambda} \frac{a+b}{ab} \frac{\delta^2}{4} = 3$,‡ which corresponds to the value

$$b = \frac{a\delta^2}{-\delta^2 + \lambda a(1{\cdot}910)}$$

from the obstructing screen.

z.	$d \cdot \frac{\delta}{\lambda b}$.	I.
1·3550	0·4313	0·0118 min.
π	1·0000	0·1942 max.
3·7710	1·203	0·1769 min.
5·7637	1·793	0·6413 max.
2π	2·000	0·6130 min.
7·2419	2·305	0·7442 max.
8·4491	2·690	0·3803 min.
3π	3·000	0·5086 max.
etc.	etc.	etc.

(2) At the distance $y = 6$ or $b = \frac{a\delta^2}{-\delta^2 + \lambda a(3{\cdot}819)}$.

1·4835	0·4723	0·0013 min.
π	1·000	0·0359 max.
4·3450	1·3833	0·0172 min.
2π	2·000	0·0914 max.
6·9276	2·205	0·0856 min.
9·1785	2·922	0·2540 max.
3π	3·000	0·2533 min.
11·1203	3·540	0·3933 max.
4π	4·000	0·2876 min.

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* Cf. paper cited in foot-note on p. 217.

† Cf. ditto, Tab. XVII.—XX., pp. 659-661.

‡ Cf. ditto, p. 606.

(3) At the distance $y=9$ or $b=\dfrac{a\delta^2}{-\delta^2+\lambda a(5\cdot730)}$.

1·5247	0·4855	0·0003 min.
π	1·0000	0·0142 max.
4·5334	1·443	0·0036 min.
2π	2·000	0·0250 max.
7·4102	2·359	0·0163 min.
3π	3·000	0·0601 max.
10·0687	3·206	0·0571 min.
12·4745	3·972	0·1477 (28) max.
4π	4·000	0·1477 (13) min.

And (4) at the distance $y=12$ or $b=\dfrac{a\delta^2}{-\delta^2+\lambda a(7\cdot640)}$.

1·5426	0·4911	0·0001 min.
π	1·0000	0·0076 max.
4·6103	1·468	0·0011 min.
2π	2·000	0·0109 max.
7·6163	2·425	0·0045 min.
3π	3·000	0·0200 max.
10·4953	3·342	0·0150 min.
4π	4·000	0·0450 max.,

where we have retained only the first four decimals.

We observe that the results obtained above by our approximate method agree comparatively well with these exacter ones of Lommel, especially as we approach the obstructing screen, that is, as y increases in value (cf. values of z for $y=12$).

Frauenhofer's Diffraction Phenomena.—Frauenhofer's phenomena of diffraction are known as those that appear, when both source of disturbance and screen of observation are removed to infinite distance from the diffracting screen. To obtain these phenomena, we place the source to be employed at the focus of an ordinary lens, so that the waves emerging from the same will be propagated along parallel lines, and observe the (light) effect on any plane parallel to the obstructing screen and behind it by means of a telescope adjusted at infinite distance. Frauenhofer's diffraction phenomena are evidently a particular case of Fresnel's and can thus be deduced from his (Fresnel's) formulae, if we put there $\rho_1=\rho_0=\infty$. Let us first establish these formulae and then apply them to the various problems on diffraction.

For $\rho_1=\rho_0=\infty$ formula (14) for $f(x, y)$ will assume the form

$$f(x, y)=\frac{2\pi}{\lambda}[(\alpha_1+\alpha_0)x+(\beta_1+\beta_0)y]=\mu x+\nu y, \quad \text{............}(53)$$

where $$\mu=\frac{2\pi}{\lambda}(\alpha_1+\alpha_0) \text{ and } \nu=\frac{2\pi}{\lambda}(\beta_1+\beta_0), \quad\ldots\ldots\ldots\ldots\ldots(53\text{A})$$

and hence formula (15) for the light-vector s the form

$$\left.\begin{aligned}s=A\Big\{\cos\frac{2\pi}{\lambda}[vt-(\rho_1+\rho_0)]\int\cos(\mu x+\nu y)ds&\\ -\sin\frac{2\pi}{\lambda}[vt-(\rho_1+\rho_0)]\int\sin(\mu x+\nu y)ds\Big\}&\end{aligned}\right\},\ldots(54)$$

where A is given by formulae (18A) and the integration is to be extended over the aperture s in the diffracting screen.

According to the principle of superposition the intensity I produced by any light-vector of the form (54) is now proportional to the expression

$$A^2\left\{\left[\int\cos(\mu x+\nu y)ds\right]^2+\left[\int\sin(\mu x+\nu y)ds\right]^2\right\} \text{ prop. to } I, \quad (55)$$

or
$$\left.\begin{aligned}&A^2(C^2+S^2) \text{ prop. to } I,\\ \text{where}\quad &C=\int\cos(\mu x+\nu y)ds \text{ and } S=\int\sin(\mu x+\nu y)ds\end{aligned}\right\}\ldots\ldots(55\text{A})$$

(cf. p. 213). If we lay the origin of our coordinates at any point of the aperture s (cf. also p. 203) and place the telescope parallel to the direction of propagation of the incident waves, then not only α_1 and α_0 but also β_1 and β_0 will evidently differ from each other only in sign (cf. Fig. 24), and hence the quantities μ and ν of formulae (53A) vanish at all points in the (small) aperture. If we denote the intensity in this particular position of the telescope by I', the above general formula (55) will evidently give

$$A^2s^2 \text{ prop. to } I', \text{ hence } A^2 \text{ prop. to } \frac{I'}{s^2}, \quad\ldots\ldots\ldots\ldots\ldots(55\text{B})$$

where s denotes the area of the given aperture. By means of this formula for the determination of the constant A, we can now write the general formula (55) for I, when the telescope (point of observation) makes an angle with the direction of propagation of the incident waves, in the form

$$I=\frac{I'}{s^2}\left\{\left[\int\cos(\mu x+\nu y)ds\right]^2+\left[\int\sin(\mu x+\nu y)ds\right]^2\right\}=\frac{I'}{s^2}(C^2+S^2)\ \Big\}. \ldots(56)$$

Diffraction on Rectangular Aperture.—Let us, first, apply formula (56) to the case, where the aperture s has the form of a rectangle. For this purpose we choose the centre of the rectangle as origin of our coordinates and lay the x and y-axes parallel to its two sides, whose lengths we shall denote by a and b respectively. Formula (56) will then assume the form

$$I=\frac{I'}{ab}\left\{\left[\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\cos(\mu x+\nu y)dx\,dy\right]^2+\left[\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\sin(\mu x+\nu y)dx\,dy\right]^2\right\}, \quad ..(57)$$

where the integration is to be extended over the rectangle ab only.

For small rectangular apertures x and y will be small, and hence $\sin(\mu x+\nu y)$ approximately vanish at all points of the same. Here we can, therefore, reject the second integral of this expression (57) for I in comparison to the first, and we have most approximately

$$I=\frac{I'}{ab}\left[\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\cos(\mu x+\nu y)dx\,dy\right]^2, \quad \text{.......................}(57\text{A})$$

which we can evaluate as follows:

$$I=\frac{I'}{ab}\left[\int_{-\frac{a}{2}}^{\frac{a}{2}}dx\int_{-\frac{b}{2}}^{\frac{b}{2}}(\cos\mu x\cos\nu y-\sin\mu x\sin\nu y)dy\right]^2$$

$$=\frac{I'}{ab}\left[\int_{-\frac{a}{2}}^{\frac{a}{2}}dx\left|\frac{\cos\mu x\sin\nu y}{\nu}+\frac{\sin\mu x\cos\nu y}{\nu}\right|_{-\frac{b}{2}}^{\frac{b}{2}}\right]^2$$

$$=\frac{I'}{ab}\left[\int_{-\frac{a}{2}}^{\frac{a}{2}}dx\frac{2\sin\frac{\nu b}{2}\cos\mu x}{\nu}\right]^2$$

$$=\frac{I'}{ab}\left[\frac{2\sin\frac{\nu b}{2}}{\nu}\left|\frac{\sin\mu x}{\mu}\right|_{-\frac{a}{2}}^{\frac{a}{2}}\right]^2=\frac{I'}{ab}\left[\frac{2\sin\frac{\nu b}{2}}{\nu}\cdot\frac{2\sin\frac{\mu a}{2}}{\mu}\right]^2,$$

hence

$$I=I'\left(\frac{\sin\frac{\mu a}{2}}{\frac{\mu a}{2}}\right)^2\cdot\left(\frac{\sin\frac{\nu b}{2}}{\frac{\nu b}{2}}\right)^2. \quad \text{.................................}(58)$$

This expression for I will now vanish, when either of its last two factors vanishes, that is, when

$$\left.\begin{aligned}&\pm\frac{\mu a}{2}=\pi,\ 2\pi,\ 3\pi,\ \ldots\\ \text{or}\quad&\pm\frac{\nu b}{2}=\pi,\ 2\pi,\ 3\pi,\ \ldots\end{aligned}\right\} \quad \text{.......................}(59)$$

If the diffracting screen is so placed that the direction of propagation of the incident waves is normal to the same, then $\alpha_1=\beta_1=0$ and the intensity I in any direction α_0, β_0 can be observed at that point of the focal plane in the object glass of the telescope, when placed parallel to the incident waves, whose coordinates are

$$x'=f\alpha_0 \text{ and } y'=f\beta_0, \text{ ....(60)}$$

where f denotes the focal distance of the object glass and the coordinates x', y' are taken parallel to the coordinates x, y (a, b) and their origin at the focus of the object glass.

Replace x' and y' by their values (60) in formulae (53A), and we have

$$\mu=\frac{2\pi}{\lambda}\cdot\frac{x'}{f} \text{ and } \nu'=\frac{2\pi}{\lambda}\,\frac{y'}{f}.$$

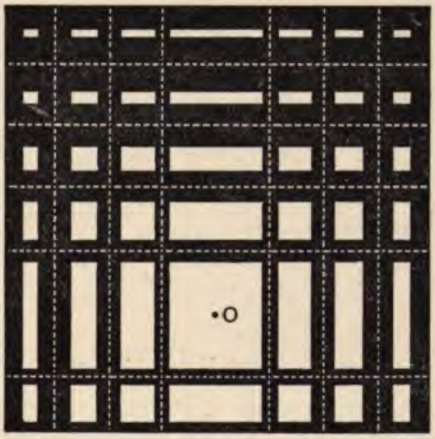

FIG. 30.

By formulae (59) the intensity I will, therefore, vanish, when

$$\pm\frac{\mu a}{2}=\pm\frac{2\pi}{\lambda}\,\frac{x'}{f}\,\frac{a}{2}=\pi,\ 2\pi,\ \ldots=h\pi,\ h=1,\ 2,\ 3,\ \ldots,$$

or

$$\pm\frac{\nu b}{2}=\pm\frac{2\pi}{\lambda}\,\frac{y'}{f}\,\frac{b}{2}=\pi,\ 2\pi,\ \ldots=k\pi,\ k=1,\ 2,\ 3,\ \ldots,$$

that is, for the following values of x' and y':

$$x'=\pm h\frac{\lambda f}{a}, \text{ or } y'=\pm k\frac{\lambda f}{b}. \text{ ....(61)}$$

It follows from these formulae that the focal plane of the object glass will exhibit two series of dark parallel bands, the one parallel to the x' (a) axis and the other to the y' (b) axis; these bands are indicated roughly in the above figure by the dotted white lines.

The bands of each series will evidently be equidistant, except at the focus O of the object glass and along its focal axes x' and y', where the distance between the first dark band on the one side of either focal axis and that on its other will be double the distance between the other bands of the respective series; this follows, since the expressions

$$\frac{\sin\frac{\mu a}{2}}{\frac{\mu a}{2}} \quad \text{and} \quad \frac{\sin\frac{\nu b}{2}}{\frac{\nu b}{2}}$$

do not vanish for $\mu=\nu=0$, but evidently assume each the value unity. With the exception of these maxima of intensity in place of minima, the two series of dark parallel bands will form a system of small equal rectangles similar to the rectangular aperture but turned through 90° (cf. Fig. 30); that these rectangles are turned through 90° is evident from the above expressions (61) for x' and y'.

Aside from the maxima of intensity at the focus of the object glass and along its focal axes, other less brilliant maxima will evidently appear at the centres of the small rectangles formed by the two series of dark equidistant bands, that is, for the values of x' and y' determined by the relations

$$\pm\frac{\mu a}{2}=\pm\frac{2\pi}{\lambda}\frac{x'}{f}\frac{a}{2}=\frac{\pi}{2}, \frac{3\pi}{2}, \ldots=(2h+1)\frac{\pi}{2}, h=1, 2, 3,\ldots$$

and $$\pm\frac{\nu b}{2}=\pm\frac{2\pi}{\lambda}\frac{y'}{f}\frac{b}{2}=\frac{\pi}{2}, \frac{3\pi}{2}, \ldots=(2k+1)\frac{\pi}{2}, k=1, 2, 3,\ldots;$$

the maximum at any such point x', y' will, by formula (58), be given by the expression

$$I=I'\left[\frac{1}{(2h+1)\frac{\pi}{2}}\right]^2\cdot\left[\frac{1}{(2k+1)\frac{\pi}{2}}\right]^2=I'\frac{16}{\pi^4(2h+1)^2(2k+1)^2}. \quad \ldots.(62)$$

Along either focal axis, that is, for $\mu=0$ or $\nu=0$, the corresponding factor in the expression (58) for I will assume the value unity, and hence the maximum intensity itself the value

$$I_{x'}=I'\frac{4}{\pi^2(2h+1)^2} \text{ or } I_{y'}=I'\frac{4}{\pi^2(2k+1)^2}. \ldots\ldots\ldots\ldots\ldots(62\text{A})$$

At the focus of the object glass $I=I'$ (cf. p. 231).

Since the maxima at the focus and along the focal axes are evidently appreciably brighter than the other maxima, determined by formula (62)—the intensity of the maxima is indicated roughly in Fig. 30 by the size of the white rectangles—the general effect or pattern produced in the focal plane of the object glass will be a bright cross parallel to the sides of the given aperture and with its centre at focus (cf. Fig. 30).

Diffraction on Narrow Slit.—If we replace the rectangular aperture ab in the preceding problem by a narrow slit of width a and length $b=\infty$, the above formula (58) for I will evidently reduce to

$$I=I'\left(\frac{\sin\frac{\mu a}{2}}{\frac{\mu a}{2}}\right)^2. \quad\text{..........................(63)}$$

If the diffracting screen is so placed that the direction of propagation of the incident waves is normal to it, then

$$\alpha_1=\beta_1=0,$$

and hence by formulae (53A)

$$\mu=\frac{2\pi}{\lambda}\alpha_0=\frac{2\pi}{\lambda}\sin\phi \quad (\nu=0),$$

where ϕ denotes the "angle of diffraction," that is, the angle, which the vector from the centre of the slit to the point of observation in the focal plane makes with the direction of propagation of the incident waves. By formula (58) the intensity at any point (ϕ) can thus be written

$$I=I'\left[\frac{\sin\left(\frac{\pi a}{\lambda}\sin\phi\right)}{\frac{\pi a}{\lambda}\sin\phi}\right]^2. \quad\text{......................(63A)}$$

I will, therefore, vanish, when

$$\frac{\pi a}{\lambda}\sin\phi=\pm h\pi,\ h=1,\ 2,\ 3\ \ldots,$$

hence $$\sin\phi=\pm\frac{h\lambda}{a},$$

that is, we obtain here a single series (cf. Ex. 12 at end of chapter) of dark equidistant bands parallel to the edge b of the slit. If $a<\lambda$, then $\sin\phi$ will be larger than unity for all integers h, that is, there will be no angle ϕ, for which I will vanish, and the waves will be diffused (cf. p. 221).

Diffraction produced by a Number of Equal Apertures.—Let us, next, examine the (Frauenhofer's) diffraction pattern (source and observation screen at infinite distance) that is produced by several small equal apertures or holes, as pin holes, in a large opaque screen. We denote the coordinates of the centres of the apertures or of points similarly situated in the same, referred to any given system of rectangular coordinates $x'y'$, by $x_1'y_1'$, $x_2'y_2'$, etc., and the coordinates of any other point in any aperture i, referred to a system of coordinates parallel to the system $x'y'$ and with origin at the centre of or at the point chosen in that aperture $(x_i'y_i')$, by x, y; the origin and axes of the system of coordinates $x'y'$ shall be so chosen in the obstructing screen as best suited for the treatment of the problem in question. The coordinates of any point in any aperture i referred to the coordinates $x'y'$ will then be

$$x_i'+x \text{ and } y_i'+y.$$

The resultant intensity I at any point of the object glass will then, by formula (55A), be given by the expression

$$I \text{ prop. to } A^2(C^2+S^2),$$

where

$$\left.\begin{aligned} C&=\sum_i\iint\cos\left[\mu(x_i'+x)+\nu(y_i'+y)\right]dx\,dy \\ S&=\sum_i\iint\sin\left[\mu(x_i'+x)+\nu(y_i'+y)\right]dx\,dy \end{aligned}\right\}, \qquad\text{...........(64)}$$

where the integration is to be extended over any aperture i and the summation over all the apertures in the screen. Since this integration and summation are evidently entirely independent of each other, we can, therefore, write C and S in the form

$$\begin{aligned} C&=\sum_i\iint[\cos(\mu x_i'+\nu y_i')\cos(\mu x+\nu y) \\ &\qquad -\sin(\mu x_i'+\nu y_i')\sin(\mu x+\nu y)]dx\,dy \\ &=\sum_i\cos(\mu x_i'+\nu y_i')\iint\cos(\mu x+\nu y)dx\,dy \\ &\qquad -\sum_i\sin(\mu x_i'+\nu y_i')\iint\sin(\mu x+\nu y)dx\,dy, \end{aligned}$$

and, similarly,

$$\begin{aligned} S&=\sum_i\sin(\mu x_i'+\nu y_i')\iint\cos(\mu x+\nu y)dx\,dy \\ &\qquad +\sum_i\cos(\mu x_i'+\nu y_i')\iint\sin(\mu x+\nu y)dx\,dy, \end{aligned}$$

or

$$C=c'c-s's \text{ and } S=s'c+c's,$$

where

$$\left.\begin{aligned} c'&=\sum_i\cos(\mu x_i'+\nu y_i'), & s'&=\sum_i\sin(\mu x_i'+\nu y_i'), \\ c&=\iint\cos(\mu x+\nu y)dx\,dy, & s&=\iint\sin(\mu x+\nu y)dx\,dy \end{aligned}\right\}. \qquad\text{......(65)}$$

The above formula for I can therefore be written

$$I \text{ prop. to } A^2[(c'c-s's)^2+(s'c+c's)^2]=A^2(c'^2+s'^2)(c^2+s^2). \qquad\text{......(66)}$$

This expression for I for several apertures differs now from that (55A) for a single aperture only in its coefficient $(c'^2+s'^2)$; it thus follows that the intensity in the former case will be $(c'^2+s'^2)$ times that in the latter, whereas the expression itself will vanish at the same points in both cases, that is, the *position* of the dark bands of a single aperture will not be altered, when that aperture is replaced by several such (equal) apertures.

Let us now examine the coefficient $(c'^2+s'^2)$ in formula (66); for this purpose we write it in the form

$$c'^2+s'^2=\sum_i \cos^2(\mu x_i'+\nu y_i')+2\sum_i\sum_k \cos(\mu x_i'+\nu y_i')\cos(\mu x_k'+\nu y_k')$$
$$+\sum_i \sin^2(\mu x_i'+\nu y_i')+2\sum_i\sum_k \sin(\mu x_i'+\nu y_i')\sin(\mu x_k'+\nu y_k'),$$

where $\qquad i=1,\ 2,\ \dots n,\quad k=1,\ 2,\ \dots n,\quad (k\gtrless i),$

where n denotes the number of apertures in the obstructing screen, hence

$$c'^2+s'^2=n+2\sum_i\sum_k \cos[\mu(x_i'-x_k')+\nu(y_i'-y_k')]. \qquad \text{.........(67)}$$

If there are many apertures in the screen and these are irregularly distributed over the same, the second term of this expression will vanish when compared with the first, since its different members will then assume values that lie irregularly but in mean equally distributed between $+2$ and -2 and hence will approximately cancel one another. In this case the intensity at any point will, therefore, be approximately n-times that produced by a single aperture, that is, it will be proportional to the number of apertures in the screen. On the other hand, if the apertures are distributed regularly or according to any law, the second term of the given expression will not, in general, vanish, when compared with the first, but it will assume a finite value determined by the law of distribution chosen. Take, for example, the simple case of two equal apertures at the distance d apart; if we choose the line joining them (their centres) as x'-axis and lay the origin of our coordinates $x'y'$ in one of the apertures, we can then put

$$x_1'=y_1'=0,\quad x_2'=d,\quad y_2'=0;$$

the quantities c' and s' of formulae (65) will then assume the simple form

$$c'=1+\cos\mu d,\quad s'=\sin\mu d,$$

and hence the coefficient $c'^2+s'^2$ the form

$$c'^2+s'^2=2+2\cos\mu d,$$

that, in which formula (67) is written. It is evident that the second term of this expression for the given coefficient cannot be neglected in comparison to the first.

We can also write the coefficient $c'^2+s'^2$ here in the form

$$c'^2+s'^2=4\cos^2\frac{\mu d}{2}.$$

It will thus evidently vanish, when

$$\mu l=(2h+1)\pi, \quad h-1, 2, 3, \dots,$$

that is, for those values of $\mu(x)$ that are determined by this relation. The given diffraction pattern will, therefore, exhibit a third series of equidistant dark bands running at right angles to the line joining the two apertures in addition to the two series of dark bands produced by the single (rectangular) aperture (cf. formulae (56) and (66)).

For another example, where the second term of the general expression (67) for the coefficient $c_2'+s_2'$ cannot be rejected, see Ex. 13 at end of chapter.

Babinet's Principle.—Let us now compare the diffraction pattern produced by a small screen s_1 with any number of very small apertures with that produced by its so-called "complementary" screen s_2 or that formed by the apertures of the former as opaque portions, the opaque portions of the former becoming the transparent ones or apertures in the latter. The intensity I_1 at any point behind the screen s_1 is now, by formula (55A), proportional to the expression $C_1^2+S_1^2$, where the integrals C_1 and S_1 (cf. also formulae (65)) are to be extended over the apertures in that screen; similarly the intensity I_2 behind the complementary screen s_2 will be proportional to the expression $C_2^2+S_2^2$, where the integrations are to be extended over its transparent portions. If now the small screen s_1 or s_2 is replaced by a single aperture, the intensity I_0 behind it will evidently be proportional to the expression $C_0^2+S_0^2$, where the integrals C_0 and S_0 are to be extended over the single aperture previously occupied by the screens s_1 or s_2; this surface of integration can now be replaced by the transparent portions of both screens together, and we can thus write I_0 in the form

$$I_0 \text{ prop. to } (C_1+C_2)^2+(S_1+S_2)^2,$$

where C_1, C_2 and S_1, S_2 are the integrals employed above.

Outside the geometrical image, that is, within the region of the diffraction pattern proper cast by either screen s_1 or s_2, I_0 will evidently vanish (cf. p. 193), and hence

$$(C_1+C_2)^2+(S_1+S_2)^2=0.$$

This condition can now be satisfied only when

$$C_1=-C_2 \text{ and } S_1=-S_2;$$

in which case $$I_1=I_2. \qquad \text{(68)}$$

In this formula is embodied Babinet's principle, which we can state as follows: "The intensity at all points of the diffraction pattern proper will be the same for the complementary screen as for the screen itself." According to this principle the diffraction pattern produced by any number of arbitrarily situated small screens of equal size will be the same as that produced by the same number of similar and similarly situated apertures of that size in a large screen, which problem has just been treated above.

Diffraction Gratings and their Patterns.—An opaque screen with a system of very narrow, equal and equidistant rectangular apertures or slits is called a "grating" or "diffraction grating." Diffraction gratings are usually formed by tracing a system of parallel equidistant straight lines on a glass plate with a diamond; these lines act like a system of narrow opaque screens, reflecting back the incident waves in all directions without allowing any to pass through, whereas the transparent spaces or strips between the lines allow the waves free transmission through the glass; these gratings are thus known as "reflection gratings." Let us now consider the diffraction pattern produced by such a grating; the given problem is evidently a particular case of the above general one, where we had a system of arbitrarily situated apertures of equal size in a large opaque screen. For gratings the above general formulae will evidently assume a much simpler form: the system of coordinates x', y' can evidently be so chosen that the coordinates $x_1'y_1'$, $x_2'y_2'$,... of given points in the different transparent strips of the grating may be written

$$x_1'=0,\ x_2'=d,\ x_3'=2d,\ \text{etc.},$$

and

$$y_1'=y_2'=y_3'=\ldots=0,$$

where d, the so-called "grating constant," denotes the distance between those points. By formulae (65) the quantities c' and s' will then assume the form

$$\left.\begin{aligned} c' &= 1+\cos\mu d+\cos 2\mu d+\cos 3\mu d+\ldots \\ s' &= \phantom{1+{}}\sin\mu d+\sin 2\mu d+\sin 3\mu d+\ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(69)$$

To determine the coefficient $c'^2+s'^2$ in the expression (66) for the resultant intensity, we write it in the form

$$c'^2+s'^2=(c'+is')(c'-is'),$$

where i is the imaginary unit $\sqrt{-1}$, replace here c' and s' by their values (69), and we have

$$\begin{aligned} c'^2+s'^2 = \{&1+(\cos\mu d+i\sin\mu d)+(\cos 2\mu d+i\sin 2\mu d)+\ldots \\ &+[\cos(n-1)\mu d+i\sin(n-1)\mu d]\} \\ \times\{&1+(\cos\mu d-i\sin\mu d)+(\cos 2\mu d-i\sin 2\mu d)+\ldots \\ &+[\cos(n-1)\mu d-i\sin(n-1)\mu d]\}, \end{aligned}$$

where n denotes the number of transparent strips in the given grating, hence

$$\begin{aligned} c'^2+s'^2 &= \left[1+e^{i\mu d}+e^{i2\mu d}+e^{i3\mu d}+\ldots e^{i(n-1)\mu d}\right] \\ &\quad\times\left[1+e^{-i\mu d}+e^{-i2\mu d}+e^{-i3\mu d}+\ldots e^{-i(n-1)\mu d}\right] \\ &= \frac{e^{in\mu d}-1}{e^{i\mu d}-1}\cdot\frac{e^{-in\mu d}-1}{e^{-i\mu d}-1}=\frac{2-e^{in\mu d}-e^{-in\mu d}}{2-e^{i\mu d}-e^{-i\mu d}} \\ &= \frac{2-(\cos n\mu d+i\sin n\mu d)-(\cos n\mu d-i\sin n\mu d)}{2-(\cos\mu d+i\sin\mu d)-(\cos\mu d-i\sin\mu d)} \\ &= \frac{1-\cos n\mu d}{1-\cos\mu d}=\left(\frac{\sin\frac{n\mu d}{2}}{\sin\frac{\mu d}{2}}\right)^2. \end{aligned}$$

By formula (66) the resultant intensity I at any point of the given diffraction pattern will, therefore, be given by the expression

$$I=A^2(c^2+s^2)\left(\frac{\sin\frac{n\mu d}{2}}{\sin\frac{\mu d}{2}}\right)^2,$$

which, by formula (63), can evidently be written in the form

$$I=I'\left(\frac{\sin\frac{\mu a}{2}}{\frac{\mu a}{2}}\right)\left(\frac{\sin\frac{n\mu d}{2}}{\sin\frac{\mu d}{2}}\right)^2, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70)$$

where I' denotes the intensity at the centre of the diffraction pattern produced by a *single* narrow slit of breadth a, here the breadth of the narrow transparent strips of the given grating.

We can now interpret the expression (70) for I as follows: its first two factors represent the diffraction pattern produced by a single narrow slit of breadth a (cf. formula (63)), whereas its last factor will vanish, when

$$\frac{n\mu d}{2}=h\pi,\quad h=1,\ 2,\ \ldots;$$

the diffraction pattern produced by the given grating will, therefore, be that of the narrow slit with its bright bands traversed by a series of dark bands or lines, whose positions are determined by the following values of μ:

$$\mu_1=\frac{2\pi}{nd},\quad \mu_2=\frac{4\pi}{nd},\quad \mu_3=\frac{6\pi}{nd},\ \ldots.$$

These bands evidently run parallel to one another and at right angles to the transparent strips or grooves of the grating, and are also equidistant. The greater the number n of the grooves, the closer together are these bands. It is also evident that the intensities of the intermediate maxima will, in general, be smaller than those of the maxima of the single slit. On the other hand, we obtain maxima of much greater intensity than those of the single slit at those points (lines) of the diffraction pattern, where $\frac{\mu d}{2} = \pi h$, for then the last factor of the expression (70) for I will evidently assume the value n^2, and hence the resultant intensity be n^2 times that of the single slit. For large values of n these bright maxima will alone be observable, the intermediate ones being so much fainter and closer together that they will escape observation.

It is now possible that for given large values of h the positions of some of the bright bands determined by the values $\mu = \frac{2\pi h}{d}$ will coincide with some of those of the dark bands in the pattern produced by the single slit and determined by the values $\mu = \frac{2\pi h}{a}$; in this particular case the second factor of the expression (70) for I will now vanish and hence I itself; for such values of μ these bright bands will not, therefore, appear, but will be replaced by dark ones. The condition for the appearance of these dark bands is evidently that the breadth a of the transparent strips and the grating constant d stand in a rational ratio to each other. For reflecting gratings it is easy to show that these dark bands, provided they appear, will be separated from one another by a considerable number of bright ones, for replace the breadth a of the transparent strips by that b of the fine grooves of the grating, as allowed by Babinet's principle, and the distance between two such consecutive dark bands will be given by

$$\mu_i - \mu_{i-1} = \frac{2\pi k}{b},$$

where i and k are integers and b is a small quantity compared with the grating constant d, whereas the distance between two consecutive bright bands will be given by

$$\mu_h - \mu_{h-1} = \frac{2\pi}{d}.$$

As we recede from the centre of the diffraction pattern, we encounter, therefore, a (great) number of equidistant and equally bright bands, before we reach the first dark band of the series in question.

If the grating is so placed that the incident waves strike it at right angles, then

$$\mu = \frac{2\pi}{\lambda} \sin \phi,$$

where ϕ denotes the angle of diffraction (cf. p. 235), and the positions of the bright bands, whose intensities are n^2 times those produced by the single slit, will be determined by

$$\mu = \frac{2\pi}{\lambda} \sin\phi = \frac{2\pi h}{d},$$

hence

$$\sin\phi = \frac{h\lambda}{d},$$

that is,

$$\sin\phi_1 = \frac{\lambda}{d}, \quad \sin\phi_2 = \frac{2\lambda}{d}, \quad \sin\phi_3 = \frac{3\lambda}{d}, \dots .$$

The Diffraction Spectra of White Light.—It follows from the last relations that the bright bands, whose intensities are n^2 times those produced by the single slit, will be equidistant, at least, for small values of ϕ, where $\sin\phi$ can be replaced by ϕ itself, and that the diffraction angle ϕ will then be directly proportional to the wave-length λ of the waves employed and indirectly proportional to the grating constant d. If we let white light pass through a diffraction grating, the waves of different wave-length or colour contained in it will, therefore, all be diffracted according to their wave-length, and thus produce spectra; these spectra are, therefore, known as "normal" ones, to distinguish them from the refraction spectra formed by glass prisms. Of these the first ($h=1$) or so-called "spectrum of the first order" will be absolutely pure, that is, there will be no overlapping of the waves of different colour in it, the second ($h=2$) or "spectrum of the second order" will be only partially pure, whereas that of the third order ($h=3$) will include the red rays of the further end of the second spectrum overlapping its own violet rays; as we continue to recede from the centre of the pattern, the overlapping of the waves of different colour from the different spectra proper will evidently increase and the spectra themselves thus become less and less pure.

Sommerfeld's Theory of Diffraction; Diffraction on Straight Edge of Large Reflecting Screen.—Fresnel's modified theory of diffraction was based on the assumption that the light-vector s vanished directly behind opaque bodies placed in the course of the waves and assumed its natural value in all regions that were illuminated directly from the source (cf. p. 199); as this assumption is now only approximately realized, the given theory can be regarded as only an approximate one, except at short distances from the obstructing body and its geometrical shadow (cf. assumption made on p. 200), as we shall see below. A more rigorous treatment of, at least, one problem on diffraction, that on the straight edge of a large screen, has now been effected by Sommerfeld;* his treatment of this problem enables us to examine the behaviour of the light-vector not only in the neighbourhood of the geometrical shadow but at any distance from it.

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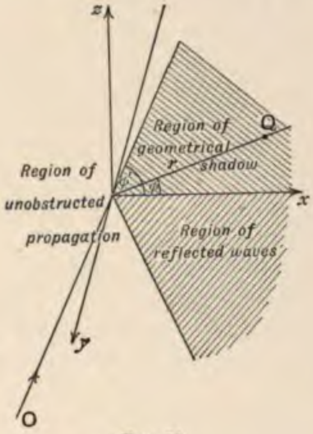
* *Math. Annalen*, Band 47, p. 317, 1895.

Sommerfeld starts from the differential equation for the light-vector s at any point

$$\frac{\partial^2 s}{\partial t^2}=v^2\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}\right) \quad\text{..........................(71)}$$

(cf. formula (48, V)) and seeks a solution for the same, which shall satisfy the surface-conditions on the obstructing screen. For simplicity let us assume that the source of light O be an infinitely long straight line parallel to the obstructing edge of the screen, which we shall choose as y-axis of a system of rectangular coordinates; the x-axis shall lie in the obstructing screen and the positive z-axis be directed away from the source O, as indicated in the annexed figure. We denote the angle, which the direction of propagation of the waves from the source O makes with the x-axis, by ϕ' (cf. Fig. 31). The light-vector s at any point Q will then evidently be a function only of x and z, that is, it will be defined by the differential equation



z
x
y
O
Q
r
φ′
φ
Region of geometrical shadow
Region of unobstructed propagation
Region of reflected waves

FIG. 31.

$$\frac{\partial^2 s}{\partial t^2}=v^2\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial z^2}\right), \quad\text{..........................(71A)}$$

or, if we replace the rectangular coordinates x, z by the polars r, ϕ, where

$$x=r\cos\phi$$
$$z=r\sin\phi$$

(cf. Fig. 31), by

$$\frac{\partial^2 s}{\partial t^2}=v^2\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}+\frac{1}{r^2}\frac{\partial^2 s}{\partial \phi^2}\right). \quad\text{.....................(72)}$$

Sommerfeld now assumes that the waves that strike the obstructing screen are all reflected and not, as assumed above by Fresnel, absorbed. There will then be three characteristic regions, instead of two as above, that will come into consideration here, the region of the geometrical shadow, that of the reflected waves and that of unobstructed propagation, as indicated in Fig. 31. The surface-conditions on such a (highly polished metallic) screen are, as we shall see in Chapters VII. and VIII. (cf. also below),

$$s=0, \qquad\qquad (73)$$

when the incident waves are polarized at right angles to the edge of the obstructing screen, and

$$\frac{\partial s}{\partial z}=0, \qquad\qquad (74)$$

when they are polarized parallel to that edge.

A solution of the differential equation (72), which will also satisfy these surface-conditions, is now

$$s=a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}\left\{e^{-i\gamma}\int_{-\infty}^{\sigma}e^{-i\frac{\pi v^2}{2}}dv \mp e^{-i\gamma'}\int_{-\infty}^{\sigma'}e^{-i\frac{\pi v^2}{2}}dv\right\}, \qquad (75)$$

where

$$\left.\begin{aligned} \gamma&=\frac{2\pi r}{\lambda}\cos(\phi-\phi'), & \gamma'&=\frac{2\pi r}{\lambda}\cos(\phi+\phi') \\ \sigma&=\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi-\psi'), & \sigma'&=-\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi+\phi') \end{aligned}\right\}, \qquad (76)$$

and the minus-sign before the second or last integral is to be chosen for the surface-condition (73) and the plus sign for the condition (74).

Solution for Light-Vector in Form of Complex Quantity; Expression for Intensity.—The solution (75) for the light vector s is a complex quantity of the form

$$\begin{aligned} s&=(A+iB)e^{i\frac{2\pi vt}{\lambda}} \\ &=(A+iB)\left(\cos\frac{2\pi vt}{\lambda}+i\sin\frac{2\pi vt}{\lambda}\right) \\ &=\left(A\cos\frac{2\pi vt}{\lambda}-B\sin\frac{2\pi vt}{\lambda}\right)+i\left(A\sin\frac{2\pi vt}{\lambda}+B\cos\frac{2\pi vt}{\lambda}\right). \end{aligned}$$

The physical meaning of such a solution is now to be sought either in its real part or in the real factor of its imaginary one.

The intensity I at any point Q produced by either of these physical expressions or solutions for s will now, by formula (4, IV.), be proportional to the expression

$$A^2+B^2 \text{ prop. to } I.$$

We also obtain this expression for I, on multiplying the given solution (75) expressed in the above complex form by its conjugate complex quantity

$$A - iBe^{-i\frac{2\pi vt}{\lambda}};$$

thus

$$(A+iB)\,e^{i\frac{2\pi vt}{\lambda}}.(A-iB)\,e^{-i\frac{2\pi vt}{\lambda}} = A^2 + B^2 \text{ prop. to } I, \ldots\ldots\ldots(77)$$

which can be interpreted as follows: "When the solution for the light-vector s is a complex quantity or expression, the resultant intensity will be proportional to the product of that quantity and its conjugate complex quantity."

Confirmation of given Solution.—We can confirm that the expression (75) for s is a solution of the differential equation (72), on replacing there s by that expression and performing the differentiations indicated. To confirm that the surface condition (73) is fulfilled, we put $\phi = 0$ and 2π in the respective solution (75) for s, and we have, since then $\gamma = \gamma'$ and $\sigma = \sigma'$,

$$s = a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}\left\{e^{-i\gamma}\int_{-\infty}^{\sigma} e^{-i\frac{\pi v^2}{2}}\,dv - e^{-i\gamma}\int_{-\infty}^{\sigma} e^{-i\frac{\pi v^2}{2}}\,dv\right\} = 0.$$

To confirm that the other surface condition (74) is fulfilled, we first form $\frac{\partial s}{\partial z}$; we have

$$\frac{\partial s}{\partial z} = \frac{\partial s}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial s}{\partial \phi}\frac{\partial \phi}{\partial z};$$

since now $r^2 = x^2 + z^2$ and $\phi = \arctan z/x$ (cf. the above relations between x, z and r, ϕ), we have

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

and $$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}(\arctan z/x) = \frac{x}{x^2+z^2},$$

which expressions evidently reduce to the following on the obstructing screen, that is, for $\phi = 0$ and 2π or $z = 0$:

$$\frac{\partial r}{\partial z} = 0 \text{ and } \frac{\partial \phi}{\partial z} = \frac{1}{x} = \frac{1}{r}.$$

The above differential quotient will, therefore, assume the following particular form on the obstructing screen:

$$\frac{\partial s}{\partial z} = \frac{1}{r}\frac{\partial s}{\partial \phi}.$$

Replace here s by its value (75) for the given surface-condition, and we have

$$\frac{\partial s}{\partial z}=\frac{1}{r}\frac{\partial s}{\partial \phi}=\frac{1}{r}a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}\frac{\partial}{\partial \phi}\left\{e^{-i\gamma}\int_{-\infty}^{\sigma}e^{-i\frac{\pi v^2}{2}}dv+e^{-i\gamma'}\int_{-\infty}^{\sigma'}e^{-i\frac{\pi v^2}{2}}dv\right\},$$

or, on replacing the γ's and the σ's by their values (76) and performing the differentiation indicated,

$$\begin{aligned}\frac{\partial s}{\partial z}=C\Bigg\{&e^{-i\frac{2\pi r}{\lambda}\cos(\phi-\phi')}\frac{\partial}{\partial \phi}\left[\Big|f(v)\Big|_{-\infty}^{\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi-\phi')}\right]\\ &+i\frac{2\pi r}{\lambda}\sin(\phi-\phi')e^{-i\frac{2\pi r}{\lambda}\cos(\phi-\phi')}\Big|f(v)\Big|_{-\infty}^{\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi-\phi')}\\ &+e^{-i\frac{2\pi r}{\lambda}\cos(\phi+\phi')}\frac{\partial}{\partial \phi}\left[\Big|f(v)\Big|_{-\infty}^{-\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi+\phi')}\right]\\ &+i\frac{2\pi r}{\lambda}\sin(\phi+\phi')e^{-i\frac{2\pi}{r}\cos(\phi+\phi')}\Big|f(v)\Big|_{-\infty}^{-\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi+\phi')}\Bigg\},\end{aligned}$$

where
$$C=\frac{1}{r}a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}$$

and
$$f(v)=\int e^{-i\frac{\pi v^2}{2}}dv.$$

On the obstructing screen, $\phi=0$ or 2π, the second and last terms of this expression for $\frac{\partial s}{\partial z}$ will now cancel one another, since the upper limits of the function $f(v)$ in both terms then assume one and the same value, whereas their coefficients differ only in sign. We can thus write the expression for $\frac{\partial s}{\partial z}$ in the form

$$\begin{aligned}\frac{\partial s}{\partial z}=C\Bigg\{&e^{-i\frac{2\pi r}{\lambda}\cos(\phi-\phi')}\left[\frac{\partial}{\partial v_1}f(v_1)\frac{\partial v_1}{\partial \phi}-\frac{\partial}{\partial \phi}f(-\infty)\right]\\ &+e^{-i\frac{2\pi r}{\lambda}\cos(\phi+\phi')}\left[\frac{\partial}{\partial v_2}f(v_2)\frac{\partial v_2}{\partial \phi}-\frac{\partial}{\partial \phi}f(-\infty)\right]\Bigg\},\end{aligned}$$

where $v_1=\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi-\phi')$, and $v_2=-\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi+\phi')$,

or, since

$$\frac{\partial v_1}{\partial \phi}=\sqrt{\frac{8r}{\lambda}}\frac{\cos\frac{1}{2}(\phi-\phi')}{2},\quad \frac{\partial v_2}{\partial \phi}=-\sqrt{\frac{8r}{\lambda}}\frac{\cos\frac{1}{2}(\phi+\phi')}{2},$$

and
$$\frac{\partial}{\partial \phi}f(-\infty)=0,$$

$$\frac{\partial s}{\partial z}=C\left\{e^{-i\frac{2\pi r}{\lambda}\cos(\phi-\phi')}\sqrt{\frac{8r}{\lambda}}\,\frac{\cos\frac{1}{2}(\phi-\phi')}{2}\,\frac{\partial}{\partial v_1}f(v_1)\right.$$
$$\left.-e^{-i\frac{2\pi r}{\lambda}\cos(\phi+\phi')}\sqrt{\frac{8r}{\lambda}}\,\frac{\cos\frac{1}{2}(\phi+\phi')}{2}\,\frac{\partial}{\partial v_2}f(v_2)\right\},$$

which on the obstructing screen, $\phi=0$ or 2π, will evidently assume the form

$$\frac{\partial s}{\partial z}=C\left\{e^{-i\frac{2\pi r}{\lambda}\cos\phi'}\sqrt{\frac{8r}{\lambda}}\,\frac{\cos\phi'/2}{2}\,\frac{\partial}{\partial\left[-\sqrt{\frac{8r}{\lambda}}\sin\phi'\right]}f\left[-\sqrt{\frac{8r}{\lambda}}\sin\phi'\right]\right.$$
$$\left.-e^{-i\frac{2\pi r}{\lambda}\cos\phi'}\sqrt{\frac{8r}{\lambda}}\,\frac{\cos\phi'/2}{2}\,\frac{\partial}{\partial\left[-\sqrt{\frac{8r}{\lambda}}\sin\phi'\right]}f\left[-\sqrt{\frac{8r}{\lambda}}\sin\phi'\right]\right\}=0,$$

as maintained above.

Geometrical Form of Expression for Light-Vector.—Put

$$e^{-i\frac{\pi v^2}{2}}=\cos\frac{\pi v^2}{2}-i\sin\frac{\pi v^2}{2}$$

in formula (75), and we have

$$\left.\begin{aligned}s=a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}\Bigg\{&e^{-i\gamma}\left[\int_{-\infty}^{\sigma}\cos\frac{\pi v^2}{2}\,dv-i\int_{-\infty}^{\sigma}\sin\frac{\pi v^2}{2}\,dv\right]\\&\mp e^{-i\gamma'}\left[\int_{-\infty}^{\sigma'}\cos\frac{\pi v^2}{2}\,dv-i\int_{-\infty}^{\sigma'}\sin\frac{\pi v^2}{2}\,dv\right]\Bigg\}\end{aligned}\right\}\quad\dots\;..(78)$$

$$=a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}\{e^{-i\gamma}(\xi-i\eta)\mp e^{-i\gamma'}(\xi'-i\eta')\},\quad\dots\dots\dots\dots(79)$$

where ξ, η and ξ', η' denote the projections on the ξ and η axes of the vectors $A'E$ and $A'E'$ respectively from the asymptotic point A' of Cornu's spiral (cf. Figs. 26 and 29) to those points E and E' of the same, whose coordinates are determined by the values of Fresnel's fundamental integrals (22A), whose upper limits are σ and σ' respectively and lower ones zero (cf. also p. 226).

Approximate Expression for Intensity within Image near Boundary of Geometrical Shadow.—Let us now examine the expression (78) or (79) for s at any point of the region $\phi'<\phi<\pi$; here

$$0<\tfrac{1}{2}(\phi-\phi')<\tfrac{1}{2}(\phi+\phi')<\pi,$$

and hence, by formulae (76), σ always positive and σ' always negative; for waves of short wave-length λ, as those of light, σ will thus assume large positive values and σ' (very) large negative ones at finite distances r from the edge of the obstructing screen. The last two integrals of formula (78) or the projections ξ', η' of the vector $A'E'$ of formula (79) will, therefore, be (very) small compared with the values assumed by the first two integrals or the projections ξ, η of the vector $A'E$, so that we can write the expression (79) for s here most approximately

$$s = a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}e^{-i\gamma}(\xi - i\eta), \text{ ....(80)}$$

and hence, by formula (77), the resultant intensity I

$$I \text{ prop. to } \left[a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}e^{-i\gamma}(\xi - i\eta)\right]\left[a\frac{1-i}{2}e^{-i\frac{2\pi vt}{\lambda}}e^{i\gamma}(\xi + i\eta)\right],$$

$$\text{prop. to } \frac{a^2}{2}(\xi^2 + \eta^2) = \frac{a^2}{2}\overline{A'E}^2. \text{ ....(81)}$$

Comparison of Sommerfeld's Expression (81) with Fresnel's.—Let us now compare the expression (81) for I with that obtained by Fresnel's (modified) method for the particular case, where the source of disturbance is at considerable (infinite) distance from the obstructing screen; here Fresnel's formula (38) can evidently be written

$$I \text{ prop. to } \frac{a^2}{2}\left[\left(\xi' + \frac{1}{2}\right)^2 + \left(\eta' + \frac{1}{2}\right)^2\right] = \frac{a^2}{2}\overline{A'E}^2, \text{ ....(82)}$$

where, however, the point E has the coordinates ξ', η' determined by the values of the (Fresnel's) integrals, whose upper limits v' are given by formula (41) and lower ones are zero; formula (41) will now assume here the particular form

$$v' = d\sqrt{\frac{2}{\lambda b}}. \text{ ....(83)}$$

The distance d of the point of observation from the boundary of the geometrical shadow will evidently be given here, that is, in terms of the quantities employed in Fig. 31, by $r\sin(\phi - \phi')$, where $(\phi - \phi')$ is the angle the vector r makes with the direction of propagation of the incident waves (cf. Fig. 31). On the other hand, since formula (83) for v' holds only in the next proximity of the boundary of the geometrical shadow (cf. p. 200), we may replace there b by r. We can, therefore, write v' most approximately

$$v' = r\sin(\phi - \phi')\sqrt{\frac{2}{\lambda r}} = \sqrt{\frac{2r}{\lambda}}\sin(\phi - \phi')$$

$$= \sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi - \phi')\cos\frac{1}{2}(\phi - \phi'), \text{ ....(84)}$$

or, since here the angle $(\phi - \phi')$ is small, most approximately

$$v' = \sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi - \phi'), \text{ ....(84A)}$$

which is Sommerfeld's expression for σ (cf. formulae (76)). It thus follows that the expression (82) for I obtained by Fresnel's method, but which holds only in the neighbourhood of the boundary of the geometrical shadow, will differ only infinitesimally from that (81) determined by Sommerfeld's method, at least, when the source of disturbance is at considerable distance from the obstructing screen. The value (82) determined by the former method will evidently be somewhat smaller than that (81) found by Sommerfeld.

General Expression for Intensity.—The approximate formula (80) for s and (81) for I will also hold in the region of unobstructed propagation, $\phi' < \phi < 2\pi - \phi'$ (cf. also Ex. 14), but in the regions of the geometrical shadow and the reflected waves we shall be obliged to employ the explicit expression (79) for s, since the last two integrals of formula (78) will evidently assume finite values in those regions. In these regions the light-vector will, therefore, be given by formula (79), and hence, by formula (77), the resultant intensity I by

$$\left.\begin{aligned} I \text{ prop. to } &\left\{a\frac{1+i}{2}e^{i\frac{2\pi vt}{\lambda}}\left[e^{-i\gamma}(\xi - i\eta) \mp e^{-i\gamma'}(\xi' - i\eta')\right]\right\} \\ \times &\left\{a\frac{1-i}{2}e^{-i\frac{2\pi vt}{\lambda}}\left[e^{i\gamma}(\xi + i\eta) \mp e^{i\gamma'}(\xi' + i\eta')\right]\right\} \end{aligned}\right\}, \text{ ......(85)}$$

which can be written in the form

$$\left.\begin{aligned} I \text{ prop. to } \frac{a^2}{2}[(\xi^2+\eta^2) + (\xi'^2+\eta'^2) \mp 2(\xi\xi' + \eta\eta')\cos(\gamma - \gamma') \\ \mp 2(\xi\eta' - \eta\xi')\sin(\gamma - \gamma')] \end{aligned}\right\}, \text{ ......(86)}$$

$$\text{prop. to } \frac{a^2}{2}[\overline{A'E}^2 + \overline{A'E'}^2 \mp 2A'E\,.\,A'E'\cos(\gamma - \gamma' + \chi)] \text{ ....(86A)}$$

(cf. Ex. 16 at end of chapter), where χ denotes the angle included between the two vectors $A'E$ and $A'E'$ of Cornu's spiral (cf. Fig. 29). This formula states that the resultant intensity is proportional to the square of the geometrical difference or sum, according as the incident waves are polarized at right angles or parallel to the edge of the obstructing screen, of two vectors $A'E$ and $A'E'$ in Cornu's spiral, which make the angle χ with one another.

Approximate Expression for Intensity within Geometrical Shadow at considerable Distance from its Boundary.—Let us now examine the expression (86A) for I at any point within the geometrical shadow that is at considerable distance from the boundary of the same, that is, for values of ϕ that are considerably smaller than the angle of incidence ϕ' of the incident waves; $\phi - \phi'$ will then be negative, $\phi + \phi'$ positive, and hence both σ and σ' negative; for waves of short wave-length λ, σ and σ' will, therefore, assume large negative values at finite distances r from the edge of the obstructing screen. The four

(Fresnel's) integrals that come into consideration here are now similar to those examined on pp. 226-228; by the geometrical properties peculiar to these integrals the vectors $A'E$ and $A'E'$ will be very short and will thus cut the (Cornu's) spiral approximately orthogonally; hence the lengths of these vectors will be given approximately by the radii of curvature, ρ and ρ', of the spiral itself at the two points E and E' of the same; by formula (25) we have then

$$A'E = \rho = \frac{1}{\pi\sigma} \text{ and } A'E' = \rho' = \frac{1}{\pi\sigma'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (87)$$

For similar reasons the angle χ, which the two vectors $A'E$ and $A'E'$ make with each other, may be replaced here by the angle τ included between the two tangents to the spiral at the points E and E'; the latter angle is now determined by formula (24); we can thus write most approximately

$$\chi = \tau = \frac{\pi}{2}(\sigma^2 - \sigma'^2).$$

On replacing the vectors $A'E$ and $A'E'$ and the angles χ, γ and γ' by their above values in formula (86A) for I, we have

$$I \text{ prop. to } \frac{a^2}{2\pi^2}\left\{\frac{1}{\sigma^2} + \frac{1}{\sigma'^2} \mp \frac{2}{\sigma\sigma'}\cos\left[\frac{2\pi r}{\lambda}(\cos\phi - \phi') - \frac{2\pi r}{\lambda}\cos(\phi + \phi')\right.\right.$$
$$\left.\left. + \frac{4\pi r}{\lambda}\sin^2\tfrac{1}{2}(\phi - \phi') - \frac{4\pi r}{\lambda}\sin^2\tfrac{1}{2}(\phi + \phi')\right]\right\},$$

or, since the angle $(\gamma - \gamma' + \chi)$ evidently vanishes here,

$$I \text{ prop. to } \frac{a^2}{2\pi^2}\left(\frac{1}{\sigma^2} + \frac{1}{\sigma'^2} \mp \frac{2}{\sigma\sigma'}\right) = \frac{a^2}{2\pi^2}\left(\frac{1}{\sigma} \mp \frac{1}{\sigma'}\right)^2.$$

Lastly, replace here σ and σ' by their values (76), and we have

$$I \text{ prop. to } \frac{a^2}{2\pi^2}\frac{\lambda}{8r}$$
$$\times \frac{\sin^2\tfrac{1}{2}(\phi + \phi') + \sin^2\tfrac{1}{2}(\phi - \phi') \pm 2\sin\tfrac{1}{2}(\phi + \phi')\sin\tfrac{1}{2}(\phi - \phi')}{\sin^2\tfrac{1}{2}(\phi - \phi')\sin^2\tfrac{1}{2}(\phi + \phi')}$$

$$\text{prop. to } \frac{a^2}{\pi^2}\frac{\lambda}{2r}$$
$$\times \frac{(\sin^2\phi/2\cos^2\phi'/2 + \cos^2\phi/2\sin^2\phi'/2) \pm (\sin^2\phi/2\cos^2\phi'/2 - \cos^2\phi/2\sin^2\phi'/2)}{[1 - \cos(\phi - \phi')][1 - \cos(\phi + \phi')]};$$

which gives $$I \text{ prop. to } \frac{a^2}{\pi^2}\frac{\lambda}{r}\frac{\sin^2\phi/2\cos^2\phi'/2}{(\cos\phi - \cos\phi')^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (88)$$

when the waves are polarized at right angles to the edge of the obstructing screen, and

$$I \text{ prop. to } \frac{a^2}{\pi^2}\frac{\lambda}{r}\frac{\cos^2\phi/2\sin^2\phi'/2}{(\cos\phi - \cos\phi')^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (89)$$

when they are polarized parallel to that edge.

For a further examination of Sommerfeld's formulae see Exs. 17-19 at end of chapter.

Shortcomings of Sommerfeld's Theory.—It follows from formulae (88) and (89) that along the screen, $\phi=0$, the waves will be polarized parallel to its edge, and that as ϕ increases both intensities, that parallel to its edge and that at right angles to the same, will increase, but their difference will diminish, whereby, however, the latter intensity will always remain smaller than the former; moreover, both intensities will be directly proportional to the wave-length λ of the waves employed. If we employ white light, the waves of greater wave-length should, therefore, predominate well within the geometrical shadow, that is, for small values of ϕ (cf. also formulae (81)-(84A)). Observation* now shows that the distribution of the waves of different wave-length or colour within the geometrical shadow is not according to this law, but that it depends on the nature of the screen, whether its diffracting edge be sharp or rounded, etc.; certain screen constants would, therefore, have to be introduced into the surface conditions (73) and (74) in order that the results obtained should agree with observation; not only the determination of these constants but also the integration of our differential equation (72) for other surface-conditions than the above (73) and (74) offers now unsurmountable difficulties. In consideration of these shortcomings it is evident that little has been gained by the introduction of the "totally reflecting" diffraction screen assumed by Sommerfeld in place of the "opaque" screen employed above in Fresnel's (modified) theory of diffraction. Whether the inaccuracies due to the introduction of the totally reflecting screen in the above form, where the screen constants have been entirely neglected, are of greater moment than those that arise from the assumption made in Fresnel's (modified) theory that, namely, the light-vector s vanishes directly behind opaque obstacles but assumes its natural value at all other points on the surface of integration, cannot well be decided except by experiment.

Form of Maxwell's Fundamental Equations for Sommerfeld's Light-Vector s.—We have seen in Chapter I. that the electromagnetic state of Maxwell's ether is defined by differential equations of the form (71), where s is to be replaced by the component electric or magnetic moments X, Y, Z, or a, b, c respectively (cf. formulae (16, I.) and (17, I.)). These components will now be the derivatives of the light-vector s with regard to x, y, and z, and Maxwell's fundamental equations (12, I.) and (13, I.) must evidently hold for those components (cf. Ex. 20). These

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*Gouy, *Ann. de Chim. et de Phys.*, (6) 8, p. 145, 1886.

equations will evidently assume here, where the electromagnetic state has been assumed to be a function of x and z only, the form

$$\left.\begin{array}{lll} \dfrac{M}{v_0}\dfrac{dX}{dt}=\dfrac{db}{dt}, & \dfrac{M}{v_0}\dfrac{dY}{dt}=\dfrac{dc}{dx}-\dfrac{da}{dz}, & \dfrac{M}{v_0}\dfrac{dZ}{dt}=-\dfrac{db}{dx} \\ \text{and}\quad \dfrac{D}{v_0}\dfrac{da}{dt}=-\dfrac{dY}{dz}, & \dfrac{D}{v_0}\dfrac{db}{dt}=\dfrac{dX}{dz}-\dfrac{dZ}{dx}, & \dfrac{D}{v_0}\dfrac{dc}{dt}=\dfrac{dY}{dx} \end{array}\right\} \ldots\ldots\ldots(90)$$

The Primary and Secondary Waves and the Different Theories of Diffraction.—We have already observed on pp. 195-196 that both the primary and secondary waves may give rise to phenomena of diffraction. A treatment of these waves according to Fresnel's modified theory of diffraction meets at the outset with unsurmountable difficulties; the first such is to establish a formula similar to formula (55, V.), which shall express the light-vector s_0 at any point as an integral taken over any suitably chosen surface enclosing that point. In the derivation of formula (55, V.) we assumed that the light-vector at any point, as on the surface of integration, was a purely spherical wave-function, an assumption that cannot be maintained for the primary or secondary waves and of which we made free use in the derivation not only of formula (55, V.) itself, but also of the fundamental formulae employed at the very start. On the other hand, a treatment of the diffraction phenomena of the primary and secondary waves according to Sommerfeld's theory would be possible, provided we could find solutions for s that would represent primary and secondary waves and also fulfil the surface-conditions on the obstructing screen; but in consideration of the complicated form of the vector s for the simple case examined by Sommerfeld, we could hardly expect much success in that direction.

The Roentgen Rays as Impulses and Sommerfeld's Theory of Diffraction.—Sommerfeld has also developed a theory of diffraction for waves that consist of a succession of short and violent impulses, to which class the Roentgen rays are often assumed to belong;* as an exposition of his theory would throw little light on the theory of diffraction proper, we refer here to Sommerfeld's paper† on the subject. We may remark, however, that the same scruples, which we had in accepting conclusively his above theory on diffraction, also arise here.

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* Cf. E. Wiechert, *Abhandlungen der Phys.-Oekon. Gesellschaft zu Königsberg*, 1896, pp. 1 and 45; also Wiedemann's *Annalen*, Bd. 59, 1896 (§ 6).

† Cf. also Sir George Stokes, *Proceedings of the Cambridge Philosophical Society*, vol. 9, p. 215, 1896, and *Proceedings of the Manchester Lit. and Phil. Society*, 1897.

EXAMPLES.

1. The breadth of the nth exterior band or fringe produced on the screen MN by the straight edge A of the opaque obstacle AB of Fig. 21 is determined by the expression

$$[\sqrt{2n+1}-\sqrt{2n-1}]\sqrt{\lambda\frac{b(a+b)}{a}}\ \text{for a bright band,}$$

and

$$[\sqrt{n}-\sqrt{n-1}]\sqrt{2\lambda\frac{b(a+b)}{a}}\ \text{for a dark band,}$$

where a denotes the distance of the edge of the obstacle from the source O and b its distance from the screen MN.

By Fig. 21 the following geometrical relations evidently hold between the distance x of any band n at Q from the boundary of the geometrical shadow P_0 and the other distances:

$$OQ=\sqrt{(a+b)^2+x^2}=(a+b)\sqrt{1+\frac{x^2}{(a+b)^2}}$$

and

$$AQ=\sqrt{b^2+x^2}=b\sqrt{1+\frac{x^2}{b^2}}.$$

For light-waves the second term of either expression under these square-root signs will be very small compared with the first or unity, so that, by the binomial theorem, the expressions for OQ and AQ can be written approximately,

$$\left.\begin{aligned}OQ&=(a+b)\left[1+\frac{x^2}{2(a+b)^2}\right]=a+b+\frac{x^2}{2(a+b)}\\ \text{and}\quad AQ&=b\left[1+\frac{x^2}{2b^2}\right]=b+\frac{x^2}{2b}\end{aligned}\right\}.\ \ldots\ldots\ldots\ldots(a)$$

For a bright band or maximum of intensity the difference in the paths AQ and PQ traversed by the elementary waves from the last unscreened half-period element of the given wave-front and those from the pole of the same must now, as we have seen on p. 187, be an odd multiple of the wave-length λ halved of the waves employed, that is, we must have

$$AQ-PQ=(2n+1)\lambda/2\ \ldots\ldots\ldots\ldots\ldots\ldots(b)$$

(cf. formula on p. 187), where n is an integer.

Similarly, for a dark band, we must have

$$AQ-PQ=2n\lambda/2.\ \ldots\ldots\ldots\ldots\ldots\ldots(c)$$

Since now $PQ=OQ-OP=OQ-a,$

we can write the condition (b) for a bright band,

$$AQ-(OQ-a)=(2n+1)\lambda/2,$$

and that (c) for a dark band,

$$AQ-(OQ-a)=n\lambda.$$

Replace here AQ and OQ by their approximate values (a), and we have

$$b+\frac{x^2}{2b}-\left[a+b+\frac{x^2}{2(a+b)}-a\right]=\frac{a}{2b(a+b)}x^2=(2n+1)\lambda/2$$

and

$$\frac{a}{2b(a+b)}x^2=n\lambda;$$

hence $$x=\sqrt{(2n+1)\lambda\frac{b(a+b)}{a}}\text{ for a bright band}$$

and $$x=\sqrt{2n\lambda\frac{b(a+b)}{a}}\text{ for a dark band.}$$

The breadth of a dark band will, therefore, be given by the expression

$$\sqrt{(2n+1)\lambda\frac{b(a+b)}{a}}-\sqrt{(2n-1)\lambda\frac{b(a+b)}{a}}=[\sqrt{2n+1}-\sqrt{2n-1}]\sqrt{\lambda\frac{b(a+b)}{a}}$$

and that of a bright band by

$$[\sqrt{n}-\sqrt{n-1}]\sqrt{2\lambda\frac{b(a+b)}{a}}.$$

Q.E.D.

2. The diffraction bands on the exterior of the geometrical shadow of a large opaque obstacle decrease at first rapidly in breadth, as we recede from the boundary of the shadow. Show, on accepting the formulae established in the preceding example, that the breadths of the following bright bands are:

Band (n),	1	2	3	4	5	9	10	25	26
Breadth,	1	0·4142	0·3179	0·2679	0·2361	0·1716	0·1623	0·1010	0·099

where the breadth of the first band is taken as unity.

3. Show that the breadth of any diffraction band within the geometrical shadow of a small opaque obstacle (wire) is according to the (Fresnel's) methods employed on pp. 188-192 given by the expression

$$\frac{\lambda b}{c},$$

where b denotes the distance between the screen of observation and the obstacle, and c the distance (breadth) between the two diffracting edges of the latter (cf. formula (7, IV.) and Fig. 12).

4. Examine by the (Fresnel's) methods employed on pp. 185-195 the diffraction phenomena produced by a very small circular aperture in a large opaque screen. Show that along the central axis of the image the intensity passes through a succession of maxima and minima, and determine approximately the distances of these maxima and minima from the aperture.

We divide the unscreened portion of the wave-front that passes through the edge of the circular aperture up into circular half-period elements with respect to the point, at which the intensity is sought. For a very small aperture the area of any such half-period element will now be given approximately by

$$2\pi\frac{r_1}{r_2}(r_2-r_1)\frac{\lambda}{2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

(cf. formula (5, V.) and Fig. 17), where r_1 denotes the distance of the aperture and r_2 that of the point of observation from the source.

The area of the whole unscreened portion of the given wave-front or the circular aperture itself is evidently πr^2, where r denotes the radius of that aperture. Since now for a very small aperture the different half-period elements will have approximately one and the same area (cf. the expression (a)), we may evidently put

$$n\pi\frac{r_1}{r_2}(r_2-r_1)\lambda=\pi r^2,$$

where n shall denote the exact number of half-period elements, into which the unscreened portion of the given wave-front may be divided, corresponding to the points of maximum and minimum intensity sought on the central axis of the given image. This relation gives

$$r_2-r_1=\frac{r_2r^2}{n\lambda r_1},$$

or

$$r_2-r_1=\frac{[(r_2-r_1)+r_1]r^2}{n\lambda r_1}$$

(cf. Fig. 17), hence

$$r_2-r_1=\frac{r_1r^2}{n\lambda r_1-r^2}; \quad \text{……………………………………}(b)$$

by which the distances of the maxima and minima of intensity from the given aperture are determined according as n is odd or even respectively.

5. Determine the formulae for the vector s_0 and intensity I of waves of long wave-length, as the electric waves, corresponding to formulae (13), (15), (18), (18A), (19), (29), (35) and (38) in text for light-waves.

By formulae (54, V.), which holds for waves of long wave-length, the expression for the vector sought can evidently be written here, where the source of disturbance O and the point of observation Q are both supposed to be at considerable distance from the aperture s in the large obstructing screen (cf. Fig. 25),

$$\left.\begin{aligned}&\frac{2\pi a}{\lambda}\,\frac{1}{\rho r}[\cos(n,\rho)-\cos(n,r)]\int\frac{\cos 2\pi}{\lambda}[vt-(\rho+r)]\,ds\\&\qquad+\frac{a}{\rho r}\left[\frac{\cos(n,\rho)}{\rho}-\frac{\cos(n,r)}{r}\right]\int\sin\frac{2\pi}{\lambda}[vt-(\rho+r)]\,ds=4\pi s_0\end{aligned}\right\}\text{.....}(13'\text{A})$$

(cf. p. 202).

Replace here ρ and r by their approximate values (12A), and we find

$$\left.\begin{aligned}4\pi s_0=\frac{2\pi a}{\lambda}\,\frac{1}{\rho r}[\cos(n,\rho)-\cos(n,r)]&\left\{\cos\frac{2\pi}{\lambda}[vt-(\rho_1+\rho_0)]\int\cos[f(x,y)]\,ds\right.\\&\left.\quad-\sin\frac{2\pi}{\lambda}[vt-(\rho_1+\rho_0)]\int\sin[f(x,y)]\,ds\right\}\\+\frac{a}{\rho r}\left[\frac{\cos(n,\rho)}{\rho}-\frac{\cos(n,r)}{r}\right]&\left\{\sin\frac{2\pi}{\lambda}[vt-(\rho_1+\rho_0)]\int\cos[f(x,y)]\,ds\right.\\&\left.\quad+\cos\frac{2\pi}{\lambda}[vt-(\rho_1+\rho_0)]\int\sin[f(x,y)]\,ds\right\}\end{aligned}\right\},(15')$$

which we can write in the form

$$\left.\begin{aligned}s_0=\sin\omega&\left\{-A_1\int\sin[f(x,y)]\,ds+A_2\int\cos[f(x,y)]\,ds\right\}\\&+\cos\omega\left\{A_1\int\cos[f(x,y)]\,ds+A_2\int\sin[f(x,y)]\,ds\right\}\end{aligned}\right\},\ \text{………}(18')$$

where

$$\left.\begin{aligned}A_1&=\frac{a}{2\lambda\rho r}[\cos(n,\rho)-\cos(n,r)]\\A_2&=\frac{a}{4\pi\rho r}\left[\frac{\cos(n,\rho)}{\rho}-\frac{\cos(n,r)}{r}\right]\\\omega&=\frac{2\pi}{\lambda}[vt-(\rho_1+\rho_0)]\end{aligned}\right\},\ \text{……………………}(18'\text{A})$$

and $f(x,y)$ is given by formula (14) or (17) according to the system of coordinates employed.

The vector s_0 can thus be conceived as due to the mutual action of two systems of (elementary) waves emitted from the given aperture, whose difference in phase is $\pi/2$ and whose amplitudes are determined by the expressions

$$-A_1\int\sin[f(x,y)]ds+A_2\int\cos[f(x,y)]ds$$

and

$$A_1\int\cos[f(x,y)]ds+A_2\int\sin[f(x,y)]ds$$

(cf. pp. 203 and 213).

For the diffraction on the straight edge of the large opaque screen of Fig. 25, formula (18′) will assume the form

$$\left.\begin{aligned}s_0=\sin\omega\Big\{&A_1\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\sin\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dx\,dy\\&+A_2\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\cos\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dx\,dy\Big\}\\+\cos\omega\Big\{&A_1\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\cos\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dx\,dy\\&-A_2\int_{-\infty}^{x'}\int_{-\infty}^{\infty}\sin\left[\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)(x^2\cos^2\phi+y^2)\right]dx\,dy\Big\}\end{aligned}\right\}\quad\text{.........}(19')$$

(cf. p. 205).

On replacing here x and y by the variables v and u respectively employed on p. 205, we find, by formulae (32), the following expression for s_0:

$$\left.\begin{aligned}s_0=\frac{\sin\omega}{\cos\phi\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\Big\{&A_1\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv+\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right]\\&+A_2\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv-\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right]\Big\}\\+\frac{\cos\omega}{\cos\phi\frac{2}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\Big\{&A_1\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv-\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right]\\&-A_2\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv+\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right]\Big\}\end{aligned}\right\}\quad\text{..........}(29')$$

The resultant intensity I at any point Q will, therefore, be proportional to the expression

$$\begin{aligned}\frac{\lambda^2}{4\cos^2\phi\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)^2}\Big\{&A_1\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv+\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right]\\&+A_2\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv-\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right]\Big\}^2\\+\frac{\lambda^2}{4\cos^2\phi\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)^2}\Big\{&A_1\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv-\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right]\\&-A_2\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv+\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right]\Big\}^2\end{aligned}$$

$$=\frac{\lambda^2}{4\cos^2\phi\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)^2}\left\{\left[(A_1-A_2)\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv+(A_1+A_2)\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right]^2\right.$$

$$\left.+\left[(A_1-A_2)\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv-(A_1+A_2)\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right]^2\right\}$$

$$=\frac{(A_1{}^2+A_2{}^2)\lambda^2}{2\cos^2\phi\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)^2}\left\{\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right]^2+\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right]^2\right\},$$

or, if we replace here A_1 and A_2 by their values (18′A) and ρ and r by their approximate values ρ_1 and ρ_0 respectively (cf. p. 202),

$$=\left[\frac{a^2}{2(\rho_1+\rho_0)^2}+\frac{a^2\lambda^2}{8\pi^2\rho_1{}^2\rho_0{}^2}\right]\left\{\left[\int_{-\infty}^{v'}\sin\frac{\pi v^2}{2}dv\right]^2+\left[\int_{-\infty}^{v'}\cos\frac{\pi v^2}{2}dv\right]^2\right\}\text{ prop. to } I, \quad\ldots(35')$$

which referred to Cornu's spiral can be written in the form

$$I\text{ prop. to }\left[\frac{a^2}{2(\rho_1+\rho_0)^2}+\frac{a^2\lambda^2}{8\pi^2\rho_1{}^2\rho_0{}^2}\right]\cdot[(\xi'+\tfrac{1}{2})^2+(\eta'+\tfrac{1}{2})^2]. \quad\ldots\ldots\ldots\ldots\ldots(38')$$

The above formulae differ from those in the text for light-waves only in the values of their coefficients; the diffraction phenomena of waves of long wave-length, as the electric waves, will, therefore, be similar to those produced by light-waves, differing only quantitatively from the latter.

6. Show that $$\int_i^{i+u}\cos\frac{\pi v^2}{2}dv=\frac{1}{\pi i}\left[\sin\frac{\pi}{2}(i^2+2iu)-\sin\frac{\pi i^2}{2}\right]$$

and $$\int_i^{i+u}\sin\frac{\pi v^2}{2}=dv\frac{1}{\pi i}\left[-\cos\frac{\pi}{2}(i^2+2iu)+\cos\frac{\pi i^2}{2}\right],*$$

where i is given and u is so small a quantity that its square (u^2) may be neglected compared with u itself.

We replace the variable v in the given integrals by $i+u$, and we have

$$\int_i^{i+u}\cos\frac{\pi v^2}{2}dv=\int_0^u\cos\frac{\pi}{2}(i+u)^2du$$

and $$\int_i^{i+u}\sin\frac{\pi v^2}{2}dv=\int_0^u\sin\frac{\pi}{2}(i+u)^2du,$$

or, on rejecting the terms in u^2,

$$\int_i^{i+u}\cos\frac{\pi v^2}{2}dv=\int_0^u\cos\frac{\pi}{2}(i^2+2iu)du=\cos\frac{\pi i^2}{2}\int_0^u\cos\pi iu\,du-\sin\frac{\pi i^2}{2}\int_0^u\sin\pi iu\,du$$

and $$\int_i^{i+u}\sin\frac{\pi v^2}{2}dv=\int_0^u\sin\frac{\pi}{2}(i^2+2iu)du=\sin\frac{\pi i^2}{2}\int_0^u\cos\pi iu\,du+\cos\frac{\pi i^2}{2}\int_0^u\sin\pi iu\,du;$$

which integrated give

$$\int_i^{i+u}\cos\frac{\pi v^2}{2}dv=\frac{1}{\pi i}\cos\frac{\pi i^2}{2}\sin\pi iu+\frac{1}{\pi i}\sin\frac{\pi i^2}{2}[\cos\pi iu-1]$$

$$=\frac{1}{\pi i}\left[\sin\frac{\pi}{2}(i^2+2iu)-\sin\frac{\pi i^2}{2}\right]$$

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* Cf. Fresnel, *Oeuvres*, tom. 1, p. 319; also Preston's *Theory of Light*, p. 275.

and
$$\int_i^{i+u} \sin\frac{\pi v^2}{2}\,dv = \frac{1}{\pi i}\sin\frac{\pi i^2}{2}\sin \pi i u - \frac{1}{\pi i}\cos\frac{\pi i^2}{2}[\cos \pi i u - 1]$$
$$= \frac{1}{\pi i}\left[-\cos\frac{\pi}{2}(i^2+2iu) + \cos\frac{\pi i^2}{2}\right].$$
Q. E. D.

Fresnel employed these integrals for calculating the values of his integrals (22), taking $u=0{\cdot}1$ and i successively equal to $0{\cdot}0$, $0{\cdot}1$, $0{\cdot}2$, etc.; by this method he was enabled to plot the curve (spiral) ξ, η, determining successively the values of ξ and η for every $0{\cdot}1$ of the quantity $u(v)$ (cf. p. 211).

7. Fresnel's integrals (22A) can be written in the form
$$\int_0^v \cos\frac{\pi v^2}{2}\,dv = M\cos\frac{\pi v^2}{2} + N\sin\frac{\pi v^2}{2},$$
and
$$\int_0^v \sin\frac{\pi v^2}{2}\,dv = M\sin\frac{\pi v^2}{2} - N\cos\frac{\pi v^2}{2},^*$$
where
$$M = v\left[1 - \frac{(\pi v^2)^2}{1\,.\,3\,.\,5} + \frac{(\pi v^2)^4}{1\,.\,3\,.\,5\,.\,7\,.\,9} - + \ldots\right]$$
and
$$N = v\left[\frac{\pi v^2}{1\,.\,3} - \frac{(\pi v^2)^3}{1\,.\,3\,.\,5\,.\,7} + \frac{(\pi v^2)^5}{1\,.\,3\,.\,5\,.\,7\,.\,9\,.\,11} - + \ldots\right].$$

Integrate the given integrals by parts, and we have
$$\int_0^v \cos\frac{\pi v^2}{2}\,dv = v\cos\frac{\pi v^2}{2} + \pi\int_0^v v^2\sin\frac{\pi v^2}{2}dv$$
$$= v\cos\frac{\pi v^2}{2} + \pi\left\{\frac{v^3}{3}\sin\frac{\pi v^2}{2} - \frac{\pi}{3}\int_0^v v^4\cos\frac{\pi v^2}{2}\,dv\right\}$$
$$= v\cos\frac{\pi v^2}{2} + \pi\left\{\frac{v^2}{3}\sin\frac{\pi v^2}{2} - \frac{\pi}{3}\left\{\frac{v^5}{5}\cos\frac{\pi v^2}{2} + \frac{\pi}{5}\int_0^v v^6\sin\frac{\pi v^2}{2}\,dv\right\}\right\}$$
etc.
$$= v\left[1 - \frac{(\pi v^2)^2}{3\,.\,5} + - \ldots\right]\cos\frac{\pi v^2}{2} + v\left[\frac{\pi v^2}{3} - \frac{(\pi v^2)^3}{3\,.\,5\,.\,7} + - \ldots\right]\sin\frac{\pi v^2}{2}$$
and
$$\int_0^v \sin\frac{\pi v^2}{2}\,dv = v\sin\frac{\pi v^2}{2} - \pi\int_0^v v^2\cos\frac{\pi v^2}{2}\,dv$$
$$= v\sin\frac{\pi v^2}{2} - \pi\left\{\frac{v^3}{3}\cos\frac{\pi v^2}{2} + \frac{\pi}{3}\int_0^v v^2\sin\frac{\pi v^2}{2}\,dv\right\}$$
$$= v\sin\frac{\pi v^2}{2} - \pi\left\{\frac{v^3}{3}\cos\frac{\pi v^2}{2} + \frac{\pi}{3}\left\{\frac{v^5}{5}\sin\frac{\pi v^2}{2} - \frac{\pi}{5}\int_0^v v^6\cos\frac{\pi v^2}{2}dv\right\}\right\}$$
etc.
$$= v\left[1 - \frac{(\pi v^2)^2}{3\,.\,5} + - \ldots\right]\sin\frac{\pi v^2}{2} - v\left[\frac{\pi v^2}{3} - \frac{(\pi v^2)^3}{3\,.\,5\,.\,7} + - \ldots\right]\cos\frac{\pi v}{2}.$$

8. Show that the following (Fresnel's) integrals can be written in the form:
$$\int_v^\infty \cos\frac{\pi v^2}{2}\,dv = P\cos\frac{\pi v^2}{2} - Q\sin\frac{\pi v^2}{2}$$
and
$$\int_v^\infty \sin\frac{\pi v^2}{2}\,dv = P\sin\frac{\pi v^2}{2} + Q\cos\frac{\pi v^2}{2},$$

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* Cf. Knochenhauer, *Die Undulationstheorie des Lichts*, p. 36.

where
$$P=v\left[\frac{1}{(\pi v^2)^2}-\frac{1.3.5}{(\pi v^2)^4}+\frac{1.3.5.7.9}{(\pi v^2)^6}-+\dots\right]$$
and
$$Q=v\left[\frac{1}{\pi v^2}-\frac{1.3}{(\pi v^2)^3}+\frac{1.3.5.7}{(\pi v^2)^5}-+\dots\right].^*$$

To evaluate these integrals, we start from the two expressions
$$-\frac{1}{\pi v^{\kappa+1}}\cos\frac{\pi v^2}{2} \text{ and } \frac{1}{\pi v^{\kappa+1}}\sin\frac{\pi v^2}{2},$$
where κ is an integer, differentiate them with regard to v, and we have
$$\frac{d}{dv}\left[-\frac{1}{\pi v^{\kappa+1}}\cos\frac{\pi v^2}{2}\right]=\frac{1}{v^\kappa}\sin\frac{\pi v^2}{2}+\frac{\kappa+1}{\pi}\frac{1}{v^{\kappa+2}}\cos\frac{\pi v^2}{2}$$
and
$$\frac{d}{dv}\left[\frac{1}{\pi v^{\kappa+1}}\sin\frac{\pi v^2}{2}\right]=\frac{1}{v^\kappa}\cos\frac{\pi v^2}{2}-\frac{\kappa+1}{\pi}\frac{1}{v^{\kappa+2}}\sin\frac{\pi v^2}{2};$$
we then multiply these expressions by dv and integrate from $v=v$ to $v=\infty$, and we find
$$\left|-\frac{1}{\pi v^{\kappa+1}}\cos\frac{\pi v^2}{2}\right|_v^\infty=\int_v^\infty\frac{1}{v^\kappa}\sin\frac{\pi v^2}{2}\,dv+\frac{\kappa+1}{\pi}\int_v^\infty\frac{1}{v^{\kappa+2}}\cos\frac{\pi v^2}{2}\,dv$$
and
$$\left|\frac{1}{\pi v^{\kappa+1}}\sin\frac{\pi v^2}{2}\right|_v^\infty=\int_v^\infty\frac{1}{v^\kappa}\cos\frac{\pi v^2}{2}\,dv-\frac{\kappa+1}{\pi}\int_v^\infty\frac{1}{v^{\kappa+2}}\sin\frac{\pi v^2}{2}\,dv,$$
hence
$$\int_v^\infty\frac{1}{v^\kappa}\sin\frac{\pi v^2}{2}\,dv=\frac{1}{\pi v^{\kappa+1}}\cos\frac{\pi v^2}{2}-\frac{\kappa+1}{\pi}\int_v^\infty\frac{1}{v^{\kappa+2}}\cos\frac{\pi v^2}{2}\,dv$$
and
$$\int_v^\infty\frac{1}{v^\kappa}\cos\frac{\pi v^2}{2}\,dv=-\frac{1}{\pi v^{\kappa+1}}\sin\frac{\pi v^2}{2}+\frac{\kappa+1}{\pi}\int_v^\infty\frac{1}{v^{\kappa+2}}\sin\frac{\pi v^2}{2}\,dv;$$
by the repeated application of these reduction-formulae ($\kappa=0, 1, 2, 3\dots$) we can evidently write the given integrals
$$\begin{aligned}\int_v^\infty\cos\frac{\pi v^2}{2}\,dv&=-\frac{1}{\pi v}\sin\frac{\pi v^2}{2}+\frac{1}{\pi}\int_v^\infty\frac{1}{v^2}\sin\frac{\pi v^2}{2}\,dv\\
&=-\frac{1}{\pi v}\sin\frac{\pi v^2}{2}+\frac{1}{\pi}\left\{\frac{1}{\pi v^3}\cos\frac{\pi v^2}{2}-\frac{3}{\pi}\int_v^\infty\frac{1}{v^4}\cos\frac{\pi v^2}{2}\,dv\right\}\\
&=-\frac{1}{\pi v}\sin\frac{\pi v^2}{2}+\frac{1}{\pi^2 v^3}\cos\frac{\pi v^2}{2}-\frac{3}{\pi^2}\left\{-\frac{1}{\pi v^5}\sin\frac{\pi v^2}{2}+\frac{5}{\pi}\int_v^\infty\frac{1}{v^6}\sin\frac{\pi v^2}{2}\,dv\right\}\\
&=-\frac{1}{\pi v}\sin\frac{\pi v^2}{2}+\frac{1}{\pi^2 v^3}\cos\frac{\pi v^2}{2}+\frac{3}{\pi^3 v^5}\sin\frac{\pi v^2}{2}-\frac{3.5}{\pi^3}\\
&\qquad\times\left\{\frac{1}{\pi v^7}\cos\frac{\pi v^2}{2}-\frac{7}{\pi}\int_v^\infty\frac{1}{v^8}\cos\frac{\pi v^2}{2}\,dv\right\}\end{aligned}$$

etc.

$$\begin{aligned}&=\left\{\frac{1}{\pi^2 v^3}-\frac{3.5}{\pi^4 v^7}+-\dots\right\}\cos\frac{\pi v^2}{2}\\
&\qquad-\left\{\frac{1}{\pi v}-\frac{3}{\pi^3 v^5}+\frac{3.5.7}{\pi^5 v^9}-+\dots\right\}\sin\frac{\pi v^2}{2};\end{aligned}$$

---

* Cf. Cauchy, *Comptes Rendus*, tom. 15, pp. 534 and 573.

and, similarly,

$$\int_v^\infty \sin\frac{\pi v^2}{2}dv = \frac{1}{\pi v}\cos\frac{\pi v^2}{2} - \frac{1}{\pi}\left\{-\frac{1}{\pi v^3}\sin\frac{\pi v^2}{2} + \frac{3}{\pi}\left\{\frac{1}{\pi v^5}\cos\frac{\pi v^2}{2} - \frac{5}{\pi}\left\{-\frac{1}{\pi v^7}\sin\frac{\pi v^2}{2} + \dots\right.\right.\right.$$

$$= \left\{\frac{1}{\pi^2 v^3} - \frac{1\,.\,3\,.\,5}{\pi^4 v^7} + - \dots\right\}\sin\frac{\pi v^2}{2} + \left\{\frac{1}{\pi v} - \frac{3}{\pi^3 v^5} + - \dots\right\}\cos\frac{\pi v^2}{2}.$$

Q. E. D.

9. Show that the following (Fresnel) integrals can be expressed in the form

$$\int_0^v \cos\frac{\pi v^2}{2}dv = \frac{1}{2} - P\cos\frac{\pi v^2}{2} + Q\sin\frac{\pi v^2}{2}$$

and

$$\int_0^v \sin\frac{\pi v^2}{2}dv = \frac{1}{2} - P\sin\frac{\pi v^2}{2} - Q\cos\frac{\pi v^2}{2},$$

where P and Q are given by the series employed in the preceding example.

We have

$$\int_0^v \cos\frac{\pi v^2}{2}dv = \int_0^\infty \cos\frac{\pi v^2}{2}dv - \int_v^\infty \cos\frac{\pi v^2}{2}dv$$

and

$$\int_0^v \sin\frac{\pi v^2}{2}dv = \int_0^\infty \sin\frac{\pi v^2}{2}dv - \int_v^\infty \sin\frac{\pi v^2}{2}dv,$$

which by formulae (31) and Ex. 8 can evidently be written in the above form.

10. Examine, as in text pp. 218-224, the diffraction of light on a slit (in a large opaque screen) of such breadth that $v_1' - v_2'$ is always very large, and show that bright and dark (coloured) bands or fringes will appear in its geometrical image. Show also, as the breadth of the slit is increased, that the diffraction phenomena will resemble more and more those produced on the straight edge of a large opaque screen.

11. Examine, as in text pp. 226-231, the distribution of the intensity outside the geometrical shadow of a narrow screen or wire; show that the diffraction bands, which appear there, will obey no simple law of distribution.*

12. Show that formula (58) will hold at any point (μ and $\nu \gtrless 0$), when the rectangular aperture in the obstructing screen is large. By formulae (61) the dark bands of both series will then be correspondingly near together, and hence for large values of a and b inobservable (cf. problem on diffraction on narrow slit ($b = \infty$), p. 235).

13. Determine the form of the coefficient $c'^2 + s'^2$ in formula (66) for four equal apertures in a large opaque screen situated at the four corners of a square.

If we lay the origin of our coordinates x' y' (cf. p. 236) at one of the four corners of the square, and choose the two sides of the square that meet at that corner as x' and y' axes, we can then put

$$x_1' = 0, \quad x_2' = 0, \quad x_3' = d, \quad x_4' = d,$$
$$y_1' = 0, \quad y_2' = d, \quad y_3' = 0, \quad y_4' = d,$$

---

* Cf. also paper by Lommel, cited in foot-note on p. 215.

where d denotes the length of the sides of the square. The quantities c' and s' of formulae (65) will then assume the particular form

$$\begin{aligned} c' &= 1+\cos\mu d+\cos\nu d+\cos(\mu+\nu)d, \\ s' &= \qquad \sin\mu d+\sin\nu d+\sin(\mu+\nu)d, \end{aligned}$$

and hence the coefficient in question

$$c'^2+s'^2=4+4[\cos\mu d+\cos\nu d+\cos\mu d\cos\nu d],\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

the form, in which the general formula (67) is written. It is evident that the second term of this expression cannot be neglected in comparison to the first (cf. also p. 238).

We can also write the expression (a) for $c'^2+s'^2$

$$\begin{aligned} c'^2+s'^2 &= (2+2\cos\mu d)+(2+2\cos\nu d)+2\cos\mu d(1+\cos\nu d)+2\cos\nu d(1+\cos\mu d) \\ &= 4\cos^2\frac{\mu d}{2}+4\cos^2\frac{\nu d}{2}+4\cos\mu d\cos^2\frac{\nu d}{2}+4\cos\nu d\cos^2\frac{\mu d}{2} \\ &= 4\cos^2\frac{\mu d}{2}(1+\cos\nu d)+4\cos^2\frac{\nu d}{2}(1+\cos\mu d) \\ &= 16\cos^2\frac{\mu d}{2}\cos^2\frac{\nu d}{2}. \end{aligned}$$

Show that the given diffraction pattern will thus exhibit two series of dark equidistant bands running parallel to the two pairs of parallel sides of the square in addition to the two series produced by the single (rectangular) aperture (cf. p. 234).

14. Sommerfeld's expression (75) for s assumes the following familiar form in the region of unobstructed propagation, $\phi'<\phi<2\pi-\phi'$ (cf. Fig. 31), for large values of r:

$$s=ae^{i\frac{2\pi}{\lambda}\left[vt-r\cos(\phi-\phi')\right]}.$$

Here σ will evidently assume large positive and σ' large negative values, so that the second integral of the general expression (75) for s may be rejected when compared with the first, whereas the latter may be replaced approximately by the integral

$$\int_{-\infty}^{\infty} e^{-i\frac{\pi v^2}{2}}\,dv=2\int_{0}^{\infty} e^{-i\frac{\pi v^2}{2}}\,dv=1-i$$

(cf. formula (30A)). We thus have, by formulae (75) and (76),

$$s=ae^{i\frac{2\pi}{\lambda}\left[vt-r\cos(\phi-\phi')\right]}.$$

Observe that for $r=\infty$ the real part of this expression represents plane-waves of amplitude a and whose direction of propagation makes the angle ϕ' with the x-axis.

15. Show that Sommerfeld's expression (75) for s assumes the following form in the region of the reflected waves, $2\pi-\phi'<\phi<2\pi$ (cf. Fig. 31), for large values of r:

$$s=a\left\{e^{i\frac{2\pi}{\lambda}\left[vt-r\cos(\phi-\phi')\right]}\mp e^{i\frac{2\pi}{\lambda}\left[vt-r\cos(\phi+\phi')\right]}\right\}.$$

Observe that for $r=\infty$ the real part of this expression represents the resultant of two plane waves, any incident wave of amplitude a and direction of propagation ϕ' with regard to x-axis, and that wave reflected (according to law of reflection).

16. Prove that Sommerfeld's general expression (85) for the intensity I can be written in the form

$$I \text{ prop. to } \frac{a^2}{2}\left\{\overline{A'E}^2+\overline{A'E'}^2\mp 2A'E\,.\,A'E'\cos(\gamma-\gamma'+\chi)\right\}$$

(cf. formula (86A)).

On performing the multiplication of the two conjugate complex quantities for s in the expression (85) for I, we have

$$\begin{aligned} I \text{ prop. to } & \frac{a}{2}\{(\xi-i\eta)(\xi+i\eta)+(\xi'-i\eta')(\xi'+i\eta')\mp[e^{i(\gamma-\gamma')}(\xi+i\eta)(\xi'-i\eta') \\ & \qquad +e^{-i(\gamma-\gamma')}(\xi-i\eta)(\xi'+i\eta')]\} \\ \text{prop. to } & \frac{a^2}{2}\{(\xi^2+\eta^2)+(\xi'^2+\eta'^2)\mp(\xi\xi'+\eta\eta')[e^{i(\gamma-\gamma')}+e^{-i(\gamma-\gamma')}] \\ & \qquad \pm i(\xi\eta'-\xi'\eta)[e^{i(\gamma-\gamma')}-e^{-i(\gamma-\gamma')}]\}, \end{aligned}$$

or, on expressing the exponential base as function of the sine and cosine,

$$I \text{ prop. to } \frac{a^2}{2}\{\overline{A'E}^2+\overline{A'E'}^2\mp 2[(\xi\xi'+\eta\eta')\cos(\gamma-\gamma')+(\xi\eta'-\xi'\eta)\sin(\gamma-\gamma')]\}$$

(cf. formula (86)).

If now we denote the angles, which the vectors $A'E$ and $A'E'$ make with the ξ-axis, by α and α' respectively, we can put

$$\begin{aligned} \xi &= A'E\cos\alpha, \quad \eta = A'E\sin\alpha, \\ \xi' &= A'E'\cos\alpha', \quad \eta' = A'E'\sin\alpha', \end{aligned}$$

and hence write the expression for I

$$\begin{aligned} I \text{ prop. to } & \frac{a^2}{2}\{\overline{A'E}^2+\overline{A'E'}^2\mp 2A'E\,.\,A'E'[(\cos\alpha\cos\alpha'+\sin\alpha\sin\alpha')\cos(\gamma-\gamma') \\ & \qquad +(\cos\alpha\sin\alpha'-\cos\alpha\sin\alpha')\sin(\gamma-\gamma')]\}, \\ \text{prop. to } & \frac{a^2}{2}\{\overline{A'E}^2+\overline{A'E'}^2\pm 2A'E\,.\,A'E'[\cos(\alpha-\alpha')\cos(\gamma-\gamma')-\sin(\alpha-\alpha')\sin(\gamma-\gamma')]\}, \\ \text{prop. to } & \frac{a^2}{2}\{\overline{A'E}^2+\overline{A'E'}^2\mp 2A'E\,.\,A'E'\cos(\gamma-\gamma'+\chi)\}, \end{aligned}$$

where $\chi=\alpha-\alpha'$. Q.E.D.

17. By Sommerfeld's theory of diffraction the intensity I at any point of the region $\phi'<\phi<2\pi-\phi'$ (cf. Fig. 31) is determined approximately by the expression

$$I=a^2\left\{1\pm\frac{1}{\pi}\sqrt{\frac{\lambda}{4r}}\frac{\cos\left[\frac{\pi}{4}+\frac{4\pi r}{\lambda}\sin^2\frac{1}{2}(\phi-\phi')\right]}{\sin\frac{1}{2}(\phi+\phi')}\right\}.$$

In the given region both $\frac{1}{2}(\phi-\phi')$ and $\frac{1}{2}(\phi+\phi')$ will be positive, and hence σ will assume large positive and σ' large negative values for finite r and waves of short wave-length λ. $A'E$ will, therefore, be large, approaching the value $\sqrt{2}$, and $A'E'$ so small that it can be replaced approximately by the radius of curvature of the spiral itself at the point E' (cf. p. 250). The vector $A'E$ will make an angle of approximately $\pi/4$ with the ξ-axis of the spiral, whereas the angle made by the tangent to the spiral at the point E' with the ξ-axis will be given approximately by the expression $\frac{\pi}{2}\sigma'^2$ (cf. p. 250 and formula (24)); the angle χ can thus be written approximately

$$\chi=\left(\frac{\pi}{4}-\frac{\pi}{2}\right)-\frac{\pi}{2}\sigma'^2,$$

or, by formulae (76),

$$=-\frac{\pi}{4}-\frac{4\pi r}{\lambda}\sin^2\tfrac{1}{2}(\phi+\phi').$$

The angle $\gamma-\gamma'+\chi$ will thus assume here the form

$$\begin{aligned}\gamma-\gamma'+\chi&=\frac{2\pi r}{\lambda}[\cos(\phi-\phi')-\cos(\phi+\phi')]-\frac{\pi}{4}-\frac{4\pi r}{\lambda}\sin^2\tfrac{1}{2}(\phi+\phi')\\&=-\frac{\pi}{4}+\frac{4\pi r}{\lambda}\sin\phi\sin\phi'-\frac{4\pi r}{\lambda}[\sin\phi/2\cos\phi'/2+\cos\phi/2\sin\phi'/2]\\&=-\frac{\pi}{4}+\frac{4\pi r}{\lambda}[4\sin\phi/2\cos\phi/2\sin\phi'/2\cos\phi'/2\\&\quad-(\sin\phi/2\cos\phi'/2+\cos\phi/2\sin\phi'/2)^2]\\&=-\frac{\pi}{4}-\frac{4\pi r}{\lambda}[\sin\phi/2\cos\phi'/2-\cos\phi/2\sin\phi\,/2]^2\\&=-\frac{\pi}{4}-\frac{4\pi r}{\lambda}\sin^2\tfrac{1}{2}(\phi-\phi').\end{aligned}$$

On replacing the angle $\gamma-\gamma'+\chi$ by this value in formula (86A), we can write the expression for I

$$I\text{ prop. to }\frac{a^2}{2}\left\{\overline{A'E}^2+\overline{A'E'}^2\mp 2A'E\ A'E'\cos\left[\frac{\pi}{4}+\frac{4\pi r}{\lambda}\sin^2\tfrac{1}{2}(\phi-\phi')\right]\right\},$$

or, if we replace here $A'E$ and $A'E'$ by their approximate values

$$A'E=\sqrt{2}$$

and

$$A'E'=\rho'=\frac{1}{\pi\sigma'},=-\frac{1}{\pi\sqrt{\frac{8r}{\lambda}}\sin\frac{1}{2}(\phi+\phi')}$$

and retain terms of only the null and first orders of magnitude in the small quantity ρ', most approximately

$$I\text{ prop. to }a^2\left\{1\pm\frac{1}{\pi}\sqrt{\frac{\lambda}{4r}}\,\frac{\cos\left[\frac{\pi}{4}+\frac{4\pi r}{\lambda}\sin^2\frac{1}{2}(\phi-\phi')\right]}{\sin\frac{1}{2}(\phi+\phi')}\right\}.$$

Show that as ϕ varies this expression for I will pass through a series of maxima and minima, and that the nearer ϕ approaches the boundary $\phi=2\pi-\phi'$ of the region of the reflected waves (cf. Fig. 31) the more marked will be the diffraction bands corresponding to these maxima and minima.

18. Show according to Sommerfeld's theory of diffraction that well within the region of the reflected waves (cf. Fig. 31) $A'E$ and $A'E'$ will each assume approximately the value $\sqrt{2}$ and χ vanish, and hence that I will be given approximately by the expression

$$I\text{ prop. to }a^2\left\{2\mp 2\cos\left[\frac{4\pi r}{\lambda}\sin\phi\sin\phi'\right]\right\};$$

that is, if we choose the minus sign, I will vanish, when

$$\frac{2r}{\lambda}\sin\phi\sin\phi'=1,\ 2,\ 3,\ \ldots,$$

whereas it will assume four times the natural intensity, that of the unobstructed incident waves, when

$$\frac{2r}{\lambda}\sin\phi\sin\phi' = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \cdots;$$

the resultant waves are evidently the so-called "stationary" waves (cf. pp. 13-14).

19. Examine Sommerfeld's expression (86A) for I in the region of unobstructed propagation (cf. Fig. 31).

20. Show that the solution (75) for s will satisfy the particular form (90) assumed in the case in question by Maxwell's fundamental equations, when X, Y, Z or a, b, c denote the components of that light-vector s.

CHAPTER VII.

REFLECTION AND REFRACTION ON SURFACE OF ISOTROPIC INSULATORS; TOTAL REFLECTION. REFLECTION AND REFRACTION OF THE PRIMARY AND SECONDARY WAVES.

Reflection and Refraction and Maxwell's Equations for Electromagnetic Disturbances.—We know, when light falls on the surface of a second medium, that part of it is turned back or "reflected" and part admitted into that medium or "refracted." This reflection and refraction of light rays on the surface of a second medium and the phenomena arising therefrom are now, according to our conception of the nature of light, electromagnetic phenomena, and we should, therefore, be able not only to explain them as such, but also to derive the empirical laws on reflection and refraction from our (Maxwell's) equations for electricity and magnetism. We shall confine ourselves here, as in the preceding chapters (cf. p. 6), to the behaviour of electromagnetic disturbances in homogeneous non-conducting media, and, in the present chapter, to their behaviour in such isotropic media. The electromagnetic state of an homogeneous non-conducting isotropic medium is now according to Maxwell defined by the differential equations

$$\left.\begin{aligned} \frac{D}{v_0}\frac{dP}{dt} &= \frac{d\beta}{dz} - \frac{d\gamma}{dy} \\ \frac{D}{v_0}\frac{dQ}{dt} &= \frac{d\gamma}{dx} - \frac{d\alpha}{dz} \\ \frac{D}{v_0}\frac{dR}{dt} &= \frac{d\alpha}{dy} - \frac{d\beta}{dx} \end{aligned}\right\} \quad \text{..........................} (1)$$

and

$$\left.\begin{aligned} \frac{M}{v_0}\frac{d\alpha}{dt} &= \frac{dR}{dy} - \frac{dQ}{dz} \\ \frac{M}{v_0}\frac{d\beta}{dt} &= \frac{dP}{dz} - \frac{dR}{dx} \\ \frac{M}{v_0}\frac{d\gamma}{dt} &= \frac{dQ}{dx} - \frac{dP}{dy} \end{aligned}\right\} \quad \text{..........................} (2)$$

(cf. formulae (5) and (6), I.), where P, Q, R and α, β, γ denote the components of the electric and magnetic forces respectively; v_0 denotes the velocity of propagation of electromagnetic disturbances in the standard medium (vacuum).

The Surface Conditions.—On the dividing surface of two non-conducting isotropic media the equations (1) and (2) assume the following particular form, when the normal to that surface is chosen as x-axis:

$$\frac{d}{dt}\left(\frac{D_1}{4\pi}P_1 - \frac{D_0}{4\pi}P_0\right)=0,\quad \text{hence}\quad D_1P_1 - D_0P_0=\text{const.}$$

$$\gamma_1=\gamma_0,\quad \beta_1=\beta_0,$$

and
$$\frac{d}{dt}\left(\frac{M^*}{4\pi}\alpha_1 - \frac{M^*}{4\pi}\alpha_0\right)=0,\quad \text{hence}\quad \alpha_1-\alpha_0=\text{const.},$$

$$R_1=R_0,\quad Q_1=Q_0$$

(cf. formulae (3), (4), (10), and (11), I.), where the index 0 or 1 denotes that the compound force to be taken is that or the sum of those acting in the first (0) or second (1) medium respectively (cf. Fig. 32). On the assumption that no electromagnetic forces were acting in the film between the two media before the passage of the disturbance in question, these so-called "surface-conditions" can evidently be written in the simpler form

$$\left.\begin{array}{r} D_1P_1-D_0P_0=0,\quad Q_1=Q_0,\quad R_1=R_0, \\ \alpha_1-\alpha_0=0,\quad \beta_1=\beta_0,\quad \gamma_1=\gamma_0 \end{array}\right\} \quad \text{.................}(3)$$

We have chosen the above form of our differential equations, where the forces and not, as in the preceding chapters, the moments appear as variables; for the present purposes this form is somewhat more convenient than the other, since the forces acting in adjacent media, and not, in general, their respective moments, may be compared or superposed directly, as is evident from our surface conditions (3) (cf. also below). The surface conditions (3) have been referred to a system of rectangular coordinates, whose x-axis has been so chosen that it coincides with the normal to the given dividing surface. Let now the y- and z-axes be so situated in this dividing surface that the normal to the wave-fronts of the given (transverse) waves incident at any point, the origin O, on that surface lie in the xy plane, as indicated in Fig. 32;

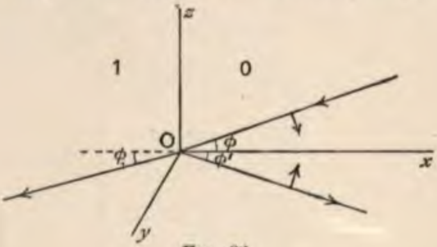

FIG. 32.

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*Cf. p. 6.

the normals to the wave-fronts of the reflected and refracted waves will then evidently lie in that same (xy) plane, which is then known as the "plane of incidence." We denote the angles, which the normals to the wave-fronts of the incident, reflected and refracted waves, make with the x-axis, by ϕ, ϕ', and ϕ_1 respectively (cf. Fig. 32).

Linearly Polarized Plane Waves.—Let us first examine the case where the incident waves are ordinary linearly polarized **plane** waves. We can now represent such waves by the function

$$ae^{in\left(t+\frac{r}{v}\right)},\dagger \qquad\qquad (4)$$

where $i=\sqrt{-1}$,

$$n=\frac{2\pi v}{\lambda}=\frac{2\pi}{T}^{*}, \qquad\qquad (5)$$

T being the period of oscillation, and r denotes the distance of any incident wave from the point O, where it strikes the dividing surface (cf. Fig. 32). Here the quantity n is not that (n) employed in the preceding chapters (cf. formulae (31), II.); we have chosen the given form for n in the ensuing investigations, since it will then assume one and the same value in all (both) media, for just as many impulses or waves per unit-time will be imparted to and transmitted through any adjacent medium as there are impulses or waves per unit-time in the given medium, that is, the period of oscillation of any given oscillation will not change upon its entering a second medium. That we choose the complex form (4) for representing the given waves is only a matter of taste; this complex function contains, in fact, two systems of waves, the one represented by its real term and the other by the real factor of its imaginary term.

The Reflected and Refracted Waves.—If the incident waves are represented by the function (4), the respective reflected and refracted waves will evidently have the form

$$a'e^{in\left(t-\frac{r'}{v'}\right)} \text{ and } a_1e^{in\left(t-\frac{r_1}{v_1}\right)},\dagger \qquad\qquad (6)$$

where a' and a_1 denote the amplitudes, v' and v_1 the velocities of propagation, and r' and r_1 the distances from the point O on the dividing surface, where the given incident waves strike the same, of the reflected and refracted waves respectively (cf. Fig. 32); the quantities a', a_1, v' and v_1 are to be regarded here as unknown and to be sought.

If we refer the incident waves (4) and the reflected and refracted ones (6), to which the former give rise, to the above system of coordinates

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† For waves *approaching* the point O, as the incident ones, the plus-sign must evidently be chosen before $\frac{r}{v}$, whereas for those that are *receding* from that point, as the reflected and refracted waves, the minus-sign must be taken.

* Cf. p. 12.

(cf. Fig. 32), the distances r, r' and r_1 can then evidently be written in the form

$$\left.\begin{aligned} r &= -y\sin\phi + x\cos\phi \\ r' &= y\sin\phi' + x\cos\phi' \\ r_1 &= y\sin\phi_1 - x\cos\phi_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

(cf. Fig. 32), and hence the three waves themselves in the form

$$\left.\begin{aligned} &ae^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)} \\ &a'e^{in\left(t-\frac{y\sin\phi' + x\cos\phi'}{v'}\right)} \\ &a_1e^{in\left(t-\frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)} \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Before we consider the general case, where the direction of oscillation in the incident waves makes an arbitrary angle with the plane of the incidence, the xy coordinate-plane, let us examine the following two particular cases:

I. The electric oscillations in the incident waves take place at right angles to the plane of incidence; and

II. The electric oscillations in the incident waves take place in the plane of incidence.

Case I.: The Electric Oscillations at ⊥ to Plane of Incidence.—Here

$$\left.\begin{aligned} &P = Q = 0, \\ \text{and hence} \quad &P' = Q' = P_1 = Q_1 = 0 \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

that is, the oscillations in the reflected and refracted waves take place at right angles to the plane of incidence.

The components R, R' and R_1 parallel to the z-axis of the electric forces acting in the incident, reflected and refracted waves respectively, will then, by formulae (4) and (6), be represented by the functions

$$\left.\begin{aligned} R &= a\,e^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)} \\ R' &= a'e^{in\left(t-\frac{y\sin\phi' + x\cos\phi'}{v'}\right)} \\ R_1 &= a_1e^{in\left(t-\frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

To find the magnetic forces, the α's, β's and γ's, that accompany these electric ones, we replace the P's and Q's by their values (9) in our fundamental equations (1) and (2), and we have

$$\frac{D}{v_0}\frac{dR}{dt} = \frac{d\alpha}{dy} - \frac{d\beta}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1\text{A})$$

and

$$\left.\begin{aligned} \frac{M}{v_0}\frac{d\alpha}{dt} &= \frac{dR}{dy} \\ \frac{M}{v_0}\frac{d\beta}{dt} &= -\frac{dR}{dx} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2\text{A})$$

—the remaining three equations all reduce to $\gamma = 0$—where R is to be replaced by its value from formulae (10), and similar equations for R' and R_1.

The Magnetic Waves.—It follows from formulae (2A) and the analogous ones for α', β', γ' and α_1, β_1, γ_1 that all three magnetic oscillations are taking place *in* the plane of incidence, that is, at right angles to the respective electric ones; to find their two components in this plane, we replace the R's by their values (10) in the given equations, and we have

$$\begin{aligned}
\frac{M}{v_0}\frac{d\alpha}{dt} &= -ain\frac{\sin\phi}{v}e^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)},\\
\frac{M}{v_0}\frac{d\beta}{dt} &= -ain\frac{\cos\phi}{v}e^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)},\\
\frac{M}{v_0}\frac{d\alpha'}{dt} &= -a'in\frac{\sin\phi'}{v'}e^{in\left(t-\frac{y\sin\phi' + x\cos\phi'}{v'}\right)},\\
\frac{M}{v_0}\frac{d\beta'}{dt} &= a'in\frac{\cos\phi'}{v'}e^{in\left(t-\frac{y\sin\phi' + x\cos\phi'}{v'}\right)},\\
\frac{M}{v_0}\frac{d\alpha_1}{dt} &= -a_1in\frac{\sin\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)},\\
\frac{M}{v_0}\frac{d\beta_1}{dt} &= -a_1in\frac{\cos\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)};
\end{aligned}$$

integrated, these equations give the following values for the α's and β's:

$$\left.\begin{aligned}
\alpha &= -a\frac{v_0}{M}\frac{\sin\phi}{v}e^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)}\\
\beta &= -a\frac{v_0}{M}\frac{\cos\phi}{v}e^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)}\\
\alpha' &= -a'\frac{v_0}{M}\frac{\sin\phi'}{v'}e^{in\left(t-\frac{y\sin\phi' + x\cos\phi'}{v'}\right)}\\
\beta' &= a'\frac{v_0}{M}\frac{\cos\phi'}{v'}e^{in\left(t-\frac{y\sin\phi' + x\cos\phi'}{v}\right)}\\
\alpha_1 &= -a_1\frac{v_0}{M}\frac{\sin\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)}\\
\beta_1 &= -a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)}
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Replace, next, the α's and β's by these and the R's by their above values (10) in our fundamental equations (1A), and we have

$$a i n \frac{D}{v_0} e^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)} = a i n \frac{v_0}{M}\left(\frac{\sin^2\phi}{v^2}+\frac{\cos^2\phi}{v^2}\right) e^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)}$$

and similar relations for the reflected and refracted waves, which give

$$\frac{D}{v_0}=\frac{v_0}{M}\frac{1}{v^2},$$

hence and similarly

$$\left.\begin{aligned} v^2 &= \frac{v_0^2}{DM} \\ v'^2 &= \frac{v_0^2}{DM} \\ v_1^2 &= \frac{v_0^2}{D_1 M} \end{aligned}\right\}; \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

that is, these familiar relations (cf. p. 11) must hold between the velocities of propagation of the waves and the medium constants, in order that our fundamental equations (1A) may be satisfied. It follows, moreover, from these conditional relations that

$$v' = v; \quad \ldots\ldots\ldots\ldots\ldots\ldots(12\text{A})$$

that is, the velocity of propagation of the given electromagnetic disturbance undergoes no change upon reflection.

The Surface Conditions and the Laws of Reflection and Refraction.—Let us, now, examine the surface-conditions (3) for the given system of waves; they evidently assume here the simple form

$$\left.\begin{aligned} &D_1P_1 = D_0P_0 = Q_1 = Q_0 = 0, \quad R_1 = R_0 \\ \text{and} \quad &\alpha_1 = \alpha_0, \quad \beta_1 = \beta_0, \quad \gamma_1 = \gamma_0 = 0 \end{aligned}\right\}. \ldots\ldots\ldots\ldots(3\text{A})$$

The index 1 denotes that the component force to be taken is that acting in the medium 1, that is, here the component force acting in the refracted waves, whereas the index 0 refers to the component force or forces acting in the medium 0, here the component force acting in the incident and that acting in the reflected waves superposed. The given surface-conditions (3A) must, therefore, be written explicitly as follows:

$$\left.\begin{aligned} R_1 &= R + R' \\ \alpha_1 = \alpha + \alpha', &\quad \beta_1 = \beta + \beta' \end{aligned}\right\} \ldots\ldots\ldots\ldots(3\text{A}')$$

—the other component forces vanish.

Replace here the R's, α's and β's by their above values on the given dividing surface, $x=0$, and we have

$$\left.\begin{array}{ll} & a_1 e^{in\left(t-\frac{y\sin\phi_1}{v_1}\right)} = a e^{in\left(t-\frac{y\sin\phi}{v}\right)} + a' e^{in\left(t-\frac{y\sin\phi'}{v}\right)}, \\ & -a_1\frac{v_0}{M}\frac{\sin\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}\right)} = a\frac{v_0}{M}\frac{\sin\phi}{v}e^{in\left(t-\frac{y\sin\phi}{v}\right)} \\ & \qquad\qquad\qquad + a'\frac{v_0}{M}\frac{\sin\phi'}{v}e^{in\left(t-\frac{y\sin\phi'}{v}\right)}, \\ \text{and} & -a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}\right)} = -a\frac{v_0}{M}\frac{\cos\phi}{v}e^{in\left(t-\frac{y\sin\phi}{v}\right)} \\ & \qquad\qquad\qquad + a'\frac{v_0}{M}\frac{\cos\phi'}{v}e^{in\left(t-\frac{y\sin\phi'}{v}\right)} \end{array}\right\}, \quad \ldots\ldots(13)$$

where, by formula (12A), we have put $v' = v$.

The surface-conditions (13) must now hold not only for all values of t but also at all points on the given dividing surface, that is, for all values of y (and z); this is now evidently only possible, when the following relation holds between the v's and the ϕ's:

$$\frac{\sin\phi}{v} = \frac{\sin\phi'}{v} = \frac{\sin\phi_1}{v_1}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

which gives

$$\phi' = \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

(cf. Fig. 32), or the familiar law, the angle of incidence is equal to the angle of reflection.

The first and last members of relation (14) give

$$\sin\phi : \sin\phi_1 = v : v_1 = n_{01}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

or the (Snell's) law of refraction; n_{01} is known as the "index of refraction" for waves passing from the medium 0 into the medium 1.

The laws (15) and (16) follow directly from the actual *existence* of the surface conditions or from the fact that a linear relation holds between the forces acting in the film, being entirely independent of the form of those conditions or that relation, its coefficients.

Determination of Amplitudes of Reflected and Refracted Waves.—We have seen that the relation (14) must hold between the v's and the ϕ's, in order that the surface-conditions (13) *may* be satisfied; but it does not necessarily follow that they *are* satisfied; this will be, in fact, the case only when certain other relations hold between the amplitudes a, a' and a_1 and the angles ϕ, ϕ' and ϕ_1 or the medium constants. These relations may now be determined from the surface-conditions upon the assumption of the validity of the latter. By the relation (14) the three surface-conditions (13) will then reduce to the two

$$a_1 = a + a', \quad \text{.............................(17)}$$

and

$$a_1 \frac{\cos\phi_1}{v_1} = a\frac{\cos\phi}{v} - a'\frac{\cos\phi'}{v} = (a - a')\frac{\cos\phi}{v},$$

or

$$a_1 \cos\phi_1 \frac{\sin\phi}{\sin\phi_1} = (a - a')\cos\phi\,;$$

which give

$$(a + a')\cos\phi_1 \sin\phi = (a - a')\cos\phi\sin\phi_1,$$

hence

$$a' = -a\frac{\sin\phi\cos\phi_1 - \cos\phi\sin\phi_1}{\sin\phi\cos\phi_1 + \cos\phi\sin\phi_1} = -a\frac{\sin(\phi - \phi_1)}{\sin(\phi + \phi_1)}, \quad \text{.......(18)}$$

and hence, by (17),

$$a_1 = a\left[1 - \frac{\sin(\phi - \phi_1)}{\sin(\phi + \phi_1)}\right] = a\frac{2\cos\phi\sin\phi_1}{\sin(\phi + \phi_1)}. \quad \text{.............(19)}$$

The Reflected and Refracted Electric Waves.—By formulae (15), (18), and (19) the reflected and refracted electric waves, to which the given incident electric waves give rise, will have the form

$$\left.\begin{aligned}
R' &= -a\frac{\sin(\phi - \phi_1)}{\sin(\phi + \phi_1)} e^{in\left(t - \frac{y\sin\phi + x\cos\phi}{v}\right)} \\
\text{and}\quad R_1 &= a\frac{2\cos\phi\sin\phi_1}{\sin(\phi + \phi_1)} e^{in\left(t - \frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)}, \\
\text{where}\quad \phi_1 &= \arcsin\left(\frac{v_1}{v}\sin\phi\right)
\end{aligned}\right\}. \quad \text{.............(20)}$$

The accompanying Magnetic Waves.—The component forces acting in the magnetic waves that accompany the electric waves (20) will evidently be given by the expressions

$$\left.\begin{aligned}
\alpha &= -a\frac{v_0}{M}\frac{\sin\phi}{v} e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)} \\
\beta &= -a\frac{v_0}{M}\frac{\cos\phi}{v} e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)} \\
\alpha' &= a\frac{v_0}{M}\frac{\sin\phi}{v}\frac{\sin(\phi - \phi_1)}{\sin(\phi + \phi_1)} e^{in\left(t - \frac{y\sin\phi + x\cos\phi}{v}\right)} \\
\beta' &= -a\frac{v_0}{M}\frac{\cos\phi}{v}\frac{\sin(\phi - \phi_1)}{\sin(\phi + \phi_1)} e^{in\left(t - \frac{y\sin\phi + x\cos\phi}{v}\right)} \\
\alpha_1 &= -a\frac{v_0}{M}\frac{\sin\phi}{v}\frac{2\cos\phi\sin\phi_1}{\sin(\phi + \phi_1)} e^{in\left(t - \frac{y\sin\phi_1 - x\cos\phi_1}{v}\right)} \\
\beta_1 &= -a\frac{v_0}{M}\frac{\cos\phi_1}{v_1}\frac{2\cos\phi\sin\phi_1}{\sin(\phi + \phi_1)} e^{in\left(t - \frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)}, \\
\text{where}\quad \phi_1 &= \arcsin\left(\frac{v_1}{v}\sin\phi\right)
\end{aligned}\right\}. \quad \text{....(21)}$$

Case II.: The Electric Oscillations in xy-Plane.—Here

$$R = R' = R_1 = 0,$$

and our fundamental equations (2) evidently assume the form

$$\frac{M}{v_0}\frac{d\alpha}{dt} = 0, \quad \frac{M}{v_0}\frac{d\beta}{dt} = 0, \quad \frac{M}{v_0}\frac{d\gamma}{dt} = \frac{dQ}{dx} - \frac{dP}{dy}; \quad \text{(2B)}$$

the first two of which give

$$\alpha = \beta = 0, \quad \text{(22)}$$

that is, the accompanying magnetic oscillations are taking place at right angles to the xy-plane or to the electric oscillations.

By formula (22) our fundamental equations (1) then assume the form

$$\frac{D}{v_0}\frac{dP}{dt} = -\frac{d\gamma}{dy}, \quad \frac{D}{v_0}\frac{dQ}{dt} = \frac{d\gamma}{dx}. \quad \text{(1B)}$$

As above, we can now represent the electric force acting in the incident electric waves by the function

$$be^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)},$$

where b denotes its resultant amplitude (in the xy-plane). Its component forces, P and Q parallel to the x and y-axes respectively, will then be

$$\left.\begin{aligned} P &= b\sin\phi\, e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)} \\ Q &= b\cos\phi\, e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)} \end{aligned}\right\}. \quad \text{(23)}$$

Similarly, the component forces acting in the reflected and refracted electric waves will have the form

$$\left.\begin{aligned} P' &= b'\sin\phi' e^{in\left(t - \frac{y\sin\phi' + x\cos\phi'}{v'}\right)} \\ Q' &= -b'\cos\phi' e^{in\left(t - \frac{y\sin\phi' + x\cos\phi'}{v'}\right)} \end{aligned}\right\} \quad \text{(24)}$$

(cf. Fig. 32, where oscillations of the incident and reflected waves corresponding to the given case are represented graphically by arrows) and

$$\left.\begin{aligned} P_1 &= b_1\sin\phi_1 e^{in\left(t - \frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)} \\ Q_1 &= b_1\cos\phi_1 e^{in\left(t - \frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)} \end{aligned}\right\}, \quad \text{(25)}$$

where b' and b_1 denote the resultant amplitudes in the reflected and refracted waves respectively; as in case I., b', b_1, ϕ', ϕ_1, v' and v_1 are to be regarded here as unknown and to be sought.

To determine the magnetic forces acting in the incident, reflected and refracted waves, we replace the P's and Q's by their values (23), (24) and (25) respectively in formulae (1B), and we have

$$\frac{D}{v_0} b \sin\phi\, ine^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)} = -\frac{d\gamma}{dy},$$

$$\frac{D}{v_0} b \cos\phi\, ine^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)} = \frac{d\gamma}{dx},$$

and similar equations for the reflected and refracted waves; these integrated give one and the same value for γ, namely,

$$\left.\begin{aligned} &\gamma = \frac{D}{v_0} bve^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)}; \\ &\text{similarly, we find the following expressions for } \gamma' \text{ and } \gamma_1\text{:} \\ &\gamma' = \frac{D}{v_0} b'v'e^{in\left(t-\frac{y\sin\phi' + x\cos\phi'}{v'}\right)} \\ \text{and} \quad &\gamma_1 = \frac{D}{v_0} b_1 v_1 e^{in\left(t-\frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(26)$$

Lastly, replace the P's, Q's and γ's by the above values in the last equation (2B), and we have

$$\frac{M}{v_0}\frac{D}{v_0} bvine^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)} = b\left(\frac{\cos^2\phi}{v} + \frac{\sin^2\phi}{v}\right) ine^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)}$$

with similar relations for the reflected and refracted waves, which gives

$$v^2 = \frac{{v_0}^2}{DM};$$

similarly, we find

$$v'^2 = \frac{{v_0}^2}{DM},$$

$$v_1{}^2 = \frac{{v_0}^2}{D_1 M},$$

hence

$$v' = v,$$

the same relations (12) and (12A) as those found in case I. By these relations, we can now write the magnetic forces (26) in the form

$$\left.\begin{aligned} &\gamma = b\frac{v_0}{M}\frac{1}{v} e^{in\left(t-\frac{y\sin\phi - x\cos\phi}{v}\right)} \\ &\gamma' = b'\frac{v_0}{M}\frac{1}{v} e^{in\left(t-\frac{y\sin\phi' + x\cos\phi'}{v}\right)} \\ \text{and} \quad &\gamma_1 = b_1\frac{v_0}{M}\frac{1}{v_1} e^{in\left(t-\frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)} \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots(27)$$

the form in which the magnetic forces appeared in case I (cf. formulae (11)).

The Surface Conditions and the Laws of Reflection and Refraction.—For the given case our surface-conditions (3) assume the form

$$\left.\begin{array}{c} D_1P_1 = D_0P_0, \quad Q_1 = Q_0, \quad R_1 = R_0 = 0 \\ \alpha_1 = \alpha_0 = \beta_1 = \beta_0 = 0, \quad \gamma_1 = \gamma_0 \end{array}\right\}, \dots\dots\dots\dots(3\text{B})$$

or explicitly

$$\left.\begin{array}{c} D_1P_1 = DP + DP', \quad Q_1 = Q + Q' \\ \gamma_1 = \gamma + \gamma' \end{array}\right\}. \dots\dots\dots\dots(3\text{B}')$$

Replace here the P's, Q's and γ's by their values on the given dividing surface, $x=0$, and we have

$$\left.\begin{array}{l} D_1b_1\sin\phi_1 e^{in\left(t-\frac{y\sin\phi_1}{v_1}\right)} = Db\sin\phi e^{in\left(t-\frac{y\sin\phi}{v}\right)} + Db'\sin\phi' e^{in\left(t-\frac{y\sin\phi'}{v}\right)}, \\ \quad b_1\cos\phi_1 e^{in\left(t-\frac{y\sin\phi_1}{v_1}\right)} = b\cos\phi e^{in\left(t-\frac{y\sin\phi}{v}\right)} - b'\cos\phi' e^{in\left(t-\frac{y\sin\phi'}{v}\right)} \\ \text{and } \; b_1\frac{v_0}{M}\frac{1}{v_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}\right)} = b\frac{v_0}{M}\frac{1}{v}e^{in\left(t-\frac{y\sin\phi}{v}\right)} + b'\frac{v_0}{M}\frac{1}{v}e^{in\left(t-\frac{y\sin\phi'}{v}\right)} \end{array}\right\}, \dots(28)$$

where we have put $v'=v$ (cf. above and formula (12A)).

As in case I., the following relation must now evidently hold, if these surface-conditions (28) are to be satisfied:

$$\frac{\sin\phi}{v} = \frac{\sin\phi'}{v} = \frac{\sin\phi_1}{v_1},$$

the same relation between the v's and the ϕ's as that (14) found in the preceding case. The same considerations as those above, and hence the same familiar laws will, therefore, also hold here.

Determination of Amplitudes of Reflected and Refracted Waves.—As above, to determine the further relations that must hold between the different quantities, in order that the surface-conditions (28) may be satisfied, we write the same by relation (14) in the simpler form

$$D_1b_1\sin\phi_1 = (Db + Db')\sin\phi,$$

$$b_1\cos\phi_1 = (b - b')\cos\phi,$$

and

$$\frac{b_1}{v_1} = \frac{b + b'}{v}.$$

By formulae (12) and (14) (cf. above), it is evident that the first and last of these conditions are identical and can be written in the form

$$b_1\frac{\sin\phi}{\sin\phi_1} = b + b'.$$

Upon the assumption of the validity of the given surface-conditions, which may be replaced by the two simpler ones

$$b_1\frac{\cos\phi_1}{\cos\phi} = b - b'$$

and

$$b_1\frac{\sin\phi}{\sin\phi_1} = b + b',$$

we can now determine the unknown quantities b' and b_1, the amplitudes of the resultant electric forces acting in the reflected and refracted waves respectively, as functions of known or given quantities. On the elimination first of b_1 and then of b' from these two conditional equations, we find the following values for b' and b_1 respectively:

$$b' = b\left(\frac{\sin\phi\cos\phi - \sin\phi_1\cos\phi_1}{\sin\phi\cos\phi + \sin\phi_1\cos\phi_1}\right) = b\frac{\tan(\phi-\phi_1)}{\tan(\phi+\phi_1)}, \quad\ldots\ldots\ldots\ldots\ldots(29)$$

and
$$b_1 = b\frac{2\cos\phi\sin\phi_1}{\sin\phi\cos\phi + \sin\phi_1\cos\phi_1} = b\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)}. \quad\ldots.(30)$$

The Reflected and Refracted Waves and the accompanying Magnetic ones.—By formulae (29) and (30), the reflected and refracted electric waves, to which the incident waves (23) give rise, will be represented by the component forces

$$\left.\begin{aligned} P' &= b\sin\phi\frac{\tan(\phi-\phi_1)}{\tan(\phi+\phi_1)}e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)}\\ Q' &= -b\cos\phi\frac{\tan(\phi-\phi_1)}{\tan(\phi+\phi_1)}e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)}\\ \text{and}\quad P_1 &= b\frac{2\cos\phi\sin^2\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)}e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)}\\ Q_1 &= b\frac{\cos\phi\sin 2\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)}e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)} \end{aligned}\right\}, \quad\ldots\ldots(31)$$

respectively, and the accompanying magnetic waves by the component forces

$$\left.\begin{aligned} \gamma &= \frac{v_0}{Mv}be^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},\\ \gamma' &= \frac{v_0}{Mv}b\frac{\tan(\phi-\phi_1)}{\tan(\phi+\phi_1)}e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)}\\ \text{and}\quad \gamma_1 &= \frac{v_0}{Mv_1}b\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)}e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)}, \end{aligned}\right\}. \quad\ldots\ldots(32)$$

where
$$\phi_1 = \arcsin\left(\frac{v_1}{v}\sin\phi\right).$$

The General Case: The Oscillations make an Arbitrary Angle with Plane of Incidence.—It is easy to derive from the two particular cases I. and II. just examined the formulae that will hold for the general case, where the electric oscillations (in the incident waves) make an arbitrary angle with the plane of incidence (the xy-plane); if we denote this angle by θ and the (resultant) amplitude of the electric force acting in the incident waves by A, we can then resolve that force into two, one at right angles to the plane of incidence, and the other *in* that plane; the component oscillation at right angles to the plane of incidence will then evidently be reflected and refracted according to formulae (14), (18) and (19) of case I., and that in the plane of incidence according to formulae (14), (29) and (30) of case II. The forces acting in these component oscillations will evidently be represented by formulae (20), (21), (31) and (32), after we have put there

$$a = A\sin\theta, \quad b = A\cos\theta \quad \text{.........................(33)}$$

(cf. Ex. 1 at end of chapter).

Reflection and Refraction of Elliptically Polarized Waves.—If the incident waves are elliptically polarized, we resolve the given elliptic oscillation (force) into two, one at right angles to the plane of incidence and the other *in* that plane, and treat each separately, as in the above general case; the same formulae will then hold as in the general case, only the phases of the oscillations taking place in the plane of incidence and those at right angles to it will differ from one another according to the degree of elliptic polarization of the incident waves. It is evident that the reflected and refracted waves will also be elliptically polarized.

Reflection and Refraction: Formulae for the Moments.—Formulae (18), (19), (29) and (30) are known as "Fresnel's reflection formulae"; they enable us to determine the amplitudes of the *forces* acting in the reflected and refracted waves from the form—amplitude, angle of incidence, etc.—of the incident waves. To obtain the amplitudes of the moments or oscillations themselves, we recall the relations that hold between the forces and the moments to which they give rise; namely the latter are proportional to the product of the former and the constant of electric induction D of the medium (cf. formulae (3) and (7), I.). In one and the same medium, the same formulae will, therefore, hold for the moments as for the forces; of the above formulae, (18) and (29) will, therefore, remain unaltered for the moments, provided the a's and b's denote there the amplitudes of the respective component moments, whereas the other two formulae, (19) and (30), will evidently assume the form

$$\left.\begin{aligned} &\frac{4\pi}{D_1}a_{1m} = \frac{4\pi}{D}a_m\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)} \\ \text{and}\qquad &\frac{4\pi}{D_1}b_{1m} = \frac{4\pi}{D}b_m\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)} \end{aligned}\right\}, \quad \text{.............(34)}$$

where the index m denotes that the amplitude to be taken is that of the moment or oscillation itself.

By formulae (12) these formulae (34) can now be written

$$\frac{v_1^2}{v_0^2}a_{1m}=\frac{v^2}{v_0^2}a_m\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)}$$

and

$$\frac{v_1^2}{v_0^2}b_{1m}=\frac{v^2}{v_0^2}b_m\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)};$$

or, by the relation (14),

$$\left.\begin{aligned} a_{1m}&=a_m\frac{\sin^2\phi}{\sin^2\phi_1}\,\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)}=a_m\frac{\sin\phi\sin 2\phi}{\sin\phi_1\sin(\phi+\phi_1)}\\ \text{and}\qquad b_{1m}&=b_m\frac{\sin^2\phi}{\sin^2\phi_1}\,\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)}\\ &=b_m\frac{\sin\phi\sin 2\phi}{\sin\phi_1\sin(\phi+\phi_1)\cos(\phi-\phi_1)}\end{aligned}\right\}. \quad \text{....(34 A)}$$

Perpendicular Incidence.—Let us, next, examine the particular case, where the incident waves strike the reflecting surface at right angles, that is, where

$$\phi=\phi'=\phi_1=0 \quad \text{..............................(35)}$$

(cf. formula (14)).

Here formulae (18), (19), (29) and (30) assume the indeterminate form

$$\frac{a'}{a}=\frac{a_1}{a}=\frac{b'}{b}=\frac{b_1}{b}=\frac{0}{0},$$

and cannot, therefore, be employed in the above form. To determine the real values of these four quotients for this limiting case, we write the given formulae in the form

$$\frac{a'}{a}=-\frac{\dfrac{\sin\phi}{\sin\phi_1}\cos\phi_1-\cos\phi}{\dfrac{\sin\phi}{\sin\phi_1}\cos\phi_1+\cos\phi}$$

$$\frac{a_1}{a}=\frac{2\cos\phi}{\dfrac{\sin\phi}{\sin\phi_1}\cos\phi_1+\cos\phi}$$

$$\frac{b'}{b}=\frac{\dfrac{\sin\phi}{\sin\phi_1}\cos\phi-\cos\phi_1}{\dfrac{\sin\phi}{\sin\phi_1}\cos\phi+\cos\phi_1}$$

and

$$\frac{b_1}{b}=\frac{2\cos\phi}{\dfrac{\sin\phi}{\sin\phi_1}\cos\phi+\cos\phi_1},$$

replace here $\dfrac{\sin\phi}{\sin\phi_1}$ by its value $\dfrac{v}{v_1}$ from relation (14), then put $\phi=\phi_1=0$, and we find

$$\left.\begin{array}{ll}\dfrac{a'}{a}=-\dfrac{v-v_1}{v+v_1}, & \dfrac{a_1}{a}=\dfrac{2v,}{v+v_1} \\ \dfrac{b'}{b}=\dfrac{v-v_1}{v+v_1}, & \dfrac{b_1}{b}=\dfrac{2v_1}{v+v_1}\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36)$$

It follows from these formulae that for $v>v_1$ not only a', the amplitude of the reflected waves at right angles to the plane of incidence, but also $b'(-b')$, their amplitude in that plane, will be oppositely directed to the respective component amplitudes a and b of the incident waves; that $b'(-b')$ is here oppositely directed to b is evident from the general expressions (24) for P' and Q', which for perpendicular incidence reduce to $P'=0$, $Q'=-b'e^{in\left(t-\frac{x}{v}\right)}$ (cf. also Fig. 32). It thus follows that the reflected waves will interfere (partially) with the incident ones, the resultant waves approaching in form stationary waves (cf. pp. 13 and 14), as the expressions for the component amplitudes a' and b' of the reflected waves approach in value those a and b of the incident waves; this limiting case could not well be realized, since then $\frac{v}{v_1}$ would have to be infinitely large, that is, the incident waves would have to be totally reflected, undergoing no change in amplitude upon reflection. The resultant waves will, therefore, be (only partially) stationary, their first (partial) node lying on the reflecting surface.

The Angle of Polarization and Common Light.—It is evident from formulae (36) and the general ones (18) and (19) that, when the oscillations in the incident electric waves are taking place at right angles to the plane of incidence, the reflected waves can never be entirely extinguished, even when the incident waves strike the reflecting surface at right angles. On the other hand, it follows from formula (29) that, when the oscillations in the incident waves are taking place in the plane of incidence, the reflected waves will be entirely extinguished, when the angle of incidence ϕ is so chosen that

$$\phi+\phi_1=\pi/2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

for then $\tan(\phi+\phi_1)$ becomes infinite and b' vanishes. When ordinary (non-polarized) light strikes a reflecting surface at this particular angle of incidence, the reflected waves will, therefore, contain oscillations that are taking place only at right angles to the plane of incidence, the component oscillations *in* the plane of incidence being entirely extinguished, that is, the reflected waves will be linearly polarized; the particular angle of incidence, for which incident waves become linearly polarized upon reflection, is thus known as the "angle of polarization." A glance at Fig. 32 shows that the angle of polarization is thereby determined that the directions of propagation of the reflected and the refracted waves make a right angle with one another (cf. formula (37)). For the angle of polarization formula (16) evidently assumes the form $\tan\phi = \frac{v}{v_1} = n_{01}$ (cf. formula (37)), which is known as Brewster's Law and states that the angle of polarization is that angle of incidence whose tangent is given by the index of refraction n_{01} of the two media. Since the index of refraction n_{01} depends on the relative constitution of the two media in question, the angle of polarization will vary for different media.

The General Case; Determination of the Planes of Oscillation of the Reflected and Refracted Waves.—According to Fresnel's theory, the plane of polarization makes an angle of 90° with the plane of oscillation; if the plane of oscillation of the incident electric waves makes an arbitrary angle θ with the plane of incidence, then their plane of polarization will make the angle $90° + \theta$ with that plane. Let us first determine the planes of oscillation of the reflected and refracted waves, to which plane waves incident at the angle ϕ and whose oscillations make an arbitrary angle θ with the plane of incidence, the xy-plane, give rise. For this purpose, we resolve the given incident oscillations (forces) * into the two component ones

$$A \sin\theta e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)}$$

at right angles to the plane of incidence and

$$A \cos\theta e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)}$$

along that plane. By formulae (18), (19), (29) and (30), these component oscillations (forces), upon striking the reflecting surface, will now give rise to the following reflected and refracted ones:

$$\left.\begin{aligned} -A\sin\theta\frac{\sin(\phi-\phi_1)}{\sin(\phi+\phi_1)}e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)} \\ A\cos\theta\frac{\tan(\phi-\phi_1)}{\tan(\phi+\phi_1)}e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)} \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots(38)$$

the components of the reflected waves, and

$$\left.\begin{aligned} A\sin\theta\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)}e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)} \\ A\cos\theta\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)}e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)} \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots(39)$$

the components of the refracted waves.

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* By "oscillation" we shall often refer to the force acting and not to the resulting moment.

On the other hand, we can now write the resultant reflected and the resultant refracted waves in the form

$$A'e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)} \text{ and } A_1e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)}$$

respectively, where A' and A_1 shall denote their resultant amplitudes. If we denote the angles these (resultant) oscillations make with the plane of incidence, the xy-plane, by θ' and θ_1 respectively, we can then replace them by their component oscillations at right angles to and along that plane, namely

$$A'\sin\theta' e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)}$$

and $$A'\cos\theta' e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)} \text{ respectively,}$$

the components of the reflected oscillations, and

$$A_1\sin\theta_1 e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)}$$

and $$A_1\cos\theta_1 e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)} \text{ respectively,}$$

the components of the refracted oscillations; here A', A_1, θ', θ_1 are to be regarded as unknown and to be sought.

On comparing these last expressions with the given ones (38) and (39) for the component-oscillations in question, we must evidently put

$$\left.\begin{aligned} A'\sin\theta' &= -A\sin\theta\frac{\sin(\phi-\phi_1)}{\sin(\phi+\phi_1)} \\ A'\cos\theta' &= A\cos\theta\frac{\tan(\phi-\phi_1)}{\tan(\phi+\phi_1)} \\ \text{and}\qquad A_1\sin\theta_1 &= A\sin\theta\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)} \\ A_1\cos\theta_1 &= A\cos\theta\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)\cos(\phi-\phi_1)} \end{aligned}\right\}; \quad \text{.........(40)}$$

which give the following values for the angles or planes of oscillation θ' and θ_1 of the reflected and refracted waves respectively:

$$\left.\begin{aligned} \tan\theta' &= -\tan\theta\frac{\cos(\phi-\phi_1)}{\cos(\phi+\phi_1)} \\ \tan\theta_1 &= \tan\theta\cos(\phi-\phi_1) \end{aligned}\right\}. \quad \text{.....................(41)}$$

Rotation of Plane of Polarization.—By formulae (41) the plane of oscillation of the above incident waves is rotated through given angles both upon reflection and upon refraction. Since now the plane of polarization, according to Fresnel, makes a right angle with the plane of oscillation, the planes of polarization of the reflected and refracted waves will make the angles $90+\theta'$ and $90+\theta_1$ respectively with the plane of incidence, where θ' and θ_1, the planes of oscillation, are determined by the formulae (41). If we denote the angles which the planes of polarization of the incident reflected and refracted waves make with the plane of incidence by Θ, Θ' and Θ_1 respectively, we find the following formulae for the determination of the two latter:

$$\left.\begin{aligned} \tan\Theta' &= \tan(90^\circ+\theta') = -\frac{1}{\tan\theta'} = \frac{1}{\tan\theta}\frac{\cos(\phi+\phi_1)}{\cos(\phi-\phi_1)} \\ &= \frac{1}{\tan(\Theta-90^\circ)}\frac{\cos(\phi+\phi_1)}{\cos(\phi-\phi_1)} = -\tan\Theta\frac{\cos(\phi+\phi_1)}{\cos(\phi-\phi_1)} \\ \text{and similarly}\qquad \tan\Theta_1 &= \tan\Theta\sec(\phi-\phi_1) \end{aligned}\right\}\quad(41\text{ A})$$

These two formulae give the following relation between Θ' and Θ_1:

$$\tan\Theta' = -\tan\Theta_1\cos(\phi+\phi_1).\ \ldots\ldots\ldots\ldots\ldots\ldots(42)$$

Summary.—It follows from the above formulae:

1. If $\phi+\phi_1=90^\circ$, then $\Theta'=0$, that is, if the incident waves strike the reflecting surface at the angle of polarization (cf. formula (37)), the reflected waves will be polarized in the plane of incidence (cf. also Exs. 3 and 5 at end of chapter).

2. For $(\phi+\phi_1)<90^\circ$, $\cos(\phi+\phi_1)<\cos(\phi-\phi_1)$, and hence $\tan\Theta'<-\tan\Theta$, that is, as ϕ increases from zero, Θ' will decrease in absolute value until $(\phi+\phi_1)=90^\circ$, where it will vanish (cf. 1); for $(\phi+\phi_1)>90^\circ$, $\cos(\phi+\phi_1)$ will, in general, assume small negative values in comparison to large positive ones assumed by $\cos(\phi-\phi_1)$, and hence $\tan\Theta'<\tan\Theta$. Θ' will, therefore, be smaller than Θ, or the effect of reflection will be to bring the plane of polarization of the reflected waves nearer to the plane of incidence, the two coinciding, when the angle of incidence becomes that of polarization (cf. also Ex. 6 at end of chapter).

3. $\sec(\phi-\phi_1)>1$ for all values of ϕ, and hence $\tan\Theta_1>\tan\Theta$, that is, the effect of refraction is to remove the plane of polarization further from the plane of incidence (cf. also Ex. 7 at end of chapter).

The Amplitudes of the Reflected and Refracted Waves.—Formulae (40) give the following expressions for the resultant amplitudes of the reflected and refracted waves (38) and (39) respectively:

$$\left.\begin{aligned} A'^2 &= A^2\left[\sin^2\theta\frac{\sin^2(\phi-\phi_1)}{\sin^2(\phi+\phi_1)}+\cos^2\theta\frac{\tan^2(\phi-\phi_1)}{\tan^2(\phi+\phi_1)}\right] \\ &= A^2\frac{\sin^2(\phi-\phi_1)}{\sin^2(\phi+\phi_1)}\left[\sin^2\theta+\cos^2\theta\frac{\cos^2(\phi+\phi_1)}{\cos^2(\phi-\phi_1)}\right] \\ &= A^2\frac{\sin^2(\phi-\phi_1)}{\sin^2(\phi+\phi_1)}\left[1-\cos^2\theta\frac{\sin 2\phi\sin 2\phi_1}{\cos^2(\phi-\phi_1)}\right] \\ \text{and}\qquad A_1{}^2 &= \frac{4A^2\cos^2\phi\sin^2\phi_1}{\sin^2(\phi+\phi_1)}\left[\sin^2\theta+\frac{\cos^2\theta}{\cos^2(\phi-\phi_1)}\right] \\ &= \frac{4A^2\cos^2\phi\sin^2\phi_1}{\sin^2(\phi+\phi_1)}\left[1+\cos^2\theta\tan^2(\phi-\phi_1)\right] \end{aligned}\right\}\ \ldots\ldots\ldots(43)$$

(cf. Ex. 1 at end of chapter), or, in terms of Θ:

$$\left.\begin{aligned} A'^2 &= A^2\frac{\sin^2(\phi-\phi_1)}{\sin^2(\phi+\phi_1)}\left[1-\sin^2\Theta\,\frac{\sin 2\phi\sin 2\phi_1}{\cos^2(\phi-\phi_1)}\right] \\ \text{and}\qquad A_1^2 &= \frac{4A^2\cos^2\phi\sin^2\phi_1}{\sin^2(\phi+\phi_1)}\left[1+\sin^2\Theta\tan^2(\phi-\phi_1)\right]\end{aligned}\right\} \text{......(43A)}$$

Total Reflection.—If the velocity of propagation in the first medium 0 or that of the incident waves is smaller than that of (the refracted) waves transmitted through the second medium 1 (cf. Fig. 32), and, if the angle of incidence is taken sufficiently large, nearly equal to 90°, no waves will enter the second medium, but all will be reflected back into the first; this familiar phenomenon is thus known as "total reflection."

For $v<v_1$ formula (14) then gives

$$\left.\begin{aligned} \sin\phi_1 &= \frac{v_1}{v}\sin\phi = \frac{\sin\phi}{N} \\ \text{where}\qquad N &= \frac{v}{v_1}<1\end{aligned}\right\}; \text{ ....(44)}$$

if now $\sin\phi>N$, $\sin\phi_1$ will be larger than unity, which is evidently only possible when ϕ_1 is a complex quantity (cf. below), that is, no waves will enter the second medium, but all will be reflected back into the first, as confirmed by observation. Let us now examine the form assumed by the formulae of partial reflection for the particular case of total reflection; for this purpose we shall first consider here as above the two particular cases where the incident electric oscillations take place at right angles to and in the plane of incidence.

Case I.: The Incident (Electric) Oscillations at $\perp$ to Plane of Incidence.—Here the amplitudes of the reflected and refracted waves were given by formulae (18) and (19); these formulae must now hold, when we replace there $\sin\phi_1$ and $\cos\phi_1$ by their values for total reflection, namely

$$\sin\phi_1 = \frac{\sin\phi}{N} \text{ ....(45)}$$

and

$$\cos\phi_1 = \sqrt{1-\frac{\sin^2\phi}{N^2}}.$$

We can regard $\sin\phi_1$ as replaced by the real quantity $\frac{\sin\phi}{N}>1$; $\cos\phi_1$ is imaginary, since $\frac{\sin^2\phi}{N^2}$ is here larger than unity; we shall, therefore, write it in the form

$$\cos\phi_1 = i\sqrt{\frac{\sin^2\phi}{N^2}-1}, \text{ ....(46)}$$

where $i=\sqrt{-1}$ is the imaginary unit; the quantity under the square root-sign then becomes positive.

Replace $\sin\phi_1$ and $\cos\phi_1$ by their values (45) and (46) in formulae (18) and (19), and we have

$$\left.\begin{aligned} a' &= -a\frac{i\sqrt{\sin^2\phi - N^2}-\cos\phi}{i\sqrt{\sin^2\phi - N^2}+\cos\phi} = a\left(\frac{N^2+\cos 2\phi}{1-N^2} - i\frac{2\cos\phi\sqrt{\sin^2\phi - N^2}}{1-N^2}\right) \\ \text{and}\qquad & \\ a_1 &= \frac{2a\cos\phi}{i\sqrt{\sin^2\phi - N^2}+\cos\phi} = a\left(\frac{2\cos^2\phi}{1-N^2} - i\frac{2\cos\phi\sqrt{\sin^2\phi - N^2}}{1-N^2}\right) \end{aligned}\right\}, \tag{47}$$

that is, the amplitudes of the reflected and refracted waves are here complex quantities, whereas that of the incident waves is real. Complex quantities as *amplitudes* can now have no physical meaning, but as expressions for the *quantities* a' and a_1 we should be able to interpret them physically. We have now obtained the expressions (47) on the assumption that the incident waves underwent no change in phase either upon reflection or upon refraction; the fact that the resulting expressions for a' and a_1 are here complex would now suggest the incorrectness of that assumption or, on the other hand, that these expressions contain, in fact, changes in phase at the reflecting surface, for the formulae themselves must be valid as such, since they follow directly from our surface conditions; in which case we should be able to write the expressions (47) for a' and a_1 in some such form as

$$\left.\begin{aligned} a' &= \bar{a}'e^{-in\delta'} \\ \text{and}\qquad a_1 &= \bar{a}_1e^{-in\delta_1} \end{aligned}\right\} \tag{48}$$

(cf. formulae (10)), where $\bar{a}'$, $\bar{a}_1$ and δ', δ_1 are **real** quantities, the former denoting the amplitudes and the latter the changes in phase sought. The component electric forces R' and R_1 acting in the reflected and refracted waves would then evidently be given by the expressions

$$R' = \bar{a}'e^{in\left(t-\frac{y\sin\phi'+x\cos\phi'}{v'}-\delta'\right)}$$

and

$$R_1 = \bar{a}_1e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}-\delta_1\right)}$$

(cf. formulae (10)).

Formulae (47) and (48) give now the two following equations for the determination of the quantities $\bar{a}'$, $\bar{a}_1$ and δ', δ_1 sought:

$$a\left(\frac{N^2+\cos 2\phi}{1-N^2} - i\frac{2\cos\phi\sqrt{\sin^2\phi - N^2}}{1-N^2}\right) = \bar{a}'e^{-in\delta'} = \bar{a}'(\cos n\delta' - i\sin n\delta')$$

and

$$a\left(\frac{2\cos^2\phi}{1-N^2} - i\frac{2\cos\phi\sqrt{\sin^2\phi - N^2}}{1-N^2}\right) = \bar{a}_1e^{-in\delta_1} = \bar{a}_1(\cos n\delta_1 - i\sin n\delta_1),$$

or, since the real and imaginary parts respectively of each equation must be equal, the four following equations:

$$a\frac{N^2+\cos 2\phi}{1-N^2}=\bar{a}'\cos n\delta',$$

$$\frac{2a\cos\phi\sqrt{\sin^2\phi-N^2}}{1-N^2}=\bar{a}'\sin n\delta',$$

$$a\frac{2\cos^2\phi}{1-N^2}=\bar{a}_1\cos n\delta_1,$$

$$\frac{2a\cos\phi\sqrt{\sin^2\phi-N^2}}{1-N^2}=\bar{a}_1\sin n\delta_1;$$

which evidently give

$$\left.\begin{array}{lll} & \bar{a}'=a & \tan n\delta'=\dfrac{2\cos\phi\sqrt{\sin^2\phi-N^2}}{N^2+\cos 2\phi} \\ \text{and} & \bar{a}_1=\dfrac{2a\cos\phi}{\sqrt{1-N^2}} & \tan n\delta_1=\dfrac{\sqrt{\sin^2\phi-N^2}}{\cos\phi}\end{array}\right\} \quad \text{..........(49)}$$

The Law of Refraction for Total Reflection.—The first two formulae (49) state that the incident waves undergo a change in phase but none in amplitude and hence in intensity upon reflection. To determine the behaviour of the refracted waves in the dividing surface or film—they cannot enter the second medium, since ϕ_1 is never less than 90°—we first examine the form assumed here by the law of refraction (14),

namely
$$\frac{\sin\phi}{v}=\frac{\sin\phi_1}{v_1}.*$$

That this law may be satisfied for total reflection, $\sin\phi_1$ must be larger than unity; this is now only possible, when the angle ϕ_1 is complex; let us, therefore, imagine it as replaced by the complex angle $\frac{\pi}{2}+i\phi_1$, and we have

$$\sin\left(\frac{\pi}{2}+i\phi_1\right)=\sin\frac{\pi}{2}\cos i\phi_1+\cos\frac{\pi}{2}\sin i\phi_1=\cos i\phi_1,$$

which can also be written in the exponential form

$$\sin\left(\frac{\pi}{2}+i\phi_1\right)=\frac{e^{\phi_1}+e^{-\phi_1}}{2}, \quad \text{......................(50)}$$

an expression that evidently assumes all values between 1 and ∞ for real values of ϕ_1. For any given case of total reflection, that is, for given v, v_1 and ϕ, we can thus imagine the angle of refraction as replaced by the complex angle $\frac{\pi}{2}+i\phi_1$, where ϕ_1 is to be determined as a function of v, v_1 and ϕ by the formula

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* Strictly speaking, ϕ_1 and v_1 denote the angle of incidence and the velocity of propagation respectively *within* the *dividing film*.

$$e^{\phi_1}+e^{-\phi_1}=\frac{2v_1}{v}\sin\phi\,;\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51)$$

in which case the above law of refraction will be satisfied.

Direct Method of Treatment of Given Problem on Total Reflection.—On writing the angle of reflection for total reflection in the form $\frac{\pi}{2}+i\phi_1$, where ϕ_1 is to be determined by formula (50), we could evidently treat the above case of total reflection similarly to that of partial reflection (cf. pp. 268-272), and we should obtain the same formulae as those deduced by the above indirect method of treatment (cf. pp. 283-285); as this method is instructive, throwing light on the behaviour of the refracted waves in the dividing film, we shall examine it briefly here.

For the given case of total reflection we evidently have

$$P=Q=P'=Q'=P_1=Q_1=0,$$

$$R=ae^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},$$

$$R'=a'e^{in\left(t-\frac{y\sin\phi'-x\cos\phi'}{v'}\right)}$$

(cf. formulae (9) and (10)), whereas R_1, according to the above, is to be written in the form

$$R_1=a_1e^{in\left[t-\frac{y\sin(\pi/2+i\phi_1)-x\cos(\pi/2+i\phi_1)}{v_1}\right]}.$$

Since now

$$\cos(\pi/2+i\phi_1)=\pm\sqrt{1-\sin^2(\pi/2+i\phi_1)}=\pm i\sqrt{\sin^2(\pi/2+i\phi_1)-1},$$

where the expression under the last square-root sign is evidently a real quantity, we can write the expression for R_1 in the form

$$R_1=a_1e^{in\left[t-\frac{y\sin(\pi/2+i\phi_1)}{v_1}\right]}e^{\mp\frac{nx\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v_1}}\ .\ \ldots\ldots\ldots\ldots(52)$$

The positive sign must evidently be chosen before the square-root sign in the last factor of this expression, for, otherwise, the further we receded from the first medium into the dividing film, where x is to be taken negative, the greater would be the force acting R_1. R_1 must, therefore, be written

$$R_1=a_1e^{in\left[t-\frac{y\sin(\pi/2+i\phi_1)}{v_1}\right]}e^{\frac{nx\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v_1}}\ .\ \ldots\ldots\ldots\ldots(52\text{A})$$

Rapid Disappearance of Refracted Waves within Film.—Within the dividing film at the distance λ_1, the wave-length of the given waves, from the boundary of the first medium, that is, for $x = -\lambda_1$, the last factor of the expression (52A) for R_1 would assume the value

$$e^{-\frac{n\lambda_1\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v_1}},$$

or, by formulae (5), (50) and (51),

$$e^{-2\pi\sqrt{\sin^2(\pi/2+i\phi_1)-1}} = e^{-2\pi\sqrt{v^2/v_1{}^2\sin^2\phi-1}},$$

which is evidently small. As we receded further into the dividing film, this factor would now decrease rapidly in value, that is, the force R_1 would vanish and the refracted waves thus be extinguished almost immediately.

Determination of the Amplitudes and Changes in Phase of the Reflected and Refracted Waves.—To obtain the formulae for total reflection by the given method, we make use, as above in the case of partial reflection, of our surface-conditions; these are given by formulae (3A′), where R_1, α_1 and β_1 are to be replaced by their values for the given case of total reflection. We obtain the values for α_1 and β_1, on replacing, as above in the expression for R_1 for partial reflection, the angle ϕ_1 by the complex angle $\pi/2 + i\phi_1$ in the former expressions (cf. formulae (11)) for those component-forces. Replace the R's, α's and β's by their values in the surface-conditions (3A′), and we have on the dividing surface ($x = 0$)

$$\left.\begin{aligned}
&a_1 e^{in\left[t-\frac{y\sin(\pi/2+i\phi)}{v_1}\right]} = ae^{in\left(t-\frac{y\sin\phi}{v}\right)} + a'e^{in\left(t-\frac{y\sin\phi'}{v'}\right)},\\
&a_1\frac{\sin(\pi/2+i\phi_1)}{v_1}e^{in\left[t-\frac{y\sin(\pi/2+i\phi_1)}{v_1}\right]}\\
&\qquad = a\frac{\sin\phi}{v}e^{in\left(t-\frac{y\sin\phi}{v}\right)} + a'\frac{\sin\phi'}{v'}e^{in\left(t-\frac{y\sin\phi'}{v'}\right)}\\
\text{and}\quad &a_1\frac{\cos(\pi/2+i\phi_1)}{v_1}e^{in\left[t-\frac{y\sin(\pi/2+i\phi_1)}{v_1}\right]}\\
&\qquad = a\frac{\cos\phi}{v}e^{in\left(t-\frac{y\sin\phi}{v}\right)} - a'\frac{\cos\phi'}{v'}e^{in\left(t-\frac{y\sin\phi'}{v'}\right)}
\end{aligned}\right\}\quad\text{.........(53)}$$

These conditions can now evidently be satisfied only when

$$\frac{\sin\phi}{v} = \frac{\sin\phi'}{v'} = \frac{\sin(\pi/2+i\phi_1)}{v_1};\quad\text{.....................(54)}$$

hence, by formulae (12), which also hold here,

$$\phi' = \phi, \quad (v' = v).$$

The conditions themselves then reduce to the following: the first two both to

$$a_1 = a + a'$$

and the third to

$$a_1\frac{\cos(\pi/2+i\phi_1)}{v_1} = (a - a')\frac{\cos\phi}{v}.$$

That these two conditional equations may be satisfied, the quantities a' and a_1 must be determined by the same; they evidently give

$$a'\left[\frac{\cos\phi}{v}+\frac{\cos(\pi/2+i\phi_1)}{v_1}\right]=a\left[\frac{\cos\phi}{v}-\frac{\cos(\pi/2+i\phi_1)}{v_1}\right]$$

and
$$a_1\left[\frac{\cos\phi}{v}+\frac{\cos(\pi/2+i\phi_1)}{v_1}\right]=2a\frac{\cos\phi}{v},$$

which can be written in the form

$$\left.\begin{aligned}&a'\left[\frac{\cos\phi}{v}-i\frac{\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v_1}\right]=a\left[\frac{\cos\phi}{v}+i\frac{\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v_1}\right]\\&\text{and}\\&a_1\left[\frac{\cos\phi}{v}-i\frac{\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v_1}\right]=2a\frac{\cos\phi}{v}\end{aligned}\right\}, \quad (55)$$

where the expression under the square-root signs is positive.

Of the equations (55) the former for the determination of the quantity a' has now the form

$$a'(\cos\omega'-i\sin\omega')=a(\cos\omega'+i\sin\omega').$$

Since the real and the imaginary terms respectively of this equation must be equal to the respective ones of the above conditional equation for a', we have

$$\frac{\cos\phi}{v}=\cos\omega' \quad\text{and}\quad \frac{\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v_1}=\sin\omega';$$

from which ω' is determined by the formula

$$\tan\omega'=\frac{v}{v_1}\,\frac{\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{\cos\phi}. \quad\text{......................(56)}$$

We can, therefore, write the former equation (55) for a' in the form

$$a'e^{-i\omega'}=ae^{i\omega'},$$

or
$$a'=ae^{2i\omega'}, \quad\text{..............................(57)}$$

where ω' is determined by formula (56).

By formula (57) and the conditional equation

$$a_1=a+a',$$

we can write the quantity a_1 in the form

$$a_1=a(1+e^{2i\omega'})=\bar{a}e^{2i\omega_1}, \quad\text{..........................(58)}$$

where $\bar{a}$ and ω_1 denote the amplitude and change in phase respectively of the refracted force. To determine $\bar{a}$ and ω_1 in terms of a and ω', we write this equation between the same explicitly

$$a(1+\cos 2\omega'+i\sin 2\omega')=\bar{a}(\cos 2\omega_1+i\sin 2\omega_1),$$

and we thus have the two equations

$$a(1+\cos 2\omega') = \bar{a}\cos 2\omega_1$$

and

$$a\sin 2\omega' = \bar{a}\sin 2\omega_1,$$

which give

$$\tan 2\omega_1 = \frac{\sin 2\omega'}{1+\cos 2\omega'}, = \tan\omega'$$

and

$$\bar{a}^2 = 2a^2(1+\cos 2\omega') = 4a^2\cos^2\omega',$$

or, if we replace here ω' by its value (56),

$$\left.\begin{aligned} \tan 2\omega_1 &= \frac{v}{v_1}\frac{\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{\cos\phi} \\ \text{and}\qquad \bar{a} &= \frac{2a\cos\phi}{\sqrt{\cos^2\phi + \dfrac{v^2}{{v_1}^2}[\sin^2(\pi/2+i\phi_1)-1]}} \end{aligned}\right\} \text{. ....(59)}$$

We observe that these formulae for the refracted waves and the above (56) and (57) for the reflected ones are identical to those (49) found by the former indirect method; to obtain the latter we replace here v/v_1 and $\sin(\pi/2+i\phi_1)$ by their values, the quantities N and $\frac{\sin\phi}{N}$ respectively employed in the previous method; the changes in phase 2ω and $2\omega_1$ here are the $n\delta'$ and $n\delta_1$ of formulae (49).

Case II.: The Incident (Electric) Oscillations in Plane of Incidence.—The treatment of this case is similar to that of case I.

By formulae (45) and (46), formulae (29) and (30), which hold for partial reflection, will assume here the form

$$\left.\begin{aligned} b' &= b\frac{\sin\phi\cos\phi - \sin\phi_1\cos\phi_1}{\sin\phi\cos\phi + \sin\phi_1\cos\phi_1} = b\frac{\cos\phi - \dfrac{i}{N}\sqrt{\dfrac{\sin^2\phi}{N^2}-1}}{\cos\phi + \dfrac{i}{N}\sqrt{\dfrac{\sin^2\phi}{N^2}-1}} \\ &= b\left[\frac{N^2(N^2\cos^2\phi+1)-\sin^2\phi}{N^2(N^2\cos^2\phi-1)+\sin^2\phi} - i\frac{2N^2\cos\phi\sqrt{\sin^2\phi-N^2}}{N^2(N^2\cos^2\phi-1)+\sin^2\phi}\right] \\ \text{and } b_1 &= \frac{2b\cos\phi\sin\phi_1}{\sin\phi\cos\phi+\sin\phi_1\cos\phi_1} = 2b\frac{\dfrac{\cos\phi}{N}}{\cos\phi + \dfrac{i}{N}\sqrt{\dfrac{\sin^2\phi}{N^2}-1}} \\ &= 2b\left[\frac{N^3\cos^2\phi}{N^2(N^2\cos^2\phi-1+\sin^2\phi} - i\frac{N\cos\phi\sqrt{\sin^2\phi-N^2}}{N^2(N^2\cos^2\phi-1)+\sin^2\phi}\right] \end{aligned}\right\} \text{. (60)}$$

Since these expressions for the amplitudes are complex ones, we must abandon here, as in case I., the assumption of no change in phase on the dividing surface, and thus write the expressions for the quantities b' and b_1 in the form

$$\left.\begin{aligned} b' &= \bar{b}' e^{-in\zeta'} \\ b_1 &= \bar{b}_1 e^{-in\zeta_1} \end{aligned}\right\}, \qquad \text{.............................(61)}$$

where $\bar{b}'$, $\bar{b}_1$ and $n\zeta'$, $n\zeta_1$ shall denote the amplitudes and changes in phase respectively sought.

By formulae (60) and (61) we have then the two following equations for the determination of the quantities $\bar{b}'$, $\bar{b}_1$, $n\zeta'$, $n\zeta_1$:

$$b' = b\left[\frac{N^2(N^2\cos^2\phi + 1) - \sin^2\phi}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi} - i\,\frac{2N^2\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi}\right]$$
$$= \bar{b}'(\cos n\zeta' - i\sin n\zeta')$$

and
$$b_1 = 2b\left[\frac{N^3\cos^2\phi}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi} - i\,\frac{N\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi}\right]$$
$$= \bar{b}_1(\cos n\zeta_1 - i\sin n\zeta_1),$$

or, on separating the real and the imaginary terms, the four equations:

$$b\,\frac{N^2(N^2\cos^2\phi + 1) - \sin^2\phi}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi} = \bar{b}'\cos n\zeta',$$

$$\frac{2bN^2\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi} = \bar{b}'\sin n\zeta',$$

$$\frac{2bN^3\cos^2\phi}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi} = \bar{b}_1\cos n\zeta_1,$$

$$\frac{2bN\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi} = \bar{b}_1\sin n\zeta_1,$$

which give
$$\left.\begin{aligned} &\bar{b}' = b, \quad \tan n\zeta' = \frac{2N^2\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2(N^2\cos^2\phi + 1) - \sin^2\phi} \\ &\bar{b}_1 = \frac{2bN\cos\phi}{\sqrt{N^2(N^2\cos^2\phi - 1) + \sin^2\phi}}, \quad \tan n\zeta_1 = \frac{\sqrt{\sin^2\phi - N^2}}{N^2\cos\phi} \end{aligned}\right\}. \quad \text{....(62)}$$

The first two formulae state that the incident waves undergo a change in phase $n\zeta'$ but none in amplitude upon reflection.

Direct Method of Treatment of Given Case.—Lastly, let us treat the given problem by the direct method applied above to case I. Here the component electric forces P, Q, P', Q' will be given by formulae (23) and (24) ($R = R' = 0$), whereas, according to the given method, the expressions for the component forces P_1, Q_1 ($R_1 = 0$) will have to be written in the form

$$P_1 = b_1\sin(\pi/2 + i\phi_1)\,e^{in\left[t - \frac{y\sin(\pi/2+i\phi_1) - x\cos(\pi/2+i\phi_1)}{v_1}\right]},$$

$$Q_1 = b_1\cos(\pi/2 + i\phi_1)\,e^{in\left[t - \frac{y\sin(\pi/2+i\phi_1) - x\cos(\pi/2+i\phi_1)}{v_1}\right]},$$

or
$$P_1 = b_1 \sin(\pi/2 + i\phi_1)\, e^{in\left[t - \frac{y \sin(\pi/2 + i\phi_1)}{v_1}\right]} e^{nx\sqrt{\sin^2(\pi/2+i\phi_1)-1}},$$
$$Q_1 = b_1 \cos(\pi/2 + i\phi_1)\, e^{in\left[t - \frac{y \sin(\pi/2 + i\phi_1)}{v_1}\right]} e^{nx\sqrt{\sin^2(\pi/2+i\phi_1)-1}}$$

(cf. p. 286); from which it is evident that both P_1 and Q_1 will vanish at a very short distance within the dividing film, that is, the refracted waves will be extinguished here, as in case I., almost immediately (cf. p. 287).

Determination of the Amplitudes and Changes in Phase of the Reflected and Refracted Waves.—To obtain the formulae for the amplitudes and the changes in phase after incidence, we replace the P's, Q's, and γ's by their values in the surface-conditions (3B′), and we have on the dividing surface ($x = 0$)

$$D_1 b_1 \sin(\pi/2 + i\phi_1)\, e^{in\left[t - \frac{y \sin(\pi/2 + i\phi_1)}{v_1}\right]}$$
$$= Db \sin\phi e^{in\left(t - \frac{y\sin\phi}{v}\right)} + Db' \sin\phi' e^{in\left(t - \frac{y\sin\phi'}{v'}\right)},$$
$$b_1 \cos(\pi/2 + i\phi_1)\, e^{in\left[t - \frac{y \sin(\pi/2 + i\phi_1)}{v'}\right]}$$
$$= b \cos\phi e^{in\left(t - \frac{y\sin\phi}{v}\right)} - b' \cos\phi' e^{in\left(t - \frac{y\sin\phi'}{v'}\right)}$$

and
$$\frac{b_1}{v_1} e^{in\left[t - \frac{y \sin(\pi/2 + i\phi_1)}{v_1}\right]}$$
$$= \frac{b}{v} e^{in\left(t - \frac{y\sin\phi}{v}\right)} + \frac{b'}{v'} e^{in\left(t - \frac{y\sin\phi'}{v'}\right)}$$

(cf. formulae (27) for the values of the γ's, where ϕ_1 is to be replaced by $\pi/2 + i\phi_1$ in the expression for γ_1).

These conditions can evidently be satisfied only when

$$\frac{\sin\phi}{v} = \frac{\sin\phi'}{v'} = \frac{\sin(\pi/2 + i\phi_1)}{v_1}; \qquad \text{......................(63)}$$

hence, by formulae (12), $\phi' = \phi$ $(v' = v)$;

the conditions themselves then reduce to

$$\left.\begin{aligned} D_1 b_1 \sin(\pi/2 + i\phi_1) &= (b + b') D \sin\phi, \\ b_1 \cos(\pi/2 + i\phi_1) &= (b - b') \cos\phi \\ \text{and} \qquad \frac{b_1}{v_1} &= \frac{b + b'}{v} \end{aligned}\right\} \quad \text{...............(64)}$$

By formulae (12) and (63) the first and last of these conditional equations will be found to be identical. That they may be satisfied, the quantities b' and b_1 must be determined by the same; they evidently give

$$b'[v\cos\phi + v_1 \cos(\pi/2 + i\phi_1)] = b[v\cos\phi - v_1\cos(\pi/2 + i\phi_1)]$$

and
$$b_1[v\cos\phi + v_1\cos(\pi/2 + i\phi_1)] = 2bv_1\cos\phi,$$

or

$$b'[v\cos\phi - iv_1\sqrt{\sin^2(\pi/2+i\phi_1)-1}\,] = b[v\cos\phi + iv_1\sqrt{\sin^2(\pi/2+i\phi_1)-1}\,]$$

and

$$b_1[v\cos\phi - iv_1\sqrt{\sin^2(\pi/2+i\phi_1)-1}\,] = 2bv_1\cos\phi,$$

where the expression under the square-root signs is positive (cf. p. 284).

The equation for b' has now the form

$$b'(\cos\chi' - i\sin\chi') = b(\cos\chi' + i\sin\chi')$$

or

$$b' = be^{2i\chi'}, \quad\ldots\ldots(65)$$

where χ' is to be determined from the two equations

$$v\cos\phi = \cos\chi' \quad\text{and}\quad v_1\sqrt{\sin^2(\pi/2+i\phi-1)} = \sin\chi'\,;$$

which give

$$\tan\chi' = \frac{v_1\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v\cos\phi}. \quad\ldots\ldots(66)$$

To obtain b_1 in the form $\qquad b_1 = \bar{b}_1 e^{2i\chi_1},$

we replace b' by its value (65) in the last conditional equation (64), and we have

$$b_1 = b\frac{v_1}{v}(1 + e^{2i\chi'}) = \bar{b}_1 e^{2i\chi_1},$$

or explicitly

$$bv_1(1 + \cos 2\chi' + i\sin 2\chi') = b_1 v(\cos 2\chi_1 + i\sin 2\chi_1),$$

hence

$$bv_1(1+\cos 2\chi') = \bar{b}_1 v\cos 2\chi_1$$

and

$$bv_1\sin 2\chi' = b_1 v\sin 2\chi_1\,;$$

which latter give

$$\tan 2\chi_1 = \frac{\sin 2\chi'}{1+\cos 2\chi'} = \tan\chi'$$

and

$$\bar{b}_1^2 = 2b\frac{v_1^2}{v^2}(1+\cos 2\chi') = 4b^2\frac{v_1^2}{v^2}\cos^2\chi'\,;$$

lastly, replace here χ' by its value (66), and we have

$$\left.\begin{aligned}\tan 2\chi_1 &= \frac{v_1\sqrt{\sin^2(\pi/2+i\phi_1)-1}}{v\cos\phi}\\ \text{and}\qquad \bar{b}_1 &= \frac{2bv_1\cos\phi}{\sqrt{v^2\cos^2\phi + v_1^2[\sin^2(\pi/2+i\phi_1)-1]}}\end{aligned}\right\} \quad\ldots\ldots(67)$$

We observe that these and formulae (65) and (66) are identical to those (62) found by the previous method; v/v_1 and $\sin(\pi/2+i\phi_1)$ are the quantities N and $\frac{\sin\phi}{N}$ respectively of formulae (62), whereas the changes in phase, the 2χ's, were denoted there by the $n\zeta$'s.

The General Case; the Reflected Waves Elliptically Polarized.— It follows from formulae (49), or (56) and (57), and (62), or (65) and (66), *that* incident waves undergo a change in phase but none in amplitude upon total reflection and in both cases I. and II.; but this change in phase is not the same in the two cases. If now the oscillations of the incident waves make an arbitrary angle θ with the plane of incidence, their two components at right angles to and in the plane of incidence will, therefore, undergo different changes in phase upon total reflection, and hence the reflected waves will, according to Chapter III., be elliptically polarized. The ellipses described by these oscillations will evidently be determined here alone—aside from the angle θ—by the difference between the changes in phase suffered by the two component oscillations at the reflecting surface, since the amplitudes of those components undergo no change upon total reflection (cf. formulae (49) and (62)). Let us next derive the expression for this difference between the changes in phase and then examine the paths or ellipses described by the oscillations of the reflected waves.

Determination of the Difference in Phase between the Component Reflected Oscillations.—By formulae (49) and (62), the difference $n\Delta'$ between the changes in phase of the two components of any linearly polarized oscillation upon total reflection will be given by the expression

$$\begin{aligned} n\Delta' &= n(\zeta' - \delta') \\ &= \arctan \frac{2N^2\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2(N^2\cos^2\phi + 1) - \sin^2\phi} - \arctan \frac{2\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2 + \cos 2\phi}, \end{aligned}$$

or, by the trigonometric formula

$$\arctan u - \arctan v = \arctan \frac{u - v}{1 + uv},$$

$$\begin{aligned} n\Delta' &= \arctan \left\{ \frac{2(N^2-1)^2\sin^2\phi\cos\phi\sqrt{\sin^2\phi - N^2}}{[N^2(N^2\cos^2\phi + 1) - \sin^2\phi] \times [N^2 + \cos 2\phi] + 4N^2\cos^2\phi(\sin^2\phi - N^2)} \right\} \\ &= \arctan \left[\frac{2(N^2-1)\sin^2\phi\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2(N^2\cos^2\phi - 1) + \sin^2\phi + 2(N^2-1)\sin^4\phi} \right] \\ &= \arctan \left(\frac{2\sin^2\phi\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2\cos^2\phi - \sin^2\phi\cos 2\phi} \right). \quad \text{.........................} (68) \end{aligned}$$

By the trigonometric formula

$$\arctan u = \arcsin \frac{u}{\sqrt{1 + u^2}} = \arccos \frac{1}{\sqrt{1 + u^2}},$$

this difference in phase $n\Delta'$ can also be expressed as an *arcsine* or *arccosine*, namely

$$\begin{aligned} n\Delta' &= \arcsin \left(\frac{2\sin^2\phi\cos\phi\sqrt{\sin^2\phi - N^2}}{N^2\cos^2\phi - \sin^2\phi} \right) \\ &= \arccos \left(\frac{N^2\cos^2\phi - \sin^2\phi\cos 2\phi}{N^2\cos^2\phi - \sin^2\phi} \right). \quad \text{.......} (68\text{A}) \end{aligned}$$

Analytic Expressions for the Incident and Reflected Oscillations.—Let us represent the force acting in any incident electric wave by the function

$$A \sin\left(t - \frac{y \sin\phi - x \cos\phi}{v}\right), \quad\text{.....................(69)}$$

where the oscillations are supposed to be taking place in any plane θ, referred to the plane of incidence; we are choosing the real part of the complex function employed above (cf. p. 280) in the ensuing investigations for simplicity. Since now the two components of oscillation, that at right angles to the plane of incidence and that in that plane, undergo only changes in phase upon total reflection, the respective component forces acting in the reflected wave will, therefore, be given by the expressions

$$\left.\begin{aligned} &A \sin\theta \sin n\left(t - \frac{y \sin\phi + x \cos\phi}{v} - \delta'\right) \\ \text{and} \qquad &A \cos\theta \sin n\left(t - \frac{y \sin\phi + x \cos\phi}{v} - \zeta'\right) \end{aligned}\right\}, \quad\text{...............(70)}$$

where the changes in phase $n\delta'$ and $n\zeta'$ are to be replaced by their values from formulae (49) and (62).

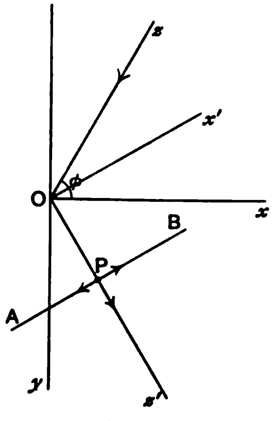

FIG. 33.

If we refer the component forces (70) to a system of rectangular co-ordinates x', y', z', in which x' shall lie in the plane of incidence, the xy-plane, and at right angles to the direction of propagation of the reflected wave, y' be normal to that plane and z' coincide with the direction of propagation of the reflected wave, as indicated in the annexed figure, we can then write the component forces P', Q', R' referred to this new system of coordinates in the form

$$\left.\begin{aligned} P' &= A\cos\theta\sin n\left(t-\frac{z'}{v}-\zeta'\right) \\ Q' &= A\sin\theta\sin n\left(t-\frac{z'}{v}-\delta'\right) \end{aligned}\right\}, \quad\text{.....................(70A)}$$

$$R' = 0.$$

In Figure 33 the y' (z)-axis is normal to the $x'z'$ (xy)-plane, the plane of the paper, and is directed towards the reader; AB is the projection of the plane of oscillation at any point P of the reflected wave upon the $x'z'$ (xy)-plane.

Determination of Path of Oscillation in Reflected Wave.—To determine the path of oscillation at any point P of the reflected wave (70A), we must now eliminate the time t from the equations (70A); for this purpose we write the same explicitly, namely

$$P' = A\cos\theta\left[\cos n\left(\frac{z'}{v}+\zeta'\right)\sin nt - \sin n\left(\frac{z'}{v}+\zeta'\right)\cos nt\right],$$

$$Q' = A\sin\theta\left[\cos n\left(\frac{z'}{v}+\delta'\right)\sin nt - \sin n\left(\frac{z'}{v}+\delta'\right)\cos nt\right],$$

or

$$\left.\begin{aligned} P' &= A_1\sin nt + A_2\cos nt \\ Q' &= B_1\sin nt + B_2\cos nt \end{aligned}\right\}, \quad\text{.........................(71)}$$

where

$$\left.\begin{aligned} A_1 &= A\cos\theta\cos n\left(\frac{z'}{v}+\zeta'\right) \\ A_2 &= -A\cos\theta\sin n\left(\frac{z'}{v}+\zeta'\right) \\ B_1 &= A\sin\theta\cos n\left(\frac{z'}{v}+\delta'\right) \\ B_2 &= -A\sin\theta\sin n\left(\frac{z'}{v}+\delta'\right) \end{aligned}\right\}. \quad\text{....................(72)}$$

On eliminating t from equations (71), we have

$$(B_1P' - A_1Q')^2 + (B_2P' - A_2Q')^2 = (A_2B_1 - A_1B_2)^2$$

or

$$(B_1^2+B_2^2)P'^2 + (A_1^2+A_2^2)Q'^2 - 2(A_1B_1+A_2B_2)P'Q'$$
$$= (A_2B_1 - A_1B_2)^2, \quad\text{............................(73)}$$

the equation of an ellipse (cf. p. 83).

On evaluating the coefficients of this equation, we find

$$\left.\begin{aligned}
&B_1^2+B_2^2=A^2\sin^2\theta, \quad A_1^2+A_2^2=A^2\cos^2\theta,\\
&-2(A_1B_1+A_2B_2)=-2A^2\sin\theta\cos\theta\Big[\cos n\left(\frac{z'}{v}+\zeta'\right)\cos n\left(\frac{z'}{v}+\delta'\right)\\
&\qquad\qquad\qquad\qquad+\sin n\left(\frac{z'}{v}+\zeta'\right)\sin n\left(\frac{z'}{v}+\delta'\right)\Big]\\
&\qquad=-2A^2\sin\theta\cos\theta\cos n(\zeta'-\delta')=-A^2\sin 2\theta\cos n\Delta'\\
&\text{and}\\
&(A_2B_1-A_1B_2)^2=\Big\{-A^2\sin\theta\cos\theta\Big[\sin n\left(\frac{z'}{v}+\zeta'\right)\cos n\left(\frac{z'}{v}+\delta'\right)\\
&\qquad\qquad\qquad\qquad-\cos n\left(\frac{z'}{v}+\zeta'\right)\sin n\left(\frac{z'}{v}+\delta'\right)\Big]\Big\}^2\\
&\qquad=[A^2\sin\theta\cos\theta\sin n(\zeta'-\delta')]^2=A^4\sin^2\theta\cos^2\theta\sin^2 n\Delta'
\end{aligned}\right\}.(74)$$

The equation (73) of the path (ellipse) of oscillation at any point of the reflected wave (70A) will, therefore, assume the definite form

$$\sin^2\theta P'^2+\cos^2\theta Q'^2-\sin 2\theta\cos n\Delta' P'Q'=A^2\sin^2\theta\cos^2\theta\sin^2 n\Delta', \ldots(75)$$

where $\cos n\Delta'$ and $\sin n\Delta'$ are to be replaced by their values from formula (68A). We observe that the coefficients of P'^2 and Q'^2 are functions only of the angle θ and not of the difference in phase $n\Delta'$.

Transformation of Ellipse of Oscillation to its Principal Axes.—Let us, next, transform the equation (75) of the ellipse of oscillation at any point of the reflected wave (70A) to its normal form

$$\frac{P''^2}{a^2}+\frac{Q''^2}{b^2}=1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(76)$$

where P'' and Q'' shall denote the component forces acting along its principal axes a and b respectively. For this purpose we make use of the same transformations as those employed in Chapter III., Exs. 20 and 21: We denote the new coordinate axes, the principal axes of the ellipse sought, in the $x'y'$ plane, by u and v respectively, and the common angle these axes make with the x' and y' axes respectively by ω; the following relations then hold between the component forces P', Q' and P'', Q'':

$$\left.\begin{aligned}P'&=P''\cos\omega-Q''\sin\omega\\ Q'&=P''\sin\omega+Q''\cos\omega\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(77)$$

(cf. Fig. 33), where ω is to be determined thereby, that the term (its coefficient) $P''Q''$ in the equation of the ellipse of oscillation sought vanish (cf. formula (76)). Replace P' and Q' by these values (77) in the equation (75) of the ellipse of oscillation, and we have

$$\begin{aligned}
&\sin^2\theta\,(P''^2\cos^2\omega-2P''Q''\sin\omega\cos\omega+Q''^2\sin^2\omega)\\
&\quad+\cos^2\theta\,(P''^2\sin^2\omega+2P''Q''\sin\omega\cos\omega+Q''^2\cos^2\omega)\\
&\quad-\sin 2\theta\cos n\Delta'\,[P''^2\sin\omega\cos\omega+P''Q''(\cos^2\omega-\sin^2\omega)-Q''\sin\omega\cos\omega]\\
&\qquad\qquad=A^2\sin^2\theta\cos^2\theta\sin^2 n\Delta',
\end{aligned}$$

or

$$\left.\begin{aligned}&P''^2(\sin^2\theta\cos^2\omega+\cos^2\theta\sin^2\omega-\sin 2\theta\cos n\Delta'\sin\omega\cos\omega)\\&\quad+Q''^2(\sin^2\theta\sin^2\omega+\cos^2\theta\cos^2\omega+\sin 2\theta\cos n\Delta'\sin\omega\cos\omega)\\&\quad-P''Q''[2\sin^2\theta\sin\omega\cos\omega-2\cos^2\theta\sin\omega\cos\omega\\&\quad+\sin 2\theta\cos n\Delta'(\cos^2\omega-\sin^2\omega)]\\&\qquad\qquad=A^2\sin^2\theta\cos^2\theta\sin^2 n\Delta'\end{aligned}\right\}. \quad (78)$$

That the term $P''Q''$ of this equation may vanish (cf. formula (76)), its coefficient must evidently vanish, that is,

$$\begin{aligned}&2\sin^2\theta\sin\omega\cos\omega-2\cos^2\theta\sin\omega\cos\omega\\&\qquad+\sin 2\theta\cos n\Delta'(\cos^2\omega-\sin^2\omega)=0, \quad\ldots\ldots.(79)\end{aligned}$$

or $$\cos 2\theta\sin 2\omega-\sin 2\theta\cos n\Delta'\cos 2\omega=0,$$

the equation for the determination of the angle ω;

which gives $$\tan 2\omega=\tan 2\theta\cos n\Delta', \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(79\text{A})$$

and hence
$$\left.\begin{aligned}\sin^2\omega&=\frac{1}{2}-\frac{1}{2}\frac{\cos 2\theta}{\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}\\\cos^2\omega&=\frac{1}{2}+\frac{1}{2}\frac{\cos 2\theta}{\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}\\\text{and}\quad\sin\omega\cos\omega&=\frac{1}{2}\frac{\sin 2\theta\cos n\Delta'}{\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}\end{aligned}\right\}, \quad\ldots\ldots\ldots\ldots(80)$$

where the signs before the square-root signs have been so chosen, that the conditional equation (79) is satisfied.

By formulae (80), we can now write the equation (78) of the ellipse of oscillation sought in the form

$$\begin{aligned}&P''^2\left(\frac{1}{2}-\frac{1}{2}\frac{\cos^2 2\theta}{\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}-\frac{1}{2}\frac{\sin^2 2\theta\cos^2 n\Delta'}{\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}\right)\\&+Q''^2\left(\frac{1}{2}+\frac{1}{2}\frac{\cos^2 2\theta}{\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}+\frac{1}{2}\frac{\sin^2 2\theta\cos^2 n\Delta'}{\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}\right)\\&\qquad\qquad=A^2\sin^2\theta\cos^2\theta\sin^2 n\Delta'\end{aligned}$$

or

$$\left.\begin{aligned}&P''^2(1-\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'})+Q''^2(1+\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'})\\&\qquad\qquad=2A^2\sin^2\theta\cos^2\theta\sin^2 n\Delta'\end{aligned}\right\}, \quad (81)$$

hence in the form (76) sought

$$\frac{P''^2}{\dfrac{2A^2\sin^2\theta\cos^2\theta\sin^2 n\Delta'}{1-\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}}+\frac{Q''^2}{\dfrac{2A^2\sin^2\theta\cos^2\theta\sin^2 n\Delta'}{1+\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'}}}=1,$$

or

$$\frac{P''^2}{\dfrac{A^2(1+\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'})}{2}}+\frac{Q''^2}{\dfrac{A^2(1-\sqrt{\cos^2 2\theta+\sin^2 2\theta\cos^2 n\Delta'})}{2}}=1. \quad (81\text{A})$$

The Refracted Waves for Total Reflection.—We have seen above that for total reflection the refracted waves were extinguished almost immediately upon their passage into the transition film, and in both cases I. and II.; in the general case, where the oscillations of the incident waves make an arbitrary angle θ with the plane of incidence, the refracted waves will, therefore, also vanish within the film, so that their further examination becomes superfluous.

Approximate Validity of Reflection Formulae; the Reflected Waves Elliptically Polarized for Angle of Polarization.—The formulae for partial reflection, (18)-(43A), are confirmed most approximatively by experiment, except when the incident waves strike the reflecting surface at an angle equal or nearly equal to the angle of polarization, when the reflected waves are found to be more or less elliptically polarized. Brewster* first observed this phenomenon; subsequently his observations were confirmed and stated in definite form by Airy† and then by Jamin,‡ the latter examining a great number of solids (metals) and fluids. The deviation from the linear polarization was found to depend not only on the reflecting surface employed but also on the condition of that surface, a polished or dirty surface exhibiting a greater deviation from the linear polarization than a perfectly clean one. The explanation of this elliptic polarization is, therefore, evidently to be sought in the constitution of the reflecting surface or, as might be supposed, in the fact that the thickness of the transition film at the reflecting surface has an effect on (the form of) the waves emitted from the same (cf. p. 301). On deducing the surface conditions employed above, we have now neglected this very thickness of the film and retained only terms of the null order of magnitude in that quantity.§

Surface Conditions of Second Order of Approximation.—Let us now examine the problem of partial reflection and refraction on the assumption that the thickness of the transition film has an effect on (the form of) the waves emitted from it. For this purpose we must first establish the surface conditions that will then hold. To obtain these conditions of the second order of approximation, we integrate our fundamental differential equations (1) and (2) through the given transition film in the direction of normal to same, here the x-axis, and we have, on the assumption that M remain constant throughout the film (cf. p. 6 and foot-note p. 299),

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* *Philosophical Transactions*, 1815, p. 125.

† ,, ,, vol. i., p. 25; and Poggendorf's *Annallen*, Bd. xxviii.

‡ *Annales der chimie et de physique*, iii. serie, tom. xxix. and xxxi.

§ Cf. also Curry, *Theory of Electricity and Magnetism*, § v., pp. 37-39.

$$\left.\begin{aligned} &\frac{1}{v_0}\frac{d}{dt}\int DPdx=\int\frac{d\beta}{dz}dx-\int\frac{d\gamma}{dy}dx\\ &\frac{1}{v_0}\frac{d}{dt}\int DQdx=\gamma_1-\gamma_0-\int\frac{d\alpha}{dz}dx\\ &\frac{1}{v_0}\frac{d}{dt}\int DRdx=\int\frac{d\alpha}{dy}dx-\beta_1+\beta_0\\ \text{and}\qquad &\frac{M}{v_0}\frac{d}{dt}\int\alpha dx\quad=\int\frac{dR}{dy}dx-\int\frac{dQ}{dz}\\ &\frac{M}{v_0}\frac{d}{dt}\int\beta dx\quad=\int\frac{dP}{dz}dx-R_1+R_0\\ &\frac{M}{v_0}\frac{d}{dt}\int\gamma dx\quad=Q_1-Q_0-\int\frac{dP}{dy}dx \end{aligned}\right\},\ \ldots\ldots\ldots\ldots(82)$$

where the index 0 or 1 denotes that the component force in question is to be taken on the right or left hand side respectively of the film, that is, in the medium 0 or 1 respectively (cf. Fig. 32).

According to the surface-conditions of the first order of approximation (3A), the quantities DP, Q, R, α, β, γ remain approximately constant throughout the film, that is, they differ by quantities of the order of magnitude of the thickness l of the film in the two adjacent media. We can, therefore, put these quantities before the sign of integration in the integrals of the surface-conditions (82) sought, assigning them the values assumed in either medium, and the values of the given terms, which are themselves of the first order of magnitude in l, will differ from the actual values by quantities of the second order, which we are rejecting here. The first and fourth conditions then lead to identities, two of the differential equations (1) and (2) themselves, whereas the other four assume the form

$$\begin{aligned} &\frac{1}{v_0}\frac{dQ_1}{dt}\int Ddx=\gamma_1-\gamma_0-\frac{d\alpha_1}{dz}\int dx,\\ &\frac{1}{v_0}\frac{dR_1}{dt}\int Ddx=\frac{d\alpha_1}{dy}\int dx-\beta_1+\beta_0,\\ &\frac{M}{v_0}\frac{d\beta_1}{dt}\int dx\quad=D_1\frac{dP_1}{dz}\int\frac{dx}{D}-R_1+R_0,\\ &\frac{M}{v_0}\frac{d\gamma_1}{dt}\int dx\quad=Q_1-Q_0-D_1\frac{dP_1}{dy}\int\frac{dx}{D}. \end{aligned}$$

To obtain the two remaining surface-conditions, we treat the two conditional equations

$$\frac{d}{dx}(DP)+\frac{d}{dy}(DQ)+\frac{d}{dz}(DR)=0\quad\text{and}\quad\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0^{*}\ldots..(83)$$

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[*] Here, as in the above and the following investigations, the constant of magnetic induction M is assumed to remain constant throughout the transition film, that is, it is assumed to have approximately one and the same value in all (transparent) insulators (crystals), as confirmed by empirical facts.

in a similar manner to the differential equations (1) and (2), and we find

$$D_1P_1 - D_0P_0 + \frac{dQ_1}{dy}\int Ddx + \frac{dR_1}{dz}\int Ddx = 0$$

and

$$\alpha_1 - \alpha_0 + \frac{d\beta_1}{dy}\int dx + \frac{d\gamma_1}{dz}\int dx = 0.$$

We obtain the conditional equations (83) on differentiating our fundamental equations (1) and (2) respectively, the first with respect to x, the second to y and the third to z, and adding.

Surface Conditions for xy-Plane as Plane of Incidence.—For plane waves propagated in the xy-plane (cf. p. 267) the above surface conditions will evidently assume the particular form

$$\left.\begin{aligned}\gamma_1 &= \gamma + \gamma' + \frac{p}{v_0}\frac{dQ_1}{dt}\\ \beta_1 &= \beta + \beta' - \frac{p}{v_0}\frac{dR_1}{dt} + l\frac{d\alpha_1}{dy}\\ R_1 &= R + R' - \frac{lM}{v_0}\frac{d\beta_1}{dt}\\ Q_1 &= Q + Q' - \frac{lM}{v_0}\frac{d\gamma_1}{dt} - qD_1\frac{dP_1}{dy}\\ D_1P_1 &= D(P+P') - p\frac{dQ_1}{dy}\\ \alpha_1 &= \alpha + \alpha' - l\frac{d\beta_1}{dy}\end{aligned}\right\}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(84)$$

where the component forces are to be replaced by their values on the dividing surface $x=0$ and

$$l = \int dx, \quad p = \int Ddx, \quad q = \int\frac{dx}{D}, \ldots\ldots\ldots\ldots\ldots\ldots(84\text{A})$$

l denoting the thickness of the film; the component forces without dash or index (1) are those acting in the incident waves and those with dash or index (1) the forces acting in the reflected or refracted waves respectively.

Problem on Reflection and Refraction.—As above, let us examine here the two particular cases, where the (incident) oscillations are taking place either at right angles to or in the plane of incidence.

Case I.: Electric Oscillations at $\perp$ to Plane of Incidence.—Here

$$P = Q = P' = Q' = P_1 = Q_1 = 0$$

(cf. p. 268). If we represent the electric force acting in the incident waves in the form

$$R = ae^{in\left(t - \frac{y \sin \phi - x \cos \phi}{v}\right)},$$

those acting in the reflected and refracted waves will evidently be given by the functions of the form

$$R' = a'e^{in\left(t - \frac{y \sin \phi' + x \cos \phi'}{v} - \delta'\right)}$$

and

$$R_1 = a_1 e^{in(t - \frac{y \sin \phi_1 - x \cos \phi_1}{v_1} - \delta_1} \qquad (v' = v) \quad \Bigg\} \cdots\cdots\cdots\cdots\cdots (85)$$

(cf. formulae (10) and p. 284), where $n\delta'$ and $n\delta_1$ denote the changes in phase of the reflected and refracted waves respectively, as they leave the film. Such changes in phase must evidently be introduced here, where we are employing surface conditions, in which the thickness of the transition film cannot be neglected. On the other hand, it is easy to show that the given surface conditions can be satisfied only when such changes in phase are assumed (cf. below). a', a_1, $n\delta'$ and $n\delta_1$ are to be regarded here as unknown; they are determined later from the surface conditions.

Replace the R's by their values (85) in formulae (2A), the particular form assumed here by our fundamental equations (2), and we find, on integrating the same with respect to t, the following expressions for the component magnetic forces acting in the incident, reflected and refracted waves respectively:

$$\left.\begin{aligned}
\alpha &= -a\,\frac{v_0}{M}\,\frac{\sin\phi}{v}\,e^{in\left(t - \frac{y \sin \phi - x \cos \phi}{v}\right)} \\
\beta &= -a\,\frac{v_0}{M}\,\frac{\cos\phi}{v}\,e^{in\left(t - \frac{y \sin \phi - x \cos \phi}{v}\right)} \\
\alpha' &= -a'\,\frac{v_0}{M}\,\frac{\sin\phi'}{v}\,e^{in\left(t - \frac{y \sin \phi' + x \cos \phi'}{v} - \delta'\right)} \\
\beta' &= a'\,\frac{v_0}{M}\,\frac{\cos\phi'}{v}\,e^{in\left(t - \frac{y \sin \phi' + x \cos \phi'}{v} - \delta'\right)} \quad * \\
\alpha_1 &= -a_1\,\frac{v_0}{M}\,\frac{\sin\phi_1}{v_1}\,e^{in\left(t - \frac{y \sin \phi_1 - x \cos \phi_1}{v_1} - \delta_1\right)} \quad * \\
\beta_1 &= -a_1\,\frac{v_0}{M}\,\frac{\cos\phi_1}{v_1}\,e^{in\left(t - \frac{y \sin \phi_1 - x \cos \phi_1}{v_1} - \delta_1\right)} \\
&(\gamma = \gamma' = \gamma_1 = 0)
\end{aligned}\right\} \cdots\cdots\cdots\cdots (86)$$

For the given system of oscillations, namely electric ones at right angles to the plane of incidence and magnetic ones in that plane, our surface conditions (84) will evidently reduce to the following three:

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*Cf. foot-note, p. 299.

$$\left.\begin{aligned}
\beta_1 &= \beta + \beta' - \frac{p}{v_0}\frac{dR_1}{dt} + l\frac{da'}{dy} \\
R_1 &= R + R' - l\frac{M}{v_0}\frac{d\beta_1}{dt} \\
a_1 &= a + a' - l\frac{d\beta_1}{dy}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(87)$$

Replace here the component forces by their values on the given dividing surface, $x=0$ (cf. formulae (85) and (86)), and we have

$$\left.\begin{aligned}
-a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\delta_1\right)} &= -a\frac{v_0}{M}\frac{\cos\phi}{v}e^{in\left(t-\frac{y\sin\phi}{v}\right)} \\
&\quad +a'\frac{v_0}{M}\frac{\cos\phi'}{v}e^{in\left(t-\frac{y\sin\phi'}{v}-\delta'\right)} - ina_1\frac{p}{v_0}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\delta_1\right)} \\
&\quad +ina_1 l\frac{v_0}{M}\frac{\sin^2\phi_1}{v_1^2}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\delta_1\right)}, \\
a_1 e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\delta_1\right)} &= ae^{in\left(t-\frac{y\sin\phi}{v}\right)} + a'e^{in\left(t-\frac{y\sin\phi'}{v}-\delta'\right)} \\
&\quad +ina_1 l\frac{\cos\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\delta_1\right)} \\
\text{and}\quad a_1\frac{\sin\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\delta_1\right)} &= a\frac{\sin\phi}{v}e^{in\left(t-\frac{y\sin\phi}{v}\right)} \\
&\quad +a'\frac{\sin\phi'}{v}e^{in\left(t-\frac{y\sin\phi'}{v}-\delta'\right)} + ina_1 l\frac{\cos\phi_1\sin\phi_1}{v_1^2}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\delta_1\right)}
\end{aligned}\right\} \quad \ldots(88)$$

That these conditions may be satisfied for all values of y and t, the familiar relation

$$\frac{\sin\phi_1}{v_1} = \frac{\sin\phi'}{v} = \frac{\sin\phi}{v}$$

(cf. p. 271) must evidently hold. By this relation the conditions themselves will evidently reduce to the following *two*:

$$\left.\begin{aligned}
-a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}(\cos\omega_1+i\sin\omega_1) &= -a\frac{v_0}{M}\frac{\cos\phi}{v}(\cos\omega+i\sin\omega) \\
&\quad +a'\frac{v_0}{M}\frac{\cos\phi'}{v}(\cos\omega'+i\sin\omega') - ina_1\frac{p}{v_0}(\cos\omega_1+i\sin\omega_1) \\
&\quad +ina_1 l\frac{v_0}{M}\frac{\sin^2\phi}{v^2}(\cos\omega_1+i\sin\omega_1) \\
\text{and}\quad a_1(\cos\omega_1+i\sin\omega_1) &= a(\cos\omega+i\sin\omega) + a'(\cos\omega'+i\sin\omega') \\
&\quad +ina_1 l\frac{\cos\phi_1}{v_1}(\cos\omega_1+i\sin\omega_1)
\end{aligned}\right\}, \quad \ldots(89)$$

where
$$\left.\begin{aligned}\omega&=n\left(t-y\frac{\sin\phi}{v}\right)\\ \omega'&=n\left(t-y\frac{\sin\phi'}{v}-\delta'\right)=n\left(t-y\frac{\sin\phi}{v}-\delta'\right)=\omega-n\delta'\\ \omega_1&=n\left(t-y\frac{\sin\phi_1}{v_1}-\delta_1\right)=n\left(t-y\frac{\sin\phi}{v}-\delta_1\right)=\omega-n\delta_1\end{aligned}\right\}.\ \ldots(90)$$

Separate the real and imaginary parts in these two conditional equations (89), and we have the following four in their place:

$$\left.\begin{aligned}-a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}\cos\omega_1&=-a\frac{v_0}{M}\frac{\cos\phi}{v}\cos\omega+a'\frac{v_0}{M}\frac{\cos\phi'}{v}\cos\omega'\\ &\qquad+na_1\frac{p}{v_0}\sin\omega_1-na_1\frac{v_0}{M}l\frac{\sin^2\phi}{v^2}\sin\omega_1,\\ -a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}\sin\omega_1&=-a\frac{v_0}{M}\frac{\cos\phi}{v}\sin\omega+a'\frac{v_0}{M}\frac{\cos\phi'}{v}\sin\omega'\\ &\qquad-na_1\frac{p}{v_0}\cos\omega_1+na_1\frac{v_0}{M}l\frac{\sin^2\phi}{v^2}\cos\omega_1\\ \text{and}\qquad a_1\cos\omega_1&=a\cos\omega+a'\cos\omega'-na_1l\frac{\cos\phi_1}{v_1}\sin\omega_1,\\ a_1\sin\omega_1&=a\sin\omega+a'\sin\omega'+na_1l\frac{\cos\phi_1}{v_1}\cos\omega_1\end{aligned}\right\}.\ \ldots(91)$$

Expand here ω' and ω_1 as functions of ω and $n\delta'$ and $n\delta_1$ respectively (cf. formulae (90)), and we have

$$\left.\begin{aligned}&-a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}(\cos\omega\cos n\delta_1+\sin\omega\sin n\delta_1)=-a\frac{v_0}{M}\frac{\cos\phi}{v}\cos\omega\\ &+a_1\frac{v_0}{M}\frac{\cos\phi'}{v}(\cos\omega\cos n\delta'+\sin\omega\sin n\delta')\\ &+na_1\left(\frac{p}{v_0}-\frac{v_0}{M}l\frac{\sin^2\phi}{v^2}\right)(\sin\omega\cos n\delta_1-\cos\omega\sin n\delta_1),\\ &-a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}(\sin\omega\cos n\delta_1-\cos\omega\sin n\delta_1)=-a\frac{v_0}{M}\frac{\cos\phi}{v}\sin\omega\\ &+a'\frac{v_0}{M}\frac{\cos\phi'}{v}(\sin\omega\cos n\delta'-\cos\omega\sin n\delta_1)\\ &-na_1\left(\frac{p}{v_0}-\frac{v_0}{M}l\frac{\sin^2\phi}{v^2}\right)(\cos\omega\cos n\delta_1+\sin\omega\sin n\delta_1),\\ &\quad a_1(\cos\omega\cos n\delta_1+\sin\omega\sin n\delta_1)\\ &\qquad\qquad=a\cos\omega+a'(\cos\omega\cos n\delta'+\sin\omega\sin n\delta')\\ &\qquad\qquad-na_1l\frac{\cos\phi_1}{v_1}(\sin\omega\cos n\delta_1-\cos\omega\sin n\delta_1)\\ &\text{and}\quad a_1(\sin\omega\cos n\delta_1-\cos\omega\sin n\delta_1)\\ &\qquad\qquad=a\sin\omega+a'(\sin\omega\cos n\delta'-\cos\omega\sin n\delta')\\ &\qquad\qquad+na_1l\frac{\cos\phi_1}{v_1}(\cos\omega\cos n\delta_1+\sin\omega\sin n\delta_1)\end{aligned}\right\},\quad(92)$$

which we can write in the form

$$\left.\begin{aligned} A\cos\omega &= B\sin\omega \\ B\cos\omega &= -A\sin\omega \\ C\cos\omega &= D\sin\omega \\ D\cos\omega &= -C\sin\omega \end{aligned}\right\}, \quad \text{......................} (92\text{A})$$

$$\left.\begin{aligned} \text{where } A &= -a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}\cos n\delta_1 + a\frac{v_0}{M}\frac{\cos\phi}{v} - a'\frac{v_0}{M}\frac{\cos\phi'}{v}\cos n\delta' \\ &\qquad + na_1\left(\frac{p}{v_0} - \frac{v_0}{M}l\frac{\sin^2\phi}{v^2}\right)\sin n\delta_1, \\ B &= a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1}\sin n\delta_1 + a'\frac{v_0}{M}\frac{\cos\phi'}{v}\sin n\delta' \\ &\qquad + na_1\left(\frac{p}{v_0} - \frac{v_0}{M}l\frac{\sin^2\phi}{v^2}\right)\cos n\delta_1, \\ C &= a_1\cos n\delta_1 - a - a'\cos n\delta' - na_1 l\frac{\cos\phi_1}{v_1}\sin n\delta_1, \\ D &= -a_1\sin n\delta_1 + a'\sin n\delta' - na_1 l\frac{\cos\phi_1}{v_1}\cos n\delta_1 \end{aligned}\right\}. \quad (93)$$

On eliminating ω from the four conditional equations (92A), we evidently find $A=0, \quad B=0, \quad C=0, \quad D=0$; ....(94) that the four conditional equations (92A) may be satisfied, the four coefficients or expressions A, B, C and D must, therefore, all vanish, that is, the conditional equations (92A) may be replaced by the simpler ones (94).

The conditional equations (94) evidently suffice for the determination of the four quantities a', a_1, $n\delta'$ and $n\delta_1$ sought; observe that these quantities are functions of the medium constants, the v's, the angle of incidence ϕ, the integrals l and p taken through the given film and n, the period of oscillation of the waves employed.

Assumption of Changes in Phase at Surface demanded by Surface Conditions.—It is evident from formulae (93) and (94) that, if we assume no changes in phase at the dividing surface, the surface conditions (87), etc., cannot be satisfied; for put $n\delta' = n\delta_1 = 0$ in the conditional equations (94), and we have

$$\begin{aligned} A &= -a_1\frac{v_0}{M}\frac{\cos\phi_1}{v_1} + a\frac{v_0}{M}\frac{\cos\phi}{v} - a'\frac{v_0}{M}\frac{\cos\phi'}{v} = 0, \\ B &= na_1\left(\frac{p}{v_0} - \frac{v_0}{M}l\frac{\sin^2\phi}{v^2}\right) = 0, \\ C &= a_1 - a - a' = 0, \\ D &= -na_1 l\frac{\cos\phi_1}{v_1} = 0, \end{aligned}$$

which evidently give $a_1 = 0, \quad a' = -a,$

that is, the given waves would not enter the second medium, but would be totally reflected at the dividing surface and for all angles of incidence ϕ, a result that is not confirmed by observation.

Case II. : Electric Oscillations in Plane of Incidence.—If we represent the electric force acting in the incident waves by the function

$$be^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},$$

its components P, Q, R will be here

$$P=b\sin\phi e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},$$
$$Q=b\cos\phi e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},$$
$$R=0$$

(cf. formulae (23)). As in case I., the component forces acting in the reflected and refracted electric waves will have to be written here, where the thickness of the transition film cannot be neglected, in the form

$$P'=b'\sin\phi' e^{in\left(t-\frac{y\sin\phi'+x\cos\phi'}{v}-\zeta'\right)},$$
$$Q'=-b'\cos\phi' e^{in\left(t-\frac{y\sin\phi'+x\cos\phi'}{v}-\zeta'\right)},$$
$$R'=0, \quad (v'=v)$$

(cf. formulae (24)) and

$$P_1=b_1\sin\phi_1 e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}-\zeta_1\right)},$$
$$Q_1=b_1\cos\phi_1 e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}-\zeta_1\right)},$$
$$R_1=0$$

respectively, where $n\zeta'$ and $n\zeta_1$ denote the changes in phase of the reflected and refracted waves respectively, as they leave the film. As in case I., such changes in phase must be assumed here in order that the given surface conditions may be satisfied (cf. below). b', b_1, $n\zeta'$ and $n\zeta_1$ are to be regarded here as unknown and to be determined from the surface conditions.

Replace the P's and Q's by their above expressions in formulae (1B), the particular form assumed here by our fundamental equations (1), and we find

$$\gamma=\frac{D}{v_0}bve^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},$$
$$\gamma'=\frac{D}{v_0}b've^{in\left(t-\frac{y\sin\phi'+x\cos\phi'}{v}-\zeta'\right)},$$
$$\gamma_1=\frac{D}{v_0}b_1v_1e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}-\zeta_1\right)}$$

(cf. p. 274). The fundamental equations (2B), which hold here, give

$$\alpha=\beta=\alpha'=\beta'=\alpha_1=\beta_1=0$$

and the familiar relations (12) between the velocities of propagation and the medium constants.

By the relations (12) we can also write the magnetic forces (γ, γ' and γ_1) in the form

$$\gamma=b\frac{v_0}{Mv}e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},$$

$$\gamma'=b'\frac{v_0}{Mv}e^{in\left(t-\frac{y\sin\phi'+x\cos\phi'}{v}-\zeta'\right)},$$

$$\gamma_1=b_1\frac{v_0}{Mv_1}e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}-\zeta_1\right)}*$$

(cf. formulae (27)).

For the given system of oscillations, namely electric ones in the plane of incidence and magnetic ones at right angles to that plane, our surface conditions (84) will evidently reduce to the following three:

$$\gamma_1=\gamma+\gamma'\frac{p}{v_0}\frac{dQ_1}{dt},$$

$$Q_1=Q+Q'-l\frac{M}{v_0}\frac{d\gamma_1}{dt}-qD_1\frac{dP_1}{dy},$$

$$D_1P_1=D(P+P')-p\frac{dQ_1}{dy}.$$

Replace here the component forces by their values on the given dividing surface, and we have

$$b_1\frac{v_0}{Mv_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\zeta_1\right)}=b\frac{v_0}{Mv}e^{in\left(t-\frac{y\sin\phi}{v}\right)}$$
$$+b'\frac{v_0}{Mv}e^{in\left(t-\frac{y\sin\phi'}{v}-\zeta'\right)}+inb_1\frac{p}{v_0}\cos\phi_1e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\zeta_1\right)},$$

$$b_1\cos\phi_1e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\zeta_1\right)}=b\cos\phi e^{in\left(t-\frac{y\sin\phi}{v}\right)}$$
$$-b'\cos\phi'e^{in\left(t-\frac{y\sin\phi'}{v}-\zeta'\right)}-inb_1l/v_1e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\zeta_1\right)}$$
$$+inb_1qD_1\frac{\sin^2\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\zeta_1\right)}$$

and $$D_1b_1\sin\phi_1e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\zeta_1\right)}=Db\sin\phi e^{in\left(t-\frac{y\sin\phi}{v}\right)}$$
$$+Db'\sin\phi'e^{in\left(t-\frac{y\sin\phi'}{v}-\zeta'\right)}+inb_1p\frac{\cos\phi_1\sin\phi_1}{v_1}e^{in\left(t-\frac{y\sin\phi_1}{v_1}-\zeta_1\right)}.$$

---

* Cf. footnote, p. 299.

That these conditions may be satisfied for all values of y and t, the familiar relation

$$\frac{\sin\phi_1}{v_1}=\frac{\sin\phi'}{v}=\frac{\sin\phi}{v}$$

must hold. By this and the relations (12) the conditions themselves will evidently reduce to the following *two*:

$$b_1\frac{v_0}{Mv_1}(\cos\omega_1+i\sin\omega_1)=b\frac{v_0}{Mv}(\cos\omega+i\sin\omega)$$
$$+b'\frac{v_0}{Mv}(\cos\omega'+i\sin\omega')+inb_1\frac{p}{v_0}\cos\phi_1(\cos\omega_1+i\sin\omega_1)$$

and
$$b_1\cos\phi_1(\cos\omega_1+i\sin\omega_1)=b\cos\phi(\cos\omega+i\sin\omega)$$
$$-b'\cos\phi'(\cos\omega'+i\sin\omega')-inb_1l/v_1(\cos\omega_1+i\sin\omega_1)$$
$$+inb_1qD_1v_1\frac{\sin^2\phi}{v^2}(\cos\omega_1+i\sin\omega_1),$$

where
$$\omega=n\left(t-\frac{y\sin\phi}{v}\right),$$
$$\omega'=n\left(t-\frac{y\sin\phi'}{v}-\zeta'\right)=\omega-n\zeta',$$
$$\omega_1=n\left(t-\frac{y\sin\phi_1}{v_1}-\zeta_1\right)=\omega-n\zeta_1.$$

On separating the real and imaginary parts in these two conditional equations, we have the following four in their place:

$$b_1\frac{v_0}{Mv_1}\cos\omega_1=b\frac{v_0}{Mv}\cos\omega+b'\frac{v_0}{Mv}\cos\omega'-nb_1\frac{p}{v_0}\cos\phi_1\sin\omega_1,$$
$$b_1\frac{v_0}{Mv_1}\sin\omega_1=b\frac{v_0}{Mv}\sin\omega+b'\frac{v_0}{Mv}\sin\omega'+nb_1\frac{p}{v_0}\cos\phi_1\cos\omega_1$$

and
$$b_1\cos\phi_1\cos\omega_1=b\cos\phi\cos\omega-b'\cos\phi'\cos\omega'$$
$$+nb_1\left(\frac{l}{v_1}-qD_1v_1\frac{\sin^2\phi}{v^2}\right)\sin\omega_1,$$
$$b_1\cos\phi_1\sin\omega_1=b\cos\phi\sin\omega-b'\cos\phi'\sin\omega'$$
$$-nb_1\left(\frac{l}{v_1}-qD_1v_1\frac{\sin^2\phi}{v^2}\right)\cos\omega_1.$$

These conditional equations are similar in form to those (91) for case I.; on expanding ω' and ω_1 as functions of ω and the ζ's and eliminating ω, we find, therefore, similar conditional equations to those (94) for the preceding case, namely

$$\left.\begin{aligned}
&b_1\frac{v_0}{Mv_1}\cos n\zeta_1 - b\frac{v_0}{Mv} - b'\frac{v_0}{Mv}\cos n\zeta' - nb_1\frac{p}{v_0}\cos\phi_1\sin n\zeta_1 = 0,\\
&\quad -b_1\frac{v_0}{Mv_1}\sin n\zeta_1 + b'\frac{v_0}{Mv}\sin n\zeta' - nb_1\frac{p}{v_0}\cos\phi_1\cos n\zeta_1 = 0,\\
&b_1\cos\phi_1\cos n\zeta_1 - b\cos\phi + b'\cos\phi'\cos n\zeta'\\
&\qquad\qquad + nb_1\left(\frac{l}{v_1} - qD_1v_1\frac{\sin^2\phi}{v^2}\right)\sin n\zeta_1 = 0,\\
&\quad -b_1\cos\phi_1\sin n\zeta_1 - b'\cos\phi'\sin n\zeta'\\
&\qquad\qquad + nb_1\left(\frac{l}{v_1} - qD_1v_1\frac{\sin^2\phi}{v^2}\right)\cos n\zeta_1 = 0.
\end{aligned}\right\}\quad\ldots(95)$$

These conditional equations evidently suffice for the determination of the four quantities b', b_1, $n\zeta'$ and $n\zeta_1$ sought; observe that these quantities are functions of the medium constants, the v's, the angle of incidence ϕ, the integrals l and q, and n, the period of oscillation of the waves employed. On the assumption of no changes in phase at the dividing surface, the conditional equations (95) would assume the form

$$\frac{b_1}{v_1} - \frac{b}{v} - \frac{b'}{v} = 0,$$
$$b_1 = 0, \quad \text{hence } b' = -b,$$
$$b_1\cos\phi_1 - b\cos\phi + b'\cos\phi = 0,$$
$$b_1 = 0, \quad \text{hence } b' = b;$$

that is, on the given assumption the above conditions could not be satisfied.

General Problem: Changes in Phase of Component Oscillations; Reflected and Refracted Waves Elliptically Polarized.—A comparison of the conditional equations (95) for the changes in amplitude and phase for electric oscillations taking place in the plane of incidence with those (94) for the changes in amplitude and phase for electric oscillations taking place at right angles to that plane shows that incident oscillations will undergo different changes both in amplitude and phase, according as they are taking place in or at right angles to the plane of incidence. If the incident oscillations make an arbitrary angle θ with the plane of incidence, the general case (cf. pp. 276-277), their component oscillations in and at right angles to that plane will, therefore, undergo different changes in amplitude and phase both upon reflection and upon refraction, and the resultant reflected and refracted oscillations will thus be both elliptically polarized, as confirmed by exacter observation (cf. p. 298). For the actual determination of the changes in phase $n\delta'$, $n\delta_1$, $n\zeta'$ and $n\zeta_1$, see Exs. 8-11 at end of chapter.

Reflection and Refraction of Purely Spherical Waves.—Before we proceed to the examination of the primary and secondary waves upon reflection and refraction, let us briefly consider the behaviour of purely spherical (electromagnetic) waves on the dividing surface of two insulators. For this purpose we choose the same system of coordinates with respect to the reflecting surface and any incident wave as that employed on pp. 266-267 (cf. also Fig. 32) and examine the two particular cases treated there. If we denote the distance of the source of disturbance P (in the xy-plane) from the origin O of our system of coordinates, the point, where any incident wave strikes the reflecting surface, by $\bar{r}$, as indicated in the annexed figure, we can then evidently

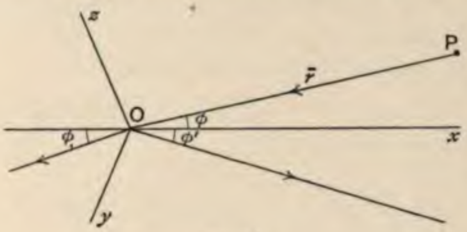

FIG. 34.

represent the incident, reflected and refracted (electric) waves referred to the given system of coordinates by the three functions

$$\frac{a}{\bar{r}-r}e^{in\left(t-\frac{\bar{r}-r}{v}\right)},$$

$$\frac{a'}{\bar{r}+r'}e^{in\left[t-\left(\frac{\bar{r}}{v}+\frac{r'}{v'}\right)\right]},$$

$$\frac{a_1}{\bar{r}+r_1}e^{in\left[t-\left(\frac{\bar{r}}{v}+\frac{r_1}{v_1}\right)\right]}$$

respectively (cf. formulae (6)), where r, r' and r_1 denote the distances from the origin O of our coordinates of the incident, reflected and refracted waves respectively (cf. figure). On replacing here the r's by their values (7) in terms of the x's, y's and ϕ's, we can write these functions for the three waves in the form

$$\left.\begin{array}{l} \dfrac{a}{\bar{r}+y\sin\phi-x\cos\phi}e^{in\left(t-\frac{\bar{r}+y\sin\phi-x\cos\phi}{v}\right)}, \\ \dfrac{a'}{\bar{r}+y\sin\phi'+x\cos\phi'}e^{in\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi'+x\cos\phi'}{v'}\right)\right]}, \\ \dfrac{a_1}{\bar{r}+y\sin\phi_1-x\cos\phi_1}e^{in\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)\right]} \end{array}\right\} \text{.........(96)}$$

(cf. Fig. 34).

We examine here, as above, the two particular cases, where the electric oscillations are taking place at right angles to and in the plane of incidence.

Case I.: Electric Oscillations at ⊥ to Plane of Incidence.—Here

$$P = Q = 0,$$

and hence

$$P' = Q' = P_1 = Q_1 = 0,$$

whereas the components R, R' and R_1 parallel to the z-axis of the electric forces acting in the incident, reflected and refracted waves may be represented by the three functions (96).

To determine the magnetic forces that accompany the electric ones (96), we employ formulae (2A), which evidently hold here; we replace there the R's by their values (96), and we have

$$\left.\begin{array}{l}
\dfrac{M}{v_0}\dfrac{d\alpha}{dt} = -\dfrac{ain}{\bar{r} + (y\sin\phi - x\cos\phi)}\dfrac{\sin\phi}{v}e^{in\left[t - \frac{\bar{r} + (y\sin\phi - x\cos\phi)}{v}\right]} \\
\text{or integrated} \\
\qquad \alpha = -\dfrac{v_0}{M}\dfrac{a}{\bar{r} + (y\sin\phi - x\cos\phi)}\dfrac{\sin\phi}{v}e^{in\left[t - \frac{\bar{r} + (y\sin\phi - x\cos\phi)}{v}\right]}, \\
\text{similarly} \\
\qquad \alpha' = -\dfrac{v_0}{M}\dfrac{a'}{\bar{r} + (y\sin\phi' + x\cos\phi')}\dfrac{\sin\phi'}{v'}e^{in\left[t - \left(\frac{\bar{r}}{v} + \frac{y\sin\phi' + x\cos\phi'}{v'}\right)\right]},^{*} \\
\qquad \alpha_1 = -\dfrac{v_0}{M}\dfrac{a_1}{\bar{r} + (y\sin\phi_1 - x\cos\phi_1)}\dfrac{\sin\phi_1}{v_1}e^{in\left[t - \left(\frac{\bar{r}}{v} + \frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)\right]} \\
\text{and} \\
\dfrac{M}{v_0}\dfrac{d\beta}{dt} = -\dfrac{ain}{\bar{r} + (y\sin\phi - x\cos\phi)}\dfrac{\cos\phi}{v}e^{in\left[t - \frac{\bar{r} + (y\sin\phi - x\cos\phi)}{v}\right]} \\
\text{or integrated} \\
\qquad \beta = -\dfrac{v_0}{M}\dfrac{a}{\bar{r} + (y\sin\phi - x\cos\phi)}\dfrac{\cos\phi}{v}e^{in\left[t - \frac{\bar{r} + (y\sin\phi - x\cos\phi)}{v}\right]}, \\
\text{similarly} \\
\qquad \beta' = +\dfrac{v_0}{M}\dfrac{a'}{\bar{r} + (y\sin\phi' + x\cos\phi')}\dfrac{\cos\phi'}{v'}e^{in\left[t - \left(\frac{\bar{r}}{v} + \frac{y\sin\phi' + x\cos\phi'}{v'}\right)\right]},^{*} \\
\qquad \beta_1 = -\dfrac{v_0}{M}\dfrac{a_1}{+(y\sin\phi_1 - x\cos\phi_1)}\dfrac{\cos\phi_1}{v_1}e^{in\left[t - \left(\frac{\bar{r}}{v} + \frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)\right]}.
\end{array}\right\} (97)$$

We observe that by the differentiations with regard to x and y (z) the amplitudes $\frac{a}{\bar{r} - r'}$, $\frac{a'}{\bar{r} + r'}$, and $\frac{a_1}{\bar{r} + r_1}$ can evidently be regarded as constant.

On replacing the R's, α's and β's by the above values, we find that formulae (1A), which must evidently hold here, will be fulfilled, provided the familiar relations (12) remain valid; these relations give $v' = v$ (cf. formula (12A).

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* Cf. foot-note, p. 299.

We, next, replace the R's, α's and β's by their values on the dividing surface $x = 0$ in the surface conditions (of the first order of approximation) (3A′), and we have

$$\left.\begin{aligned}
&\frac{a_1}{\bar{r}}\, e^{in\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]} = \frac{a}{\bar{r}}\, e^{in\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)} + \frac{a'}{\bar{r}}\, e^{in\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right)},\\
&\frac{a_1}{\bar{r}}\,\frac{\sin\phi_1}{v_1}\, e^{in\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]}\\
&\qquad\qquad = \frac{a}{\bar{r}}\,\frac{\sin\phi}{v}\, e^{in\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)} + \frac{a'}{\bar{r}}\,\frac{\sin\phi'}{v}\, e^{in\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right)}\\
&\text{and}\\
&\frac{a_1}{\bar{r}}\,\frac{\cos\phi_1}{v_1}\, e^{in\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]}\\
&\qquad\qquad = \frac{a}{\bar{r}}\,\frac{\cos\phi}{v}\, e^{in\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)} - \frac{a}{\bar{r}}\,\frac{\cos\phi'}{v}\, e^{in\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right)}
\end{aligned}\right\}, \quad\ldots(98)$$

where we have put $v' = v$ (cf. above). These conditions must now hold at all points on the given dividing surface and for all values of t; different points on the dividing surface correspond to different origins of systems of coordinates y, z in that surface; for this reason we cannot put $y = 0$, except in the expressions for the amplitudes (cf. also p. 317), on the given surface. That the conditions (98) may be satisfied for all values of y, z and t, we must evidently put

$$\left.\begin{aligned}
&\frac{\sin\phi_1}{v_1} = \frac{\sin\phi}{v} = \frac{\sin\phi'}{v},\\
\text{hence}\qquad &\phi' = \phi.
\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(99)$$

It follows from these relations that purely spherical (electromagnetic) waves will obey the same laws of reflection and refraction as plane waves do.

By the relations (99) the surface conditions (98) will reduce to those that hold for plane waves (cf. p. 272); the **quantities** a' and a_1 will, therefore, be determined by the same formulae as those (18) and (19) that hold for the reflected and refracted amplitudes of plane-waves, whereas the **amplitudes** of the given reflected and refracted (spherical) waves will evidently be given by the expressions

$$\frac{a'}{\bar{r}+r'} = \frac{a'}{\bar{r}+y\sin\phi'+x\cos\phi'}$$

and

$$\frac{a_1}{\bar{r}+r_1} = \frac{a_1}{\bar{r}+y\sin\phi_1-x\cos\phi_1},$$

where a' and a_1 are to be replaced by their values from formulae (18) and (19).

Case II.: Electric Oscillations in Plane of Incidence.—The treatment of this case is similar on the one hand to the preceding one and on the other to case II. of plane waves. We find the same familiar relations (99) for the angles of reflection and refraction as in the above case, whereas the amplitudes of the reflected and refracted waves are given by the expressions

$$\frac{b'}{\bar{r}+r'}=\frac{b'}{\bar{r}+y\sin\phi'+x\cos\phi'},$$

$$\frac{b_1}{\bar{r}+r_1}=\frac{b_1}{\bar{r}+y\sin\phi_1-x\cos\phi_1},$$

where b' and b_1 are to be replaced by their values from formulae (29) and (30).

Reflection and Refraction of the Primary and Secondary Waves.—Lastly, let us examine the behaviour of the primary and secondary waves on the dividing surface of two media (insulators); for this purpose we shall choose the most general type of such waves, those namely of Problem 3, Chapter II. These waves or, more strictly, the component forces acting in the same, which we shall consider here instead of the component moments (cf. p. 266), are evidently represented by the expressions

$$\begin{aligned}P=&\frac{4\pi n^2}{v^2r}[a_1(\beta^2+\gamma^2)-\alpha(a_2\beta+a_3\gamma)]\sin n\left(t-\frac{r}{v}\right)\\&+\frac{4\pi n}{vr^2}[2a_1-3a_1(\beta^2+\gamma^2)+3\alpha(a_2\beta+a_3\gamma)]\cos n\left(t-\frac{r}{v}\right)\\&+\frac{4\pi}{r^3}[2a_1-3a_1(\beta^2+\gamma^2)+3\alpha(a_2\beta+a_3\gamma)]\sin n\left(t-\frac{r}{v}\right),\end{aligned}$$

with analogous expressions for Q and R, and

$$\alpha^*=\frac{4\pi nv_0}{Mv^2}(a_2\gamma-a_3\beta)\left(\frac{n}{vr}\sin\omega-\frac{1}{r^2}\cos\omega\right),$$

with analogous expressions for β^* and γ^* (cf. formulae (3) and (4), I. and (43) and (44), II.), where, however,

$$n=\frac{2\pi v}{\lambda}=\frac{2\pi}{T}\quad\text{.............................}(100)$$

(cf. p. 267).

Let us denote the distance of the source of disturbance $\bar{O}$ of the given waves from the point O on the dividing surface, where any wave from that source strikes the same, by $\bar{r}$, as indicated in the annexed figure; let $\bar{x}$, $\bar{y}$, $\bar{z}$ denote the rectangular coordinates, to which the given system of waves is referred, their source $\bar{O}$ being taken as origin of those coordinates; the wave advancing along any vector $\bar{r}$ will then be characterized by the direction-cosines $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ of that vector with respect to those axes. For simplicity, let the dividing surface be taken parallel to the $\bar{y}\bar{z}$ plane. We choose the point O on this surface, where any wave from the source $\bar{O}$ strikes the same, as origin of a second system of rectangular coordinates x, y, z; let its x-axis coincide with the normal to the given surface and be directed towards the source $\bar{O}$ and its y and z axes be so chosen in that surface that its xy-plane coincide with the plane of incidence of the given wave.

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*These component magnetic forces α, β, γ are not to be confounded with the direction-cosines α, β, γ ($\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$).

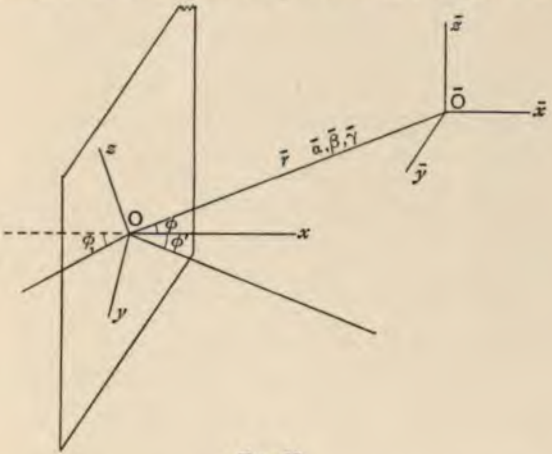

FIG. 35.

Any incident wave of the given system, referred to the system of coordinates x, y, z, will then be represented by the component forces

$$\left.\begin{aligned} P = & \frac{4\pi n^2}{v^2(\bar{r}-r)}[a_1(\bar{\beta}^2+\bar{\gamma}^2) - \bar{\alpha}(a_2\bar{\beta}+a_3\bar{\gamma})] \sin n\left(t-\frac{\bar{r}-r}{v}\right) \\ & + \frac{4\pi n}{v(\bar{r}-r)^2}[2a_1 - 3a_1(\bar{\beta}^2+\bar{\gamma}^2) + 3\bar{\alpha}(a_2\bar{\beta}+a_3\bar{\gamma})] \cos n\left(t-\frac{\bar{r}-r}{v}\right) \\ & + \frac{4\pi}{(\bar{r}-r)^3}[2a_1 - 3a_1(\bar{\beta}^2+\bar{\gamma}^2) + 3\bar{\alpha}(a_2\bar{\beta}+a_3\bar{\gamma})] \sin n\left(t-\frac{\bar{r}-r}{v}\right) \end{aligned}\right\}, \quad (101)$$

with analogous expressions for Q and R, and

$$\begin{aligned} \alpha = \frac{4\pi n v_0}{M v^2}(a_2\bar{\gamma} - a_3\bar{\beta})\Bigg[& \frac{n}{v(\bar{r}-r)} \sin n\left(t-\frac{\bar{r}-r}{v}\right) \\ & - \frac{1}{(\bar{r}-r)^2} \cos n\left(t-\frac{\bar{r}-r}{v}\right)\Bigg], \quad \ldots \; (102) \end{aligned}$$

with analogous expressions for β and γ, where r denotes the distance of the incident wave from the point O on the given dividing surface, where that wave strikes the same. It is easy to show that these expressions satisfy our fundamental differential equations (1) and (2) (cf. Ex. 12 at end of chapter).

The expressions in $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ in the coefficients of the different terms of the above expressions for the component forces will evidently assume given values along any given vector $\bar{r}$. In formulae (101) and (102) these expressions are now referred to the coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$; if we refer them to the coordinates x, y, z with origin at O on the given dividing surface, then

$$\left.\begin{aligned} \cos(\bar{r}, x) &= \cos(\phi+\pi) = -\cos\phi, \\ \cos(\bar{r}, y) &= \cos(\pi/2-\phi) = \sin\phi, \\ \cos(\bar{r}, z) &= 0, \end{aligned}\right\} \quad \text{..................(103)}$$

where ϕ denotes the angle of incidence of the given wave (cf. Fig. 35), and the expressions themselves can evidently be written

$$\left.\begin{aligned} a_1(\bar{\beta}^2+\bar{\gamma}^2) - \bar{\alpha}(a_2\bar{\beta}+a_3\bar{\gamma}) &= (b_1\sin\phi+b_2\cos\phi)\sin\phi = A_1, \\ a_2(\bar{\alpha}^2+\bar{\gamma}^2) - \bar{\beta}(a_1\bar{\alpha}+a_3\bar{\gamma}) &= (b_1\sin\phi+b_2\cos\phi)\cos\phi = A_2, \\ a_3(\bar{\alpha}^2+\bar{\beta}^2) - \bar{\gamma}(a_1\bar{\alpha}+a_2\bar{\beta}) &= b_3 = A_3, \end{aligned}\right\} \text{...(104)}$$

$$\left.\begin{aligned} &2a_1 - 3a_1(\bar{\beta}^2+\bar{\gamma}^2) + 3\bar{\alpha}(a_2\bar{\beta}+a_3\bar{\gamma}) \\ &\qquad = 2b_1 - 3(b_1\sin\phi+b_2\cos\phi)\sin\phi = B_1, \\ &2a_2 - 3a_2(\bar{\alpha}^2+\bar{\gamma}^2) + 3\bar{\beta}(a_1\bar{\alpha}+a_3\bar{\gamma}) \\ &\qquad = 2b_2 - 3(b_1\sin\phi+b_2\cos\phi)\cos\phi = B_2, \\ &2a_3 - 3a_3(\bar{\alpha}^2+\bar{\beta}^2) + 3\bar{\gamma}(a_1\bar{\alpha}+a_2\bar{\beta}) = -b_3 = B_3 \end{aligned}\right\} \text{...(105)}$$

and

$$\left.\begin{aligned} a_2\bar{\gamma} - a_3\bar{\beta} &= -b_3\sin\phi, \\ a_3\bar{\alpha} - a_1\bar{\gamma} &= -b_3\cos\phi, \\ a_1\bar{\beta} - a_2\bar{\alpha} &= b_1\sin\phi + b_2\cos\phi, \end{aligned}\right\} \text{..................(106)}$$

where b_1, b_2, b_3 denote the components of the resultant amplitude coefficient $a=\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}$ along the x, y, z axes respectively; for the values of b_1, b_2, b_3 in terms of a_1, a_2, a_3 and $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ see Ex. 13 at end of chapter. For a confirmation of formulae (104) to (106) see Ex. 14.

Expressions for Incident Waves.—Referred to the system of coordinates x, y, z, we can, therefore, write the above formulae (101) and (102) for the component forces acting in the incident waves in the form

$$\left.\begin{aligned}
P&=\frac{4\pi n^2A_1}{v^2(\bar{r}-r)}\sin\omega+\frac{4\pi nB_1}{v(\bar{r}-r)^2}\cos\omega+\frac{4\pi B_1}{(\bar{r}-r)^3}\sin\omega,\\
Q&=\frac{4\pi n^2A_2}{v^2(\bar{r}-r)}\sin\omega+\frac{4\pi nB_2}{v(\bar{r}-r)^2}\cos\omega+\frac{4\pi B_2}{(\bar{r}-r)^3}\sin\omega,\\
R&=\frac{4\pi n^2A_3}{v^2(\bar{r}-r)}\sin\omega+\frac{4\pi nB_3}{v(\bar{r}-r)^2}\cos\omega+\frac{4\pi B_3}{(\bar{r}-r)^3}\sin\omega
\end{aligned}\right\}\dots\dots(107)$$

and

$$\left.\begin{aligned}
\alpha&=-\frac{4\pi n^2v_0b_3}{Mv^3(\bar{r}-r)}\sin\phi\sin\omega+\frac{4\pi nv_0b_3}{Mv^2(\bar{r}-r)^2}\sin\phi\cos\omega,\\
\beta&=-\frac{4\pi n^2v_0b_3}{Mv^3(\bar{r}-r)}\cos\phi\sin\omega+\frac{4\pi nv_0b_3}{Mv^2(\bar{r}-r)^2}\cos\phi\cos\omega,\\
\gamma&=\quad\frac{4\pi n^2v_0}{Mv^3(\bar{r}-r)}(b_1\sin\phi+b_2\cos\phi)\sin\omega\\
&\qquad\qquad-\frac{4\pi nv_0}{Mv^2(\bar{r}-r)^2}(b_1\sin\phi+b_2\cos\phi)\cos\omega,
\end{aligned}\right\}\dots\dots(108)$$

where $$\omega=n\left(t-\frac{\bar{r}-r}{v}\right)=n\left(t-\frac{\bar{r}+y\sin\phi-x\cos\phi}{v}\right)\dots\dots\dots\dots(109)$$

(cf. formulae (7)).

Expressions for Reflected Waves.—It is evident that for any given incident wave, characterized by given $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, only the quantities b_1, b_2, b_3 (a_1, a_2, a_3) in the expressions (104)-(106) will undergo changes upon reflection and refraction (cf. Ex. 13 at end of chapter); if b_1', b_2', b_3' denote the values assumed by the components b_1, b_2, b_3 of the resultant amplitude coefficient $b=\sqrt{b_1^2+b_2^2+b_3^2}$ after reflection, the component forces acting in the reflected wave, to which any incident wave (107) and (108) gives rise, will evidently be given by the expressions

$$\left.\begin{aligned}
P'&=\frac{4\pi n^2A_1'}{v^2(\bar{r}+r')}\sin\omega'+\frac{4\pi nB_1'}{v(\bar{r}+r')^2}\cos\omega'+\frac{4\pi B_1'}{(\bar{r}+r')^3}\sin\omega',\\
Q'&=\frac{4\pi n^2A_2'}{v^2(\bar{r}+r')}\sin\omega'+\frac{4\pi nB_2'}{v(\bar{r}+r')^2}\cos\omega'+\frac{4\pi B_2'}{(\bar{r}+r')^3}\sin\omega',\\
R'&=\frac{4\pi n^2A_3'}{v^2(\bar{r}+r')}\sin\omega'+\frac{4\pi nB_3'}{v(\bar{r}+r')^2}\cos\omega'+\frac{4\pi B_3'}{(\bar{r}+r')^3}\sin\omega'
\end{aligned}\right\}\dots\dots\dots(110)$$

and
$$\left.\begin{aligned}
\alpha'&=-\frac{4\pi n^2v_0b_3'}{Mv^3(\bar{r}+r')}\sin\phi\sin\omega'+\frac{4\pi nv_0b_3'}{Mv^2(\bar{r}+r')^2}\sin\phi\cos\omega',\\
\beta'&=-\frac{4\pi n^2v_0b_3'}{Mv^3(\bar{r}+r')}\cos\phi\sin\omega'+\frac{4\pi nv_0b_3'}{Mv^2(\bar{r}+r')^2}\cos\phi\cos\omega',\\
\gamma'&=\frac{4\pi n^2v_0}{Mv^3(\bar{r}+r')}(b_1'\sin\phi-b_2'\cos\phi)\sin\omega'\\
&\qquad\qquad-\frac{4\pi nv_0}{Mv^2(\bar{r}+r_1)^2}(b_1'\sin\phi-b_2'\cos\phi)\cos\omega'
\end{aligned}\right\}\dots\dots(111)$$

where we have put $v' = v$, and where, by formulae similar to (103),

$$\left.\begin{aligned}
A_1' &= (b_1'\sin\phi - b_2'\cos\phi)\sin\phi,\\
A_2' &= (b_1'\sin\phi - b_2'\cos\phi)\cos\phi,\\
A_3' &= b_3',\\
B_1' &= 2b_1' - 3(b_1'\sin\phi - b_2'\cos\phi)\sin\phi,\\
B_2' &= 2b_2' - 3(b_1'\sin\phi - b_2'\cos\phi)\cos\phi,\\
B_3' &= -b_3'
\end{aligned}\right\} \tag{112}$$

and

$$\omega' = n\left[t - \left(\frac{\bar{r}+r'}{v}\right)\right] = n\left[t - \left(\frac{\bar{r}}{v} + \frac{y\sin\phi' + x\cos\phi'}{v}\right)\right] \tag{113}$$

(cf. formulae (7)), r' denoting the distance of the given (reflected) wave from the origin O and ϕ' the angle of reflection.

Expressions for Refracted Waves.—Similarly, the component forces acting in the refracted wave, to which any incident wave (107) and (108) gives rise, will evidently be given by the expressions

$$\left.\begin{aligned}
P_1 &= \frac{4\pi n^2 A_{11}}{v_1^2(\bar{r}+r_1)}\sin\omega_1 + \frac{4\pi n B_{11}}{v_1(\bar{r}+r_1)^2}\cos\omega_1 + \frac{4\pi B_{11}}{(\bar{r}+r_1)^3}\sin\omega_1,\\
Q_1 &= \frac{4\pi n^2 A_{21}}{v_1^2(\bar{r}+r_1)}\sin\omega_1 + \frac{4\pi n B_{21}}{v_1(\bar{r}+r_1)^2}\cos\omega_1 + \frac{4\pi B_{21}}{(\bar{r}+r_1)^3}\sin\omega_1,\\
R_1 &= \frac{4\pi n^2 A_{31}}{v_1^2(\bar{r}+r_1)}\sin\omega_1 + \frac{4\pi n B_{31}}{v_1(\bar{r}+r_1)^2}\cos\omega_1 + \frac{4\pi B_{31}}{(\bar{r}+r_1)^3}\sin\omega_1
\end{aligned}\right\} \tag{114}$$

and

$$\left.\begin{aligned}
\alpha_1 &= -\frac{4\pi n^2 v_0 b_{31}}{M v_1^3(\bar{r}+r_1)}\sin\phi\sin\omega_1 + \frac{4\pi n v_0 b_{31}}{M v_1^2(\bar{r}+r_1)^2}\sin\phi\cos\omega_1,\\
\beta_1 &= -\frac{4\pi n^2 v_0 b_{31}}{M v_1^3(\bar{r}+r_1)}\cos\phi\sin\omega_1 + \frac{4\pi n v_0 b_{31}}{M v_1^2(\bar{r}+r_1)^2}\cos\phi\cos\omega_1,\\
\gamma_1 &= \frac{4\pi n^2 v_0}{M v_1^3(\bar{r}+r_1)}(b_{11}\sin\phi + b_{21}\cos\phi)\sin\omega_1\\
&\qquad\qquad - \frac{4\pi n v_0}{M v_1^2(\bar{r}+r_1)^2}(b_{11}\sin\phi + b_{21}\cos\phi)\cos\omega_1,
\end{aligned}\right\} \tag*{(115)*}$$

where, by formulae similar to (103),

$$\left.\begin{aligned}
A_{11} &= (b_{11}\sin\phi + b_{21}\cos\phi)\sin\phi,\\
A_{21} &= (b_{11}\sin\phi + b_{21}\cos\phi)\cos\phi,\\
A_{31} &= b_{31},\\
B_{11} &= 2b_{11} - 3(b_{11}\sin\phi + b_{21}\cos\phi)\sin\phi,\\
B_{21} &= 2b_{21} - 3(b_{11}\sin\phi + b_{21}\cos\phi)\cos\phi,\\
B_{31} &= -b_{31}
\end{aligned}\right\} \tag{116}$$

and

$$\omega_1 = n\left[t - \left(\frac{\bar{r}}{v} + \frac{r_1}{v_1}\right)\right] = n\left[t - \left(\frac{\bar{r}}{v} + \frac{y\sin\phi_1 - x\cos\phi_1}{v_1}\right)\right] \tag{117}$$

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[*] Cf. foot-note, p. 299.

(cf. formulae (7)), r_1 denoting the distance of the given (refracted) wave from the origin O, ϕ_1 the angle of refraction and b_{11}, b_{21}, b_{31} the values assumed by the components b_1, b_2, b_3 of the resultant amplitude coefficient $b=\sqrt{b_1^2+b_2^2+b_3^2}$ after refraction.

The Surface Conditions of First Order of Approximation.—Let us now examine the surface conditions for the above system of waves. Since the expressions for the component forces acting in these waves satisfy our differential equations (1) and (2) (cf. Ex. 12), the surface conditions will be those already established; if we employ here the surface conditions of the first order of approximation, where the thickness of the dividing film may be neglected, the conditions in question are the familiar ones

$$\left.\begin{aligned} D_1P_1=DP+DP', \quad Q_1=Q+Q',& \\ R_1=R+R'& \\ \text{and} \qquad \alpha_1=\alpha+\alpha_1, \quad \beta_1=\beta+\beta', \quad \gamma_1=\gamma+\gamma',& \end{aligned}\right\} \text{..............(118)}$$

where we have put $M_1=M$ (cf. foot-note, p. 299).

Replace the component forces by their values on the given dividing surface, $x=0$, in the surface conditions (118) and we have, on rejecting terms of the order of magnitude of the thickness of the film (cf. above and p. 320),

$$\begin{aligned} D_1\Bigg\{&\frac{n^2A_{11}}{v_1^2\bar{r}}\sin n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]+\frac{nB_{11}}{v_1\bar{r}^2}\cos n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right] \\ &+\frac{B_{11}}{\bar{r}^3}\sin n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]\Bigg\}=D\left\{\frac{n^2A_1}{v^2\bar{r}}\sin n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)\right. \\ &+\frac{nB_1}{v\bar{r}^2}\cos n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)+\frac{B_1}{\bar{r}^3}\sin n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right) \\ &+\frac{n^2A_1'}{v^2\bar{r}}\sin n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right)+\frac{nB_1'}{v\bar{r}^2}\cos n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right) \\ &\qquad\qquad\left.+\frac{B_1'}{\bar{r}^3}\sin n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right)\right\}, \end{aligned}$$

$$\begin{aligned} &\frac{n^2A_{21}}{v_1^2\bar{r}}\sin n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]+\frac{nB_{21}}{v_1\bar{r}^2}\cos n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right] \\ &+\frac{B_{21}}{\bar{r}^3}\sin n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]=\frac{n^2A_2}{v^2\bar{r}}\sin n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right) \\ &+\frac{nB_2}{v\bar{r}^2}\cos n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)+\frac{B_2}{\bar{r}^3}\sin n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right) \\ &+\frac{n^2A_2'}{v^2\bar{r}}\sin n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right)+\frac{nB_2'}{v\bar{r}^2}\cos n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right) \\ &\qquad\qquad+\frac{B_2'}{\bar{r}^3}\sin n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right). \end{aligned}$$

with an analogous condition for the R's, and

$$-\frac{nb_{31}}{v_1^3\bar{r}}\sin n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]+\frac{b_{31}}{v_1^2\bar{r}^2}\cos n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]$$

$$=-\frac{nb_3}{v^3\bar{r}}\sin n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)+\frac{b_3}{v^2\bar{r}^2}\cos n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)$$

$$-\frac{nb_3'}{v^3\bar{r}}\sin n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right)+\frac{b_3'}{v^2\bar{r}^2}\cos n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right),$$

with an analogous condition for the β's, and

$$\frac{n(b_{11}\sin\phi+b_{21}\cos\phi)}{v_1^3\bar{r}}\sin n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]$$

$$-\frac{b_{11}\sin\phi+b_{21}\cos\phi}{v_1^2\bar{r}^2}\cos n\left[t-\left(\frac{\bar{r}}{v}+\frac{y\sin\phi_1}{v_1}\right)\right]$$

$$=\frac{n(b_1\sin\phi+b_2\cos\phi)}{v^3\bar{r}}\sin n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)$$

$$-\frac{b_1\sin\phi+b_2\cos\phi}{v^2\bar{r}^2}\cos n\left(t-\frac{\bar{r}+y\sin\phi}{v}\right)$$

$$+\frac{n(b_1'\sin\phi-b_2'\cos\phi)}{v^3\bar{r}}\sin n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right)$$

$$-\frac{b_1'\sin\phi-b_2'\cos\phi}{v^2\bar{r}^2}\cos n\left(t-\frac{\bar{r}+y\sin\phi'}{v}\right).$$

Laws of Reflection and Refraction and Determination of Component Amplitudes and Changes in Phase from Surface Conditions: Total Reflection of Incident Waves.—The above six surface conditions must now hold for all values of t and at all points on the given dividing surface; this is evidently only possible, when the familiar relation (12), namely

$$\frac{\sin\phi_1}{v_1}=\frac{\sin\phi'}{v}=\frac{\sin\phi}{v}, \quad\text{........................}(119)$$

holds between the ϕ's and the v's, and also when the coefficients of the terms with one and the same factor $\sin\omega_1=\sin\omega'=\sin\omega$ or $\cos\omega_1=\cos\omega'=\cos\omega$—by the relation (119) the three ω's then assume one and the same form—all vanish. The vanishing of these coefficients leads now to the following ten conditional equations in place of the above six surface conditions:

$$D_1\left(\frac{n^2A_{11}}{v_1^2}+\frac{B_{11}}{\bar{r}^2}\right)=D\left(\frac{n^2A_1}{v^2}+\frac{B_1}{\bar{r}^2}+\frac{n^2A_1'}{v^2}+\frac{B_1'}{\bar{r}^2}\right),$$

$$D_1\frac{B_{11}}{v_1}=D\left(\frac{B_1}{v}+\frac{B_1'}{v}\right),$$

$$\frac{n^2A_{21}}{v_1^2}+\frac{B_{21}}{\bar{r}^2}=\frac{n^2A_2}{v^2}+\frac{B_2}{\bar{r}^2}+\frac{n^2A_2'}{v^2}+\frac{B_2'}{\bar{r}^2},$$

$$\frac{B_{21}}{v_1}=\frac{B_2}{v}+\frac{B_2'}{v},$$

$$\frac{n^2A_{31}}{v_1^{\,2}}+\frac{B_{31}}{r^2}=\frac{n^2A_3}{v^2}+\frac{B_3}{r^2}+\frac{n^2A_3'}{v^2}+\frac{B_3'}{r^2},$$

$$\frac{B_{31}}{v_1}=\frac{B_3}{v}+\frac{B_3'}{v},$$

$$\frac{b_{31}}{v_1^{\,3}}=\frac{b_3}{v^3}+\frac{b_3'}{v^3},$$

$$\frac{b_{31}}{v_1^{\,2}}=\frac{b_3}{v^2}+\frac{b_3'}{v^2},$$

$$\frac{b_{11}\sin\phi+b_{21}\cos\phi}{v_1^{\,3}}=\frac{b_1\sin\phi+b_2\cos\phi}{v^3}+\frac{b_1'\sin\phi-b_2'\cos\phi}{v^3},$$

$$\frac{b_{11}\sin\phi+b_{21}\cos\phi}{v_1^{\,2}}=\frac{b_1\sin\phi+b_2\cos\phi}{v^2}+\frac{b_1'\sin\phi-b_2'\cos\phi}{v^2}.$$

The 5th, 6th, 7th and 8th of these conditional equations can evidently be satisfied only when

$$b_{31}=0 \text{ and } b_3'=-b_3,$$

and the last two only when

$$b_{11}\sin\phi+b_{21}\cos\phi=0$$

and $$b_1\sin\phi+b_2\cos\phi=-b_1'\sin\phi+b_2'\cos\phi.$$

Replace the b's by these values in formulae (112) and (116), and we have

$$A_1'=-(b_1\sin\phi+b_2\cos\phi)\sin\phi,$$
$$A_2'=-(b_1\sin\phi+b_2\cos\phi)\cos\phi,$$
$$A_3'=-b_3,$$
$$B_1'=2b_1'+3(b_1\sin\phi+b_2\cos\phi)\sin\phi,$$
$$B_2'=2b_2'+3(b_1\sin\phi+b_2\cos\phi)\cos\phi,$$
$$B_3'=b_3$$

and $$A_{11}=A_{21}=A_{31}=0,$$

$$B_{11}=2b_{11},\quad B_{21}=2b_{21},\quad B_{31}=0.$$

Lastly, replace the A's and B's by these values in the first four of the above conditional equations, and we have

$$D_1b_{11}=D(b_1+b_1'),$$

$$\frac{D_1}{v_1}b_{11}=\frac{D}{v}(b_1+b_1'),$$

$$b_{21}=b_2+b_2',$$

$$\frac{b_{21}}{v_1}=\frac{b_2+b_2'}{v};$$

that these four equations may be satisfied, we must now evidently put

$$b_{11}=b_1+b_1'=0,$$
$$b_{21}=b_2+b_2'=0.$$

It follows from these and the above relations that the given incident waves (both electric and magnetic) would be totally reflected on the surface of a second insulator, no disturbance whatever entering the second medium, a result which we could hardly expect to be confirmed by experiment.

Necessity of Assumption of Surface Conditions of Higher Order of Approximation for Primary and Secondary Waves.—In the above development we have employed surface conditions of the first order of approximation, where namely the thickness of the dividing film has been neglected; on the assumption of the validity of these simpler surface conditions we found now on pp. 268-282 that *linearly-polarized plane* waves remained linearly polarized upon reflection and refraction, a result that is also not confirmed by exacter experiment, whereas, on employing the surface conditions of the second order of approximation, where the thickness of the film was not neglected, we obtained a marked elliptic polarization (cf. Exs. 8–11 at end of chapter), as demanded by empirical facts. Aside from the above analogy, there are other reasons why the simpler surface conditions could hardly be expected to lead to correct results for the reflection and refraction of the primary and secondary waves; among others the facts that any system of primary and secondary waves is represented by the derivatives of given functions with regard to x, y, z, whereas the existence (thickness) of the dividing film itself, within which these very derivatives play a most important part, is entirely overlooked, are hardly consistent with one another.

Difficulties Encountered in Derivation of Surface Conditions of higher Order of Approximation. Elliptic Polarization of the Primary and Secondary Waves according to Ordinary Laws of Reflection and Refraction.—To ascertain the behaviour of the primary and secondary waves on the surface of a second insulator, we should evidently have to employ surface conditions of at least the second order of approximation. The actual derivation of these conditions and the determination of the component amplitudes and changes in phase at the dividing surface from the same offer serious difficulties; the former evidently demands the differentiation of the component forces with regard to the coordinates (cf. formulae (84) and Ex. 12 at end of chapter), whereby the changes in phase sought must be regarded as functions of those coordinates, being different at different points on the dividing surface, that is, for different angles of incidence ϕ. This alone so complicates the problem, that all attempts to solve it must prove fruitless. We observe only that the familiar relation

$$\frac{\sin\phi_1}{v_1}=\frac{\sin\phi'}{v}=\frac{\sin\phi}{v}$$

must hold for all surface conditions that are derived on the above principles from our differential equations (1) and (2) (cf. above), since this relation is embodied in the existence alone of such surface conditions and does not depend on the explicit form of the same (cf. p. 271). The primary and secondary waves will, therefore, obey the same laws of reflection and refraction (cf. formulae (14)), as the plane and purely spherical waves do; they will also evidently become elliptically polarized upon reflection and refraction, but the actual determination of the respective ellipses of oscillation (changes in amplitude and phase at the dividing surface) will have to be abandoned.

EXAMPLES.

1. Show, when the direction of the electric force acting in the incident wave

$$A\epsilon^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}=\sqrt{P^2+Q^2+R^2}$$

makes an arbitrary angle θ with the plane of incidence (the xy-plane), that the resultant forces acting in the reflected and refracted electric waves and in the accompanying incident, reflected and refracted magnetic ones are given by the expressions

$$A\frac{\sin(\phi-\phi_1)}{\sin(\phi+\phi_1)}\sqrt{\sin^2\theta+\cos^2\theta\frac{\cos^2(\phi+\phi_1)}{\cos^2(\phi-\phi_1)}}\,e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)},$$

$$A\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)}\sqrt{\sin^2\theta+\cos^2\theta\sec^2(\phi-\phi_1)}\,e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)}$$

and

$$A\frac{v_0}{Mv}e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},$$

$$A\frac{v_0}{Mv}\frac{\sin(\phi-\phi_1)}{\sin(\varphi+\phi_1)}\sqrt{\sin^2\theta+\cos^2\theta\frac{\cos^2(\phi+\phi_1)}{\cos^2(\phi-\phi_1)}}\,e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)},$$

$$A\frac{v_0}{Mv}\frac{2\cos\phi\sin\phi_1}{\sin(\phi+\phi_1)}\sqrt{\sin^2\theta+\cos^2\theta\sec^2(\phi-\phi_1)}\,e^{in\left(t-\frac{y\sin\phi_1-x\cos\phi_1}{v_1}\right)}$$

respectively, where $\phi'=\arctan\left(\frac{v_1}{v}\sin\phi\right)$.

2. Show, when the incident electric wave of Ex. 1 strikes the reflecting surface at right angles, that the component forces, electric and magnetic, acting in the incident, reflected and refracted waves are given by the expressions

$$P=0,\quad Q=A\cos\theta e^{in\left(t+\frac{x}{v}\right)},\qquad R=A\sin\theta e^{in\left(t+\frac{x}{v}\right)},$$

$$P'=0,\quad Q'=-A\cos\theta\frac{v-v_1}{v+v_1}e^{in\left(t-\frac{x}{v}\right)},\qquad R'=-A\sin\theta\frac{v-v_1}{v+v_1}e^{in\left(t-\frac{x}{v}\right)},$$

$$P_1=0,\quad Q_1=A\cos\theta\,\frac{2v_1}{v+v_1}\,e^{in\left(t+\frac{x}{v_1}\right)},\qquad R_1=A\sin\theta\,\frac{2v_1}{v+v_1}\,e^{in\left(t+\frac{x}{v_1}\right)},$$

$$\alpha=0,\quad \beta=-A\sin\theta\,\frac{v_0}{Mv}\,e^{in\left(t+\frac{x}{v}\right)},\qquad \gamma=A\cos\theta\,\frac{v_0}{Mv}\,e^{in\left(t+\frac{x}{v}\right)},$$

$$\alpha'=0,\quad \beta'=-A\sin\theta\,\frac{v-v_1}{v+v_1}\,\frac{v_0}{Mv}\,e^{in\left(t-\frac{x}{v}\right)},\qquad \gamma'=A\cos\theta\,\frac{v-v_1}{v+v_1}\,\frac{v_0}{Mv}\,e^{in\left(t-\frac{x}{v}\right)},$$

$$\alpha_1=0,\quad \beta_1=-A\sin\theta\,\frac{2}{v+v_1}\,\frac{v_0}{M}\,e^{in\left(t+\frac{x}{v_1}\right)},\qquad \gamma_1=A\cos\theta\,\frac{2}{v+v_1}\,\frac{v_0}{M}\,e^{in\left(t+\frac{x}{v_1}\right)}.$$

3. Show, when the electric wave of Ex. 1 is incident on the reflecting surface at the angle of polarization, that formulae (18), (19), (29) and (30) assume the form

$$a'=A\sin\theta\cos 2\phi,\qquad a_1=2A\sin\theta\cos^2\phi,$$
$$b'=0,\qquad b_1=A\cos\theta\cot\phi,$$

or, if we express the angle of polarization in terms of the velocities of propagation v and v_1, namely

$$\phi=\arctan\left(\frac{v}{v_1}\right)$$

(cf. formulae (14) and (37)),

$$a'=-2A\sin\theta\,\frac{v^2-v_1^2}{v^2+v_1^2},\qquad a_1=2A\sin\theta\,\frac{v_1^2}{v^2+v_1^2},$$

$$b'=0,\qquad b_1=A\cos\theta\,\frac{v_1}{v}.$$

The component electric and magnetic forces acting in the incident, reflected and refracted waves are then given by the expressions

$$P=A\cos\theta\sin\phi e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}=A\cos\theta\,\frac{v}{\sqrt{v^2+v_1^2}}\,e^{in\left(t-\frac{yv-xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$Q=A\cos\theta\cos\phi e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}=A\cos\theta\,\frac{v_1}{\sqrt{v^2+v_1^2}}\,e^{in\left(t-\frac{yv-xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$R=A\sin\theta e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}=A\sin\theta e^{in\left(t-\frac{yv-xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$P'=Q'=0,$$

$$R'=A\sin\theta\cos 2\phi e^{in\left(t-\frac{y\sin\phi+x\cos\phi}{v}\right)}=-A\sin\theta\,\frac{v^2-v_1^2}{v^2+v_1^2}\,e^{in\left(t-\frac{yv+xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$P_1=A\cos\theta\cot\phi\cos\phi e^{in\left(t-\frac{y\cos\phi-x\sin\phi}{v_1}\right)}=A\cos\theta\,\frac{v_1^2}{v\sqrt{v^2+v_1^2}}\,e^{in\left(t-\frac{yv_1-xv}{v_1\sqrt{v^2+v_1^2}}\right)},$$

$$Q_1=A\cos\theta\cos\phi e^{in\left(t-\frac{y\cos\phi-x\sin\phi}{v_1}\right)}=A\cos\theta\,\frac{v_1}{\sqrt{v^2+v_1^2}}\,e^{in\left(t-\frac{yv_1-xv}{v_1\sqrt{v^2+v_1^2}}\right)},$$

$$R_1=2A\sin\theta\cos^2\phi e^{in\left(t-\frac{y\cos\phi-x\sin\phi}{v_1}\right)}=2A\sin\theta\,\frac{v_1^2}{v^2+v_1^2}\,e^{in\left(t-\frac{yv_1-xv}{v_1\sqrt{v^2+v_1^2}}\right)},$$

$$\alpha=-A\sin\theta\,\frac{v_0}{M}\,\frac{\sin\phi}{v}\,e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}=-A\sin\theta\,\frac{v_0}{M}\,\frac{1}{\sqrt{v^2+v_1^2}}\,e^{in\left(t-\frac{yv-xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$\beta = -A \sin\theta \frac{v_0}{M} \frac{\cos\phi}{v} e^{in\left(t - \frac{y \sin\phi - x\cos\phi}{v}\right)}$$

$$= -A \sin\theta \frac{v_0}{M} \frac{v_1}{v\sqrt{v^2+v_1^2}} e^{in\left(t - \frac{yv - xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$\gamma = A \cos\theta \frac{v_0}{M} \frac{1}{v} e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)} = A \cos\theta \frac{v_0}{M} \frac{1}{v} e^{in\left(t - \frac{yv - xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$\alpha' = -A \sin\theta \cos 2\phi \frac{v_0}{M} \frac{\sin\phi}{v} e^{in\left(t - \frac{y\sin\phi + x\cos\phi}{v}\right)}$$

$$= A \sin\theta \frac{v_0}{M} \frac{v^2 - v_1^2}{(v^2+v_1^2)^{\frac{3}{2}}} e^{in\left(t - \frac{yv + xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$\beta' = A \sin\theta \cos 2\phi \frac{v_0}{M} \frac{\cos\phi}{v} e^{in\left(t - \frac{y\sin\phi + x\cos\phi}{v}\right)}$$

$$= -A \sin\theta \frac{v_0}{M} \frac{v_1(v^2 - v_1^2)}{v(v^2+v_1^2)^{\frac{3}{2}}} e^{in\left(t - \frac{yv + xv_1}{v\sqrt{v^2+v_1^2}}\right)},$$

$$\gamma' = 0,$$

$$\alpha_1 = -2A \sin\theta \cos^2\phi \frac{v_0}{M} \frac{\cos\phi}{v_1} e^{in\left(t - \frac{y\cos\phi - x\sin\phi}{v_1}\right)}$$

$$= -2A \sin\theta \frac{v_0}{M} \frac{v_1^2}{(v^2+v_1^2)^{\frac{3}{2}}} e^{in\left(t - \frac{yv_1 - xv}{v_1\sqrt{v^2+v_1^2}}\right)},$$

$$\beta_1 = -2A \sin\theta \cos^2\phi \frac{v_0}{M} \frac{\sin\phi}{v_1} e^{in\left(t - \frac{y\cos\phi - x\sin\phi}{v_1}\right)}$$

$$= -2A \sin\theta \frac{v_0}{M} \frac{vv_1}{(v^2+v_1^2)^{\frac{3}{2}}} e^{in\left(t - \frac{yv_1 - xv}{v_1\sqrt{v^2+v_1^2}}\right)},$$

$$\gamma_1 = A \cos\theta \cot\phi \frac{v_0}{M} \frac{1}{v_1} e^{in\left(t - \frac{y\cos\phi - x\sin\phi}{v_1}\right)} = A \cos\theta \frac{v_0}{M} \frac{1}{v} e^{in\left(t - \frac{yv_1 - xv}{v_1\sqrt{v^2+v_1^2}}\right)}.$$

4. Show that formulae (41A) and (43A) assume the following form for perpendicular incidence ($\phi = \phi' = \phi_1 = 0$):

$$\tan\Theta' = -\tan\Theta,$$

$$\tan\Theta_1 = \tan\Theta,$$

that is, the plane of polarization undergoes a change only upon reflection (cf. p. 282), and

$$A'^2 = A^2 \frac{(v - v_1)^2}{(v + v_1)^2},$$

$$A_1^2 = A^2 \frac{4v_1^2}{(v + v_1)^2}$$

(cf. p. 279), that is, the amplitudes of the reflected and refracted waves are independent of the plane of polarization Θ of the incident waves.

5. Determine the form assumed by formulae (41A) and (43A), when the given waves are incident at the angle of polarization ($\phi + \phi_1 = \pi/2$).

Formulae (41A) assume here the form

$$\tan\Theta' = 0$$

(cf. p. 282),
$$\tan\Theta_1 = \frac{\tan\Theta}{\cos(2\phi - \pi/2)} = \frac{\tan\Theta}{\sin 2\phi},$$

or, since $\tan\phi = v/v_1$ here (cf. p. 280),

$$\tan\Theta_1 = \tan\Theta\,\frac{2vv_1}{v^2+v_1^2},$$

and formulae (43A) the form

$$A'^2 = A^2\cos^2 2\phi\cos^2\Theta,$$

$$A_1^2 = 4A^2\cos^4\phi(1+\sin^2\Theta\cot^2 2\phi)$$

$$= A^2\cot^2\phi(1-\cos^2\Theta\cos^2 2\phi),$$

or, in terms of v and v_1,

$$A'^2 = A^2\cos^2\Theta\left(\frac{v^2-v_1^2}{v^2+v_1^2}\right)^2,$$

$$A_1^2 = A^2\frac{v_1^2}{v^2}\left[1-\cos^2\Theta\left(\frac{v^2-v_1^2}{v^2+v_1^2}\right)^2\right].$$

6. Show, when linearly polarized plane waves are reflected m times at the same incidence and in the same plane, that their plane of polarization after the mth reflection Θ_m' is given by

$$\tan\Theta_m' = -\tan\Theta\,\frac{\cos^m(\phi+\phi_1)}{\cos^m(\phi-\phi_1)},$$

where Θ denotes the plane of polarization of the incident waves (cf. formulae (41 A)).

The effect of such repeated reflection will, therefore, be to bring the plane of polarization nearer and nearer to the plane of incidence; if common light is employed, the reflected waves will evidently become (partially) polarized in planes that make small angles with the plane of incidence.

7. The plane of polarization $\Theta_{1_{2m}}$ of linearly polarized plane waves after their passage through m plates (of glass) placed parallel to each other is given by the expression

$$\tan\Theta_{1_{2m}} = \tan\Theta\sec^{2m}(\phi-\phi_1),$$

where Θ denotes the plane of polarization of the incident waves.

The plane of polarization of the refracted waves in the first plate will be given, by formulae (41A), by

$$\tan\Theta_{1_1} = \tan\Theta\sec(\phi-\phi_1), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

in the first layer of air between that and the second plate by

$$\tan\Theta_{1_2} = \tan\Theta_{1_1}\sec(\phi_1-\phi)$$

or, by (a), and, since $\sec(\phi_1-\phi)=\sec(\phi-\phi_1)$, by

$$\tan\Theta_{1_2} = \tan\Theta\sec^2(\phi-\phi_1),$$

and similarly in the second plate by

$$\tan\Theta_{1_3} = \tan\Theta_{1_2}\sec(\phi-\phi_1) = \tan\Theta\sec^3(\phi-\phi_1)$$

and after refraction out of that plate by

$$\tan\Theta_{1_4} = \tan\Theta_{1_3}\sec(\phi_1-\phi) = \tan\Theta\sec^4(\phi-\phi_1),$$

etc.

The effect of repeated refraction will, therefore, be a rotation of the plane of polarization further and further from the plane of incidence (cf. p. 282); common light would thus become (partially) polarized in planes that make approximately right angles with the plane of incidence upon repeated refraction.

8. Determine the changes in phase $n\delta'$ and $n\delta_1$ of the conditional equations (94).

On eliminating first a' and then a_1 from the last two equations (94), we find the following values for a_1 and a' respectively:

$$a_1 = \frac{a \sin n\delta'}{\sin n(\delta' - \delta_1) - nl\dfrac{\cos\phi_1}{v_1}\cos n(\delta' - \delta_1)}$$

and

$$a' = \frac{a\left(\sin n\delta_1 + nl\dfrac{\cos\phi_1}{v_1}\cos n\delta_1\right)}{\sin n(\delta' - \delta_1) - nl\dfrac{\cos\phi_1}{v_1}\cos n(\delta' - \delta_1)}.$$

Replace a_1 and a' by these values in the first two equations (94), and we have

$$-\frac{v_0}{M}\frac{\cos\phi_1}{v_1}\sin n\delta'\cos n\delta_1 + \frac{v_0}{M}\frac{\cos\phi}{v}\left[\sin n(\delta' - \delta_1) - nl\frac{\cos\phi_1}{v_1}\cos n(\delta' - \delta_1)\right]$$
$$-\frac{v_0}{M}\frac{\cos\phi}{v}\left(\sin n\delta_1 + nl\frac{\cos\phi_1}{v_1}\cos n\delta_1\right)\cos n\delta' + nG\sin n\delta'\sin n\delta_1 = 0$$

and

$$\frac{v_0}{M}\frac{\cos\phi_1}{v_1}\sin n\delta_1 + \frac{v_0}{M}\frac{\cos\phi}{v}\left(\sin n\delta_1 + nl\frac{\cos\phi_1}{v_1}\cos n\delta_1\right) + nG\cos n\delta_1 = 0,$$

where we have put $\phi' = \phi$ (cf. p. 302) and

$$G = \frac{p}{v_0} - \frac{v_0}{M}l\frac{\sin^2\phi}{v^2}.$$

The latter of these equations gives

$$\tan n\delta_1 = -\frac{n(v_0 l\cos\phi\cos\phi_1 + vv_1 MG)}{v_0(v\cos\phi_1 + v_1\cos\phi)}, \quad \text{........................}(a)$$

the value for $n\delta_1$ sought.

The former equation can now be written

$$\frac{v_0}{M}\frac{\cos\phi}{v}\left[1 - 2\cot n\delta'\tan n\delta_1 - nl\frac{\cos\phi_1}{v_1}(2\cot n\delta' + \tan n\delta_1)\right]$$
$$+ nG\tan n\delta_1 = \frac{v_0}{M}\frac{\cos\phi_1}{v_1},$$

hence

$$\cot n\delta' = \frac{n(vv_1 MG - v_0 l\cos\phi\cos\phi_1)\tan n\delta_1 + v_0(v_1\cos\phi - v\cos\phi_1)}{2v_0\cos\phi(v_1\tan n\delta_1 + nl\cos\phi_1)}.$$

Replace here $\tan n\delta_1$ by its value (a), and we obtain the value for $n\delta'$ sought.

9. Show that the changes in phase $n\zeta'$ and $n\zeta_1$ of the conditional equations (95) are determined by the expressions

$$\tan n\zeta_1 = \frac{n(v_0^2 v_1 H - vv_1 Mp\cos\phi\cos\phi_1)}{v_0^2(v\cos\phi + v_1\cos\phi_1)}$$

and

$$\cot n\zeta' = \frac{n(v_0^2 v_1 H + vv_1 Mp\cos\phi\cos\phi_1)\tan n\zeta_1 + v_0^2(v_1\cos\phi_1 - v\cos\phi)}{2v_0^2 v_1(\cos\phi_1\tan n\zeta_1 - nH)},$$

where $\tan n\zeta_1$ is to be replaced in the latter expression by its value, the former expression, and

$$H = \frac{l}{v_1} - qD_1 v_1\frac{\sin^2\phi}{v^2}.$$

10. Show for the angle of polarization that the changes in phase $n\delta'$ and $n\zeta'$ of formulae (94) and (95) are determined by the expressions

$$\cot n\delta' = -\frac{v_0^2\cos^2\phi - n^2 v^2 M^2 G^2}{2nv_0 vMG\cos\phi}$$

and

$$\cot n\zeta' = \frac{v^2\cos^2\phi - n^2 v_1^2 H^2}{2nvv_1 H\cos\phi},$$

where

$$G = \frac{p}{v_0} - \frac{v_0}{M}l\frac{\sin^2\phi}{v^2}. \qquad H = \frac{l}{v_1} - qD_1 v_1\frac{\sin^2\phi}{v^2}.$$

It is evident from these expressions that for the angle of polarization the reflected oscillations will be highly elliptically polarized (cf. p. 298).

11. Determine the changes in phase $n\delta'$, $n\delta_1$, $n\zeta'$ and $n\zeta_1$ of formulae (94) and (95) for perpendicular incidence.

12. Show that the expressions (101) and (102) or (107) and (108) for the component electric and magnetic forces satisfy our fundamental differential equations (1) and (2).

On differentiating the component forces with respect to the coordinates, as demanded by our differential equations (1) and (2), we must evidently employ those coordinates, whose origin is at the source of disturbance O (cf. Fig. 35); that is, the component forces are to be differentiated with respect to $\bar{x} - x$, $\bar{y} - y$, $\bar{z} - z$, the component distances from that source referred to the system of coordinates x, y, z with origin at O (cf. Fig. 35). We observe, moreover, that we cannot regard the component amplitude-coefficients, the A's, B's etc. of formulae (107) and (108), as constant by these differentiations: they can evidently be regarded as constant only along one and the same vector. In order to perform the differentiations indicated, we must, therefore, write the component forces in the explicit (variable) form

$$P = \frac{4\pi n^2}{v^2(r - r)}\{a_1[(\bar{\beta} - \beta)^2 + (\bar{\gamma} - \gamma)^2] - (\alpha - \alpha)[a_2(\bar{\beta} - \beta) + a_3(\bar{\gamma} - \gamma)]\}$$
$$\times \sin n\left(t - \frac{\bar{r} - r}{v}\right)$$
$$+ \frac{4\pi n}{v(\bar{r} - r)^2}\{2a_1 - 3a_1[(\bar{\beta} - \beta)^2 + (\bar{\gamma} - \gamma)^2] + 3(\alpha - \alpha)[a_2(\beta - \beta) + a_3(\bar{\gamma} - \gamma)]\}$$
$$\times \cos n\left(t - \frac{\bar{r} - r}{v}\right)$$
$$+ \frac{4\pi}{(\bar{r} - r)^3}\{2a_1 - 3a_1[(\bar{\beta} - \beta)^2 + (\bar{\gamma} - \gamma)^2] + 3(\dot{\alpha} - \alpha)[a_2(\bar{\beta} - \beta) + a_3(\gamma - \gamma)]\}$$
$$\times \sin n\left(t - \frac{r - r}{v}\right),$$

with analogous expressions for Q and R, and

$$\alpha = \frac{4\pi n v_0}{Mv^2}[a_2(\bar{\gamma} - \gamma) - a_3(\beta - \beta)]\left[\frac{n}{v(r - r)} \sin n\left(t - \frac{\bar{r} - r}{v}\right)\right.$$
$$\left. - \frac{1}{(r - r)^2} \cos n\left(t - \frac{\bar{r} - r}{v}\right)\right],$$

with analogous expressions for β and γ.

Replace P, Q, R and α, β, γ by these values in equations (1) and (2), for example, the first of each, perform the differentiations indicated, and we have

$$\frac{D}{v_0}\left[\frac{4\pi n^3}{v^2(\bar{r} - r)} A_1 \cos\omega - \frac{4\pi n^2}{v(\bar{r} - r)^2} B_1 \sin\omega + \frac{4\pi}{(r - r)^3} B_1 \cos\omega\right]$$
$$= \frac{4\pi n v_0}{Mv^2}\left\{\left[\frac{n}{v}\frac{\sin\omega}{(r - r)^2} - \frac{\cos\omega}{(\bar{r} - r)^3}\right]\frac{d}{d(\bar{z} - z)}[a_3(x - x) - a_1(\bar{z} - z)]\right.$$
$$\left. + [a_3(\bar{x} - x) - a_1(\bar{z} - z)]\frac{d}{d(r - r)}\left[\frac{n}{v}\frac{\sin\omega}{(\bar{r} - r)^2} - \frac{\cos\omega}{(\bar{r} - r)^3}\right]\frac{d(\bar{r} - r)}{d(\bar{z} - \bar{z})}\right\}$$
$$- \frac{4\pi n v_0}{Mv^2}\left\{\left[\frac{n}{v}\frac{\sin\omega}{(\bar{r} - r)^2} - \frac{\cos\omega}{(\bar{r} - r)^2}\right]\frac{d}{d(\bar{y} - y)}[a_1(\bar{y} - y) - a_2(\bar{x} - x)]\right.$$
$$\left. + [a_1(\dot{y} - y) - a_2(\bar{x} - x)]\frac{d}{d(\bar{r} - r)}\left[\frac{n}{v}\frac{\sin\omega}{(\bar{r} - r)^2} - \frac{\cos\omega}{(\bar{r} - r)^3}\right]\frac{d(r - r)}{d(y - y)}\right\}$$

$$
\begin{aligned}
&=\frac{4\pi n v_0}{Mv^2}\Bigg\{-\left[\frac{n}{v}\frac{\sin\omega}{(\bar{r}-r)^2}-\frac{\cos\omega}{(\bar{r}-r)^3}\right]2a_1\\
&\qquad+\frac{[\bar{a}_3(\bar{x}-x)-a_1(\bar{z}-z)](\bar{z}-z)-[a_1(\bar{y}-y)-a_2(\bar{x}-x)](\bar{y}-y)}{(\bar{r}-r)}\\
&\qquad\times\left[-\frac{n^2}{v^2}\frac{\cos\omega}{(\bar{r}-r)^2}-\frac{3n}{v}\frac{\sin\omega}{(\bar{r}-r)^3}+\frac{3\cos\omega}{(\bar{r}-r)^4}\right]\Bigg\}\\
&=\frac{4\pi n v_0}{Mv^2}\Bigg\{-2a_1\left[\frac{n}{v}\frac{\sin\omega}{(\bar{r}-r)^2}-\frac{\cos\omega}{(\bar{r}-r)^3}\right]\\
&\qquad+\{a_1[\bar{\beta}-\beta)^2+(\bar{\gamma}-\gamma)^2]-(\bar{\alpha}-\alpha)[a_2(\bar{\beta}-\beta)+a_3(\bar{\gamma}-\gamma)]\}\\
&\qquad\times\left[\frac{n^2}{v^2}\frac{\cos\omega}{(\bar{r}-r)}+\frac{3n}{v}\frac{\sin\omega}{(\bar{r}-r)^2}-\frac{3\cos\omega}{(\bar{r}-r)^3}\right]\Bigg\}\\
&=\frac{4\pi n^3 v_0}{Mv^4}A_1\frac{\cos\omega}{(\bar{r}-r)}-\frac{4\pi n^2 v_0}{Mv^3}B_1\frac{\sin\omega}{(\bar{r}-r)^2}+\frac{4\pi n v_0}{Mv^2}B_1\frac{\cos\omega}{(\bar{r}-r)^3},
\end{aligned}
$$

which gives the familiar relation (cf. formulae (12)) between the velocity of propagation and the medium constants; and (the first of equations (2))

$$
\begin{aligned}
&\frac{4\pi n^2}{v^2}\Big[a_2(\bar{\gamma}-\gamma)-a_3(\bar{\beta}-\beta)\Big]\bigg[\frac{n}{v(\bar{r}-r)}\cos n\left(t-\frac{\bar{r}-r}{v}\right)\\
&\qquad+\frac{1}{(\bar{r}-r)^2}\sin n\left(t-\frac{\bar{r}-r}{v}\right)\bigg]\\
&=\frac{4\pi n^2}{v^2}\left\{\frac{\sin\omega}{(\bar{r}-r)^3}\frac{d[A_3]}{d(\bar{y}-y)}+[A_3]\frac{d}{d(\bar{r}-r)}\left[\frac{\sin\omega}{(\bar{r}-r)^3}\right]\frac{\bar{r}-r)}{d(\bar{y}-y)}\right\}\\
&+4\pi\left\{\frac{n}{v}\left[\frac{\cos\omega}{(\bar{r}-r)^4}+\frac{\sin\omega}{(\bar{r}-r)^5}\right]\frac{d[B_3]}{d(\bar{y}-y)}+[B_3]\frac{d}{d(\bar{r}-r)}\left[\frac{n\cos\omega}{v(\bar{r}-r)^4}+\frac{\sin\omega}{(\bar{r}-r)^5}\right]\frac{d(\bar{r}-r)}{d(\bar{y}-y)}\right\}\\
&-\frac{4\pi n^2}{v^2}\left\{\frac{\sin\omega}{(r-r)^3}\frac{d[A_2]}{d(\bar{z}-z)}+[A_2]\frac{d}{d(\bar{r}-r)}\left[\frac{\sin\omega}{(\bar{r}-r)^3}\right]\frac{d(\bar{r}-r)}{d(\bar{y}-y)}\right\}\\
&+4\pi\left\{\left[\frac{n}{v}\frac{\cos\omega}{(\bar{r}-r)^4}+\frac{\sin\omega}{(\bar{r}-r)^5}\right]\frac{d[B_2]}{d(\bar{z}-z)}+[B_2]\frac{d}{d(\bar{r}-r)}\left[\frac{n\cos\omega}{v(\bar{r}-r)^4}+\frac{\sin\omega}{(\bar{r}-r)^5}\right]\frac{d(\bar{r}-r)}{d(\bar{z}-z)},\right.
\end{aligned}
$$

where

$$
\begin{aligned}
[A_3]&=a_3[(\bar{x}-x)^2+(\bar{y}-y)^2]-(\bar{z}-z)[a_1(\bar{x}-x)+a_2(\bar{y}-y)],\\
[B_3]&=2a_3[(\bar{x}-x)^2+(\bar{y}-y)^2+(\bar{z}-z)^2]-3a_3[(\bar{x}-x)^2+(\bar{y}-y)^2]+3(\bar{z}-z)\\
&\qquad\times[a_1(\bar{x}-x)+a_2(\bar{y}-y)],\\
[A_2]&=a_2[(\bar{x}-x)^2+(\bar{z}-z)^2]-(\bar{y}-y)[a_1(\bar{x}-x)+a_3(\bar{z}-z)],\\
[B_2]&=2a_2[(\bar{x}-x)^2+(\bar{y}-y)^2+(\bar{z}-z)^2]-3a_2[(\bar{x}-x)^2+(\bar{z}-z)^2]\\
&\qquad+3(\bar{y}-y)[a_1(\bar{x}-x)+a_3(\bar{z}-z)],
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{4\pi n^2}{v^2}\Bigg\{\frac{\sin\omega}{(\bar{r}-r)^3}3[a_3(\bar{y}-y)-a_2(\bar{z}-z)]+[a_2(\bar{z}-z)-a_3(\bar{y}-y)](\bar{r}-r)\\
&\qquad\qquad\times\left[\frac{n}{v}\frac{\cos\omega}{(\bar{r}-r)^3}+\frac{3\sin\omega}{(\bar{r}-r)^4}\right]\Bigg\}\\
&+4\pi\Bigg\{\left[\frac{n}{v}\frac{\cos\omega}{(\bar{r}-r)^4}+\frac{\sin\omega}{(\bar{r}-r)^5}\right]5[a_2(\bar{z}-z)-a_3(\bar{y}-y)]\\
&+[a_2(\bar{z}-z)-a_3(\bar{y}-y)](\bar{r}-r\left[\frac{n^2\sin\omega}{v^2(\bar{r}-r)^4}-\frac{5n\cos\omega}{v(\bar{r}-r)^5}-\frac{5\sin\omega}{(\bar{r}-r)^6}\right]\Bigg\}\\
&=\frac{4\pi n^2}{v^2}[a_2(\bar{\gamma}-\gamma)-a_3(\bar{\beta}-\beta)]\left\{-\frac{3\sin\omega}{(\bar{r}-r)^2}+\frac{n\cos\omega}{v(\bar{r}-r)}+\frac{3\sin\omega}{(\bar{r}-r)^2}\right\}\\
&+4\pi[a_2(\bar{\gamma}-\gamma)-a_3(\bar{\beta}-\beta)]\left\{5\left[\frac{n\cos\omega}{v(\bar{r}-r)^3}+\frac{\sin\omega}{(\bar{r}-r)^4}\right]+\frac{n^2\sin\omega}{v^2(\bar{r}-r)^2}-\frac{5n\cos\omega}{v(\bar{r}-r)^3}-\frac{5\sin\omega}{(\bar{r}-r)^4}\right\}\\
&=\frac{4\pi n^2}{v^2}[a_2(\bar{\gamma}-\gamma)-a_3(\bar{\beta}-\beta)]\left[\frac{n\cos\omega}{v(\bar{r}-r)}+\frac{\sin\omega}{(\bar{r}-r)^2}\right].
\end{aligned}
$$

Q.E.D.

13. Show that the component amplitude coefficients b_1, b_2, b_3 of formulae (104)–(106) are given by the expressions

$$\left.\begin{aligned} b_1&=a_1,\\ b_2&=a_2\cos\mu-a_3\sin\mu,\\ b_3&=a_2\sin\mu+a_3\cos\mu, \end{aligned}\right\}\quad\text{.........................}(\text{A})$$

where μ denotes the angle between the z and $\bar{z}$ (y and $\bar{y}$) axes (cf. Fig. 35). This is evident from the annexed figure.

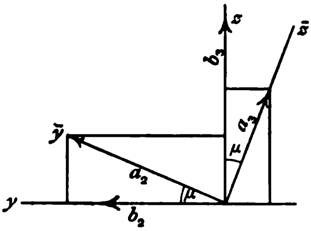

FIG. 36.

14. Confirm formulae (104)–(106).

If we denote the component amplitude coefficients of the force acting in the primary electric wave (101) along the $\bar{x}$, $\bar{y}$, $\bar{z}$, axes by $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ and those along the x, y, z axes by A_1, A_2, A_3, the following relations will evidently hold between these components:

$$\left.\begin{aligned} A_1&=\bar{A}_1,\\ A_2&=\bar{A}_2\cos\mu-\bar{A}_3\sin\mu,\\ A_3&=\bar{A}_2\sin\mu+\bar{A}_3\cos\mu, \end{aligned}\right\}\quad\text{.........................}(\text{A})$$

where μ denotes the angle between the coordinate axes $\bar{z}$ and z or $\bar{y}$ and y (cf. formulae (A), Ex. 13).

By Figure 35, we have now the following relations between the direction-cosines $\bar{a}$, $\bar{\beta}$, $\bar{\gamma}$ and a, β, γ:

$$\bar{a}=\frac{\bar{x}}{r}=-\frac{x}{r}=-a=-\cos\phi,$$

$$\bar{\beta}=\frac{\bar{y}}{r}=\frac{y\cos\mu+z\sin\mu}{-r}$$

and

$$\bar{\gamma}=\frac{\bar{z}}{r}=\frac{-y\sin\mu+z\cos\mu}{-r}$$

(cf. formulae (A), Ex. 13), or, since here $z=0$ (cf. p. 213),

$$\bar{\beta}=\frac{y\cos\mu}{-r}=-\cos(r,\,y)\cos\mu=-\cos(\pi/2+\phi)\cos\mu$$

$$=\sin\phi\cos\mu$$

and

$$\bar{\gamma}=\frac{y\sin\mu}{r}=-\sin\phi\sin\mu.$$

Replace $\bar{a}$, $\bar{\beta}$, $\bar{\gamma}$ in the expressions for the component amplitude coefficients $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ (cf. formulae (104)) by these values, and we have

$$\left.\begin{aligned} \bar{A}_1&=a_1\sin^2\phi+(a_2\sin\phi\cos\mu-a_3\sin\phi\sin\mu)\cos\phi,\\ \bar{A}_2&=a_2(\cos^2\phi+\sin^2\phi\sin^2\mu)+(a_1\cos\phi+a_3\sin\phi\sin\mu)\sin\phi\cos\mu,\\ \bar{A}_3&=a_3(\cos^2\phi+\sin^2\phi\cos^2\mu)-(a_1\cos\phi-a_2\sin\phi\cos\mu)\sin\phi\sin\mu. \end{aligned}\right\}\quad\text{.......}(\text{B})$$

On replacing the $\bar{A}$'s by these values in formulae (A), we then find the following values for the A's:

$$A_1=[a_1\sin\phi+(a_2\cos\mu-a_3\sin\mu)\cos\phi]\sin\phi,$$
$$A_2=[a_1\sin\phi+(a_2\cos\mu-a_3\sin\mu)\cos\phi]\cos\phi,$$
$$A_3=a_2\sin\mu+a_3\cos\mu,$$

which, by formulae (A), Ex. 13, can be written

$$\left.\begin{aligned}A_1&=(b_1\sin\phi+b_2\cos\phi)\sin\phi,\\A_2&=(b_1\sin\phi+b_2\cos\phi)\cos\phi,\\A_3&=b_3\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}$$

(cf. formulae (104)).

Similarly, on replacing $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ by the above values in analogous relations to (A), which will hold between $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$ and B_1, B_2, B_3, the component amplitude coefficients of the force acting in the secondary electric wave along the $\bar{x}$, $\bar{y}$, $\bar{z}$ and x, y, z axes respectively, we find (cf. formulae (104) and (105))

$$B_1=\bar{B}_1=2a_1-3\bar{A}_1,$$
$$B_2=\bar{B}_2\cos\mu-\bar{B}_3\sin\mu=(2a_2-3\bar{A}_2)\cos\mu-(2a_3-3\bar{A}_3)\sin\mu,$$
$$B_3=\bar{B}_2\sin\mu+\bar{B}_3\cos\mu=(2a_2-3\bar{A}_2)\sin\mu+(2a_3-3\bar{A}_3)\cos\mu,$$

where $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ are given by formulae (B), which we can write in the form

$$B_1=2a_1-3A_1,$$
$$B_2=2(a_2\cos\mu-a_3\sin\mu)-3(\bar{A}_2\cos\mu-\bar{A}_3\sin\mu),$$
$$B_3=2(a_2\sin\mu+a_3\cos\mu)-3(\bar{A}_2\sin\mu+\bar{A}_3\cos\mu),$$

or, by formulae (A), Ex. 13, and (A) and (C) above,

$$B_1=2b_1-3(b_1\sin\phi+b_2\cos\phi)\sin\phi,$$
$$B_2=2b_2-3(b_1\sin\phi+b_2\cos\phi)\cos\phi,$$
$$B_3=2b_3-3b_3=-b_3$$

(cf. formulae (105)).

Lastly, replace $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ by the above values in analogous relations to (A), which will hold between the component amplitude coefficients $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$ along the $\bar{x}$, $\bar{y}$, z axes and those C_1, C_2, C_3 along the x, y, z axes of the force acting in the magnetic (primary and secondary) wave, and we find, by formulae (106),

$$C_1=-(a_2\sin\mu+a_3\cos\mu)\sin\phi,$$
$$C_2=-(a_2\sin\mu+a_3\cos\mu)\cos\phi,$$
$$C_3=a_1\sin\phi+(a_2\cos\mu-a_3\sin\mu)\cos\phi,$$

which, by formulae (A), Ex. 13, can be written

$$C_1=-b_3\sin\phi,$$
$$C_2=-b_3\cos\phi,$$
$$C_3=b_1\sin\phi+b_2\cos\phi$$

(cf. formulae (106)). Q.E.D.

CHAPTER VIII.

PROPAGATION OF ELECTROMAGNETIC WAVES THROUGH CRYSTALLINE MEDIA. REFLECTION AND DOUBLE REFRACTION ON THE SURFACE OF BIAXAL AND UNIAXAL CRYSTALS; TOTAL REFLECTION.

Aeolotropic Media; the Crystals.—Isotropic media are thereby characterized that the constants of electric and magnetic induction retain one and the same values in all directions, that is, the electric and magnetic displacements or moments are independent of the directions of action of the forces, the displacements always being in the directions of action of the forces themselves (cf. formulae (3 and 4, I.)). On the other hand, media, in which the constants of electric and magnetic induction assume different values according to the directions chosen, are known as "aeolotropic" (cf. p. 7); such media will, therefore, evidently be characterized thereby, that the moments do not, in general, take place in the directions of the forces acting. The only aeolotropic media, within which light or electromagnetic phenomena can be investigated to any degree of accuracy, are now the crystalline ones; of these the best suited for investigation are the transparent crystals, all of which are known as poor conductors. Let us, therefore, confine our ensuing investigations to the behaviour of light and electromagnetic waves in aeolotropic insulators.

The Constants of Electric and Magnetic Induction.—Experiments [*] have shown that there are, in general, three directions in any aeolotropic insulator or crystal, in which the constant of electric induction assumes a maximum or minimum, and that these three principal directions or axes always stand at right angles to one another

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[*] Cf. L. Boltzmann: "Experimentaluntersuchung über das Verhalten nicht leitender Körper unter dem Einfluss elektrischer Kräfte," *Pogg. Annalen*, v. 153, 1874.

(cf. p. 7); forces acting along these axes will, therefore, evidently give rise to displacements or oscillations *along* the same. On the other hand, all attempts to detect any appreciable change in the value of the constant of magnetic induction with respect to direction have failed. In the following we may, therefore, assume that the constant of magnetic induction is independent of the direction chosen.

Maxwell's Equations for Crystalline Media.—We choose the three principal directions or axes, along which the constant of electric induction becomes a maximum or minimum, as axes of a system of rectangular coordinates x, y, z and denote the respective values of that constant D along those axes by D_1, D_2, D_3. We then make the plausible assumption that, if there be any change in the value of the constant of magnetic induction M with respect to direction, the directions of its maxima and minima coincide with those of the constant of electric induction D. Similarly, we shall denote any such principal values of M along the x, y, z axes by M_1, M_2 and M_3 respectively. Maxwell's fundamental equations for the variations of the component electric and magnetic forces P, Q, R and α, β, γ respectively acting in any aeolotropic insulator or crystal can then evidently be written

$$\left.\begin{aligned}\frac{D_1}{v_0}\frac{dP}{dt}&=\frac{d\beta}{dz}-\frac{d\gamma}{dy},\\ \frac{D_2}{v_0}\frac{dQ}{dt}&=\frac{d\gamma}{dx}-\frac{d\alpha}{dz},\\ \frac{D_3}{v_0}\frac{dR}{dt}&=\frac{d\alpha}{dy}-\frac{d\beta}{dx}\end{aligned}\right\}\dots\dots\dots\dots(1)$$

(cf. formulae (8, I.)) and

$$\left.\begin{aligned}\frac{M_1}{v_0}\frac{d\alpha}{dt}&=\frac{dR}{dy}-\frac{dQ}{dz},\\ \frac{M_2}{v_0}\frac{d\beta}{dt}&=\frac{dP}{dz}-\frac{dR}{dx},\\ \frac{M_3}{v_0}\frac{d\gamma}{dt}&=\frac{dQ}{dx}-\frac{dP}{dy}\end{aligned}\right\}\dots\dots\dots\dots(2)$$

(cf. p. 7), where v_0 denotes the velocity of propagation of electromagnetic (light) waves in any standard isotropic insulator (vacuum).

If we denote the component electric and magnetic moments, to which the component forces P, Q, R and α, β, γ give rise, by X, Y, Z and a, b, c respectively, the following relations then hold between the moments and the forces:

$$X=\frac{D_1}{4\pi}P,\quad Y=\frac{D_2}{4\pi}Q,\quad Z=\frac{D_3}{4\pi}R\dots\dots\dots\dots(3)$$

(cf. formulae (7, I.)) and

$$a=\frac{M_1}{4\pi}\alpha,\quad b=\frac{M_2}{4\pi}\beta,\quad c=\frac{M_3}{4\pi}\gamma.\qquad\qquad(4)$$

Plane-Waves.—The electromagnetic state in any crystal is now defined by the above equations (1)–(4); of the different possible states only the oscillatory one interests us here. What oscillatory states are now consistent with these equations, that is, what forms of electromagnetic waves can be propagated through the given medium? It is now possible to show that the crystalline medium defined by the equations (1)–(4) is capable of transmitting only linearly polarized plane-waves, and these, in fact, only under certain restrictions. The problem before us is, therefore, to determine the manner in which a plane-wave will travel through a crystalline medium, that is, as we shall see below, to determine the direction (directions) of oscillation and the velocity (velocities) of propagation that must prevail in such a wave, in order that it may travel in any assigned direction through that medium. We have now seen on p. 11 that plane-wave motion is always represented by a function of $\left(t\pm\frac{s}{v}\right)$, where t denotes the time, s the distance of any wave-front from any given point and v the velocity of propagation. Let us represent any system of plane electric waves that can be transmitted through the given crystal in the familiar form

$$ae^{in\left(t-\frac{s}{v}\right)},\qquad\qquad(5)$$

where a denotes their amplitude. We denote the direction-cosines of the normal to any wave-front of this system of waves at any point P on the same referred to the principal axes x, y, z of the crystal by λ, μ, ν, and those of the direction of oscillation at that point by ξ, η, ζ as indicated in Fig. 37 below; λ, μ, ν are to be regarded here as given, whereas ξ, η, ζ are to be sought. Lastly, let the point, from which the distance s to the point P is measured, be chosen as origin O of the system of coordinates x, y, z (cf. Fig. 37). The component moments or oscillations X, Y, Z at any point P on any wave-front of the waves represented by the function (5) will then evidently be

$$\left.\begin{aligned}X&=a\xi e^{in\left(t-\frac{s}{v}\right)},\\ Y&=a\eta e^{in\left(t-\frac{s}{v}\right)},\\ Z&=a\zeta e^{in\left(t-\frac{s}{v}\right)},\end{aligned}\right\}\qquad\qquad(6)$$

where

$$s=\lambda x+\mu y+\nu z.\qquad\qquad(6\text{A})$$

By formulae (3) the component electric forces P, Q, R acting at the point P will evidently be

$$\left.\begin{aligned} P &= \frac{4\pi}{D_1} a\xi e^{in\left(t-\frac{s}{v}\right)}, \\ Q &= \frac{4\pi}{D_2} a\eta e^{in\left(t-\frac{s}{v}\right)}, \\ R &= \frac{4\pi}{D_3} a\zeta e^{in\left(t-\frac{s}{v}\right)}. \end{aligned}\right\} \quad \text{..............................(7)}$$

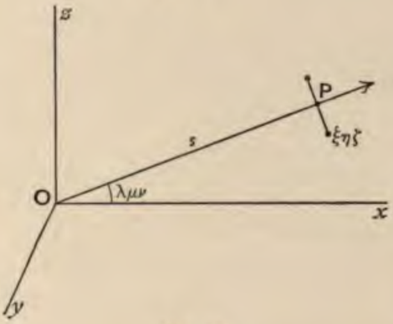

FIG. 37.

To determine the component magnetic forces α, β, γ acting at the point P, we replace P, Q, R by these values in formulae (2), and we have, on performing the differentiations indicated,

$$\frac{M_1}{v_0}\frac{d\alpha}{dt} = -4\pi a \frac{in}{v}\left(\frac{\mu\zeta}{D_3} - \frac{\nu\eta}{D_2}\right) e^{in\left(t-\frac{s}{v}\right)}$$

and analogous equations for β and γ. These equations integrated give

$$\left.\begin{aligned} \alpha &= -\frac{4\pi v_0 a}{vM_1}\left(\frac{\mu\zeta}{D_3} - \frac{\nu\eta}{D_2}\right) e^{in\left(t-\frac{s}{v}\right)}, \\ \beta &= -\frac{4\pi v_0 a}{vM_2}\left(\frac{\nu\xi}{D_1} - \frac{\lambda\zeta}{D_3}\right) e^{in\left(t-\frac{s}{v}\right)}, \\ \gamma &= -\frac{4\pi v_0 a}{vM_3}\left(\frac{\lambda\eta}{D_2} - \frac{\mu\xi}{D_1}\right) e^{in\left(t-\frac{s}{v}\right)}, \end{aligned}\right\} \quad \text{......................(8)}$$

where we have rejected the three constants of integration, which are functions of x, y, z only, since it is only the periodic or oscillatory motions that interest us here (cf. p. 29).

By formulae (4) the component magnetic moments or oscillations a, b, c, to which the component forces α, β, γ give rise, will then be represented by the expressions

$$\left.\begin{aligned} a &= -\frac{v_0 a}{v}\left(\frac{\mu\zeta}{D_3} - \frac{\nu\eta}{D_2}\right)e^{in\left(t-\frac{s}{v}\right)}, \\ b &= -\frac{v_0 a}{v}\left(\frac{\nu\xi}{D_1} - \frac{\lambda\zeta}{D_3}\right)e^{in\left(t-\frac{s}{v}\right)}, \\ c &= -\frac{v_0 a}{v}\left(\frac{\lambda\eta}{D_2} - \frac{\mu\xi}{D_1}\right)e^{in\left(t-\frac{s}{v}\right)}. \end{aligned}\right\} \quad \text{......................}(9)$$

Particular Form of Maxwell's Equations for Plane-Waves.— The expressions (7) and (8) for the component forces P, Q, R and α, β, γ respectively can now each be written in the form $e^{in\left(t-\frac{s}{v}\right)}$ times a given constant factor, this factor being, of course, different for the different components. Upon the differentiation of these expressions for the component forces with regard to the time t and the coordinates x, y, z, as demanded by our fundamental differential equations (1) and (2), the expressions for the differential quotients will, therefore, differ from those for the component forces themselves by the factors in and $-\frac{in\lambda}{v}$, $-\frac{in\mu}{v}$, $-\frac{in\nu}{v}$ respectively, so that the differential quotients of equations (1) and (2) may each be replaced there by the quantity or component force itself multiplied by that respective factor; we can, therefore, evidently replace the above differential equations (1) and (2) here by the following ordinary linear ones:

$$\left.\begin{aligned} \frac{vD_1}{v_0}P &= \mu\gamma - \nu\beta, \\ \frac{vD_2}{v_0}Q &= \nu\alpha - \lambda\gamma, \\ \frac{vD_3}{v_0}R &= \lambda\beta - \mu\alpha \end{aligned}\right\} \quad \text{............................}(10)$$

and

$$\left.\begin{aligned} \frac{vM_1}{v_0}\alpha &= \nu Q - \mu R, \\ \frac{vM_2}{v_0}\beta &= \lambda R - \nu P, \\ \frac{vM_3}{v_0}\gamma &= \mu P - \lambda Q. \end{aligned}\right\} \quad \text{............................}(11)$$

The validity of the latter equations (11) is also evident from the above values (7) and (8) for the component forces P, Q, R and α, β, γ respectively, whereas the confirmation of the former from the above values involves a knowledge of the relations that hold between the direction-cosines λ, μ, ν and ξ, η, ζ, the medium constants, the D's and M's, and the velocity of the propagation v of the waves (cf. Ex. 1 at end of chapter).

Electric and Magnetic Oscillations in Wave-Front.—Multiply equations (10), the first by λ, the second by μ and the third by ν, add, and we have

$$\lambda D_1 P + \mu D_2 Q + \nu D_3 R = 0\,;$$

equations (11), similarly treated, give

$$\lambda M_1 \alpha + \mu M_2 \beta + \nu M_3 \gamma = 0.$$

By formulae (3) and (4) these two relations can now be written

$$\lambda X + \mu Y + \nu Z = 0 \quad \text{......................} (12)$$

and

$$\lambda a + \mu b + \nu c = 0. \quad \text{......................} (13)$$

These relations, interpreted geometrically, state that both the electric and the magnetic oscillations take place at right angles to the normal (λ, μ, ν) to the given wave-front; that is, they both lie in the wave-front itself.

Electric Moment at ⊥ to Magnetic Force and Magnetic Moment at ⊥ to Electric Force.—Next, multiply equations (10), the first by α, the second by β and the third by γ, add, and we have

$$D_1 P\alpha + D_2 Q\beta + D_3 R\gamma = 0\,; \quad \text{......................} (14)$$

equations (11), similarly treated, give

$$M_1 P\alpha + M_2 Q\beta + M_3 R\gamma = 0. \quad \text{......................} (15)$$

Observe that for $M_1 = M_2 = M_3$ this last relation becomes

$$P\alpha + Q\beta + R\gamma = 0\,; \quad \text{......................} (15\text{A})$$

that is, the electric and the magnetic forces then act at right angles to each other.

By formulae (3) and (4) the relations (14) and (15) can also be written

$$X\alpha + Y\beta + Z\gamma = 0 \quad \text{......................} (16)$$

and

$$aP + bQ + cR = 0\,; \quad \text{......................} (17)$$

that is, both the electric moments and the magnetic forces and the magnetic moments and the electric forces act at right angles to one another.

Electric and Magnetic Oscillations at ⊥ for $M_1 = M_2 = M_3$.—By formulae (3) and (4) the relations (16) and (17) can also be written

$$\frac{Xa}{M_1} + \frac{Yb}{M_2} + \frac{Zc}{M_3} = 0 \quad \text{......................} (18)$$

and

$$\frac{Xa}{D_1} + \frac{Yb}{D_2} + \frac{Zc}{D_3} = 0\,; \quad \text{......................} (19)$$

which evidently give

$$Xa : Yb : Zc = \left(\frac{1}{M_3 D_2} - \frac{1}{M_2 D_3}\right) : \left(\frac{1}{M_1 D_3} - \frac{1}{M_3 D_1}\right) : \left(\frac{1}{M_2 D_1} - \frac{1}{M_1 D_2}\right).$$

Although it would follow from these relations that the electric and the magnetic oscillations do not take place at right angles to each other, we observe that this most general case, where $D_1 \gtrless D_2 \gtrless D_3$ and $M_1 \gtrless M_2 \gtrless M_3$, is only a theoretical one that is perhaps never realized (cf. above); until experiment has refuted the assumption that $M_1 = M_2 = M_3$, we are, therefore, surely justified in writing the relation (18) in the form

$$Xa + Yb + Zc = 0, \quad \text{.......................} (20)$$

and in thus maintaining that the electric and the magnetic oscillations always take place at right angles to each other.

The Particular Case $D_1 \gtrless D_2 = D_3$ and $M_1 \gtrless M_2 = M_3$.—Before we consider the most general empirical case, where $D_1 \gtrless D_2 \gtrless D_3$ and $M_1 = M_2 = M_3$, let us examine the simpler theoretical one, where $D_1 \gtrless D_2 = D_3$ and $M_1 \gtrless M_2 = M_3$, the most general form for the medium-constants in so-called "uniaxal" crystals (cf. p. 344); this particular case is not included in the most general empirical one, whereas it cannot be deduced from the most general theoretical case, where $D_1 \gtrless D_2 \gtrless D_3$ and $M_1 \gtrless M_2 \gtrless M_3$, since the latter is too complicated to admit of an explicit solution (cf. p. 341). Moreover, the brief examination of this particular case will throw light on the more complicated treatment of the most general empirical one.

For $D_1 \gtrless D_2 = D_3 = D$ and $M_1 \gtrless M_2 = M_3 = M$ the relations (18) and (19) can be written in the form

$$MXa + M_1(Yb + Zc) = 0$$

and

$$DXa + D_1(Yb + Zc) = 0;$$

which give

$$(D_1M - DM_1)Xa = 0$$

and

$$(DM_1 - D_1M)(Yb + Zc) = 0;$$

hence, since, in general, $(D_1M - DM_1) \gtrless 0$,

$$Xa = 0$$

and

$$Yb + Zc = 0$$

or

$$\left.\begin{aligned} Xa &= 0 \\ Xa + Yb + Zc &= 0 \end{aligned}\right\} \text{.............................} (21)$$

and

These relations, interpreted geometrically, state that the electric and the magnetic oscillations take place at right angles to each other and either the former or the latter in planes parallel to the yz-plane. Let us now examine these two possible cases.

Case 1: The Magnetic Oscillations Parallel to yz-Plane.—Here the direction of oscillation of the magnetic oscillations at any point P (cf. Fig. 37) will evidently be uniquely determined by the intersection of the given wave-front (cf. formula (13)) and the plane parallel to the

yz-plane passing through that point. The accompanying electric oscillations will evidently take place in the given wave-front (cf. formula (12)) at right angles to the magnetic oscillations (cf. formulae (21)).

Here $a=0$, and formulae (10) and (11) thus assume the particular form

$$\left.\begin{aligned} \frac{vD_1}{v_0}P &= \mu\gamma - \nu\beta, \\ \frac{vD}{v_0}Q = -\lambda\gamma, \quad & \frac{vD}{v_0}R = \lambda\beta \end{aligned}\right\} \qquad \text{......................(22)}$$

and

$$\left.\begin{aligned} 0 &= \nu Q - \mu R, \\ \frac{vM}{v_0}\beta = \lambda R - \nu P, \quad & \frac{vM}{v_0}\gamma = \mu P - \lambda Q. \end{aligned}\right\} \qquad \text{......................(23)}$$

The first two equations (23) give

$$\frac{vM}{v_0}\beta = \frac{\nu}{\mu}(\lambda Q - \mu P),$$

and this and the last equation:

$$\beta = -\frac{\nu}{\mu}\gamma.$$

Replace β by this value in formulae (22), and we have

$$\frac{vD_1}{v_0}P = \frac{\mu^2+\nu^2}{\mu}\gamma,$$

$$\frac{vD}{v_0}Q = -\lambda\gamma, \quad \frac{vD}{v_0}R = -\frac{\lambda\nu}{\mu}\gamma.$$

Lastly, replace P, Q, R by these values in formulae (23): the first equation then leads to an identity, whereas the other two both give one and the same conditional equation between the medium-constants, the direction-cosines λ, μ, ν of the normal to the given wave-front and the velocity of propagation v of the waves, namely

$$\frac{v^2M}{v_0^2} = \frac{\lambda^2}{D} + \frac{\mu^2+\nu^2}{D_1}.$$

Since now $\qquad \lambda^2+\mu^2+\nu^2=1,$

we can write this equation in the form

$$v^2 = \frac{v_0^2}{M}\left(\frac{\lambda^2}{D} + \frac{1-\lambda^2}{D_1}\right); \qquad \text{......................(24)}$$

that is, the velocity of propagation squared of the given waves, both electric and magnetic, will, on the assumption of the validity of Maxwell's equations (1) and (2), be given by this expression (24), which is a function of λ, D, D_1 and M (cf. also Ex. 2 at end of chapter).

It thus follows that the given crystalline medium is capable of transmitting electromagnetic plane-waves in any assigned direction (λ, μ, ν), provided, on the one hand, the magnetic oscillations take place at right angles to that direction and to the x-axis and the electric ones at right angles to that same direction and to the magnetic oscillations, whereby the directions of oscillation of both will be uniquely determined, and provided, on the other hand, both oscillations are propagated (in the assigned direction) with the velocity determined by formula (24).

Case 2: The Electric Oscillations Parallel to yz-Plane.—Here the direction of oscillation of the electric oscillations at any point will evidently be uniquely determined by the intersection of the wave-front and the plane parallel to the yz-plane passing through that point, whereas the accompanying magnetic oscillations will take place in that wave-front at right angles to the electric oscillations. The further treatment of this case is similar to that of the preceding one. In place of formulae (22) and (23) we evidently have

$$\left.\begin{aligned} 0 &= \mu\gamma - \nu\beta, \\ \frac{vD}{v_0} Q = \nu\alpha - \lambda\gamma, &\quad \frac{vD}{v_0} R = \lambda\beta - \mu\alpha \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

and

$$\left.\begin{aligned} \frac{vM_1}{v_0}\alpha &= \nu Q - \mu R, \\ \frac{vM}{v_0}\beta = \lambda R, &\quad \frac{vM}{v_0}\gamma = -\lambda Q. \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

The elimination of α, β, γ from equations (25) gives

$$Q = -\frac{\nu}{\mu} R.$$

On replacing Q by this value in formulae (26), we have

$$\frac{vM_1}{v_0}\alpha = -\frac{\mu^2 + \nu^2}{\mu} R,$$

$$\frac{vM}{v_0}\beta = \lambda R, \quad \frac{vM}{v_0}\gamma = \frac{\lambda\nu}{\mu} R.$$

Lastly, the substitution of these values for α, β, γ in formulae (25) leads to an identity and to the following conditional equation between the medium-constants, the direction-cosines λ, μ, ν and the velocity of propagation v:

$$v^2 = \frac{v_0^2}{D}\left(\frac{\lambda^2}{M} + \frac{1 - \lambda^2}{M_1}\right); \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

that is, the velocity of propagation squared of the given electric and magnetic waves will be given by this expression (27), which is a function of λ, D, M and M_1 (cf. also Ex. 3 at end of chapter).

It thus follows that the given crystalline medium is also capable of transmitting electromagnetic plane-waves in any assigned direction (λ, μ, ν), provided, on the one hand, the electric oscillations take place at right angles to that direction and to the x-axis and the magnetic ones at right angles to that same direction and to the electric oscillations, whereby their directions of oscillation will be uniquely determined, and provided, on the other hand, both oscillations are propagated with the velocity determined by formula (27).

Given Medium Capable of Transmitting Two Systems of Plane-Waves with Different Velocities of Propagation.—It follows from the above that the given medium is capable of transmitting electromagnetic plane-waves of two different directions of oscillation, as determined above, in any assigned direction, whereby the velocity of propagation will differ for those two directions of oscillation, being determined by formula (24), when the magnetic oscillations are taking place in planes parallel to the yz-plane, and by formula (27), when the electric oscillations are in those planes. We observe that for $M_1 = M_2 = M_3$ the expressions (24) and (27) for the velocities of propagation of the two possible systems of plane-waves that may be transmitted through the given medium in any assigned direction are the square roots of the quadratic equation (37) in v^2 for biaxal crystals (cf. p. 343) modified accordingly for uniaxal crystals.

For $M_1 = M$ formula (24) remains unaltered, whereas formula (27) reduces to

$$v^2 = \frac{v_0{}^2}{DM}.$$

It thus follows that for $M_1 = M$ (uniaxal crystals in current sense) the velocity of propagation of the electromagnetic waves of Case 2, where the electric oscillations are taking place in planes parallel to the yz-plane, will be entirely independent of their direction of propagation (λ, μ, ν); we observe that their velocity of propagation is then that of electromagnetic waves in a similarly constituted isotropic medium (insulator) $(D_1 = D_2 = D_3 = D)$.

For $D_1 = D$ formula (24) reduces to

$$v^2 = \frac{v_0{}^2}{DM},$$

whereas formula (27) remains unaltered. Here the velocity of propagation of the electromagnetic waves of Case 1, where the magnetic oscillations are taking place in planes parallel to the yz-plane, would also be entirely independent of their direction of propagation; that is, the given waves could be propagated in all directions through the medium with one and the same velocity, that of electromagnetic waves in a similarly constituted isotropic medium. We observe that of the two possible cases (media), $M_1 = M$, $D_1 \gtrless D$ and $D_1 = D$, $M_1 \gtrless M$, the latter is probably never realized.

Most General Case: $\mathbf{D}_1 \gtrless \mathbf{D}_2 \gtrless \mathbf{D}_3$.—Let us now consider the most general case, where

$$D_1 \gtrless D_2 \gtrless D_3 \text{ and } M_1 \gtrless M_2 \gtrless M_3,$$

examining thereby the conditions, under which electromagnetic linearly polarized plane-waves may be transmitted in any assigned direction (λ, μ, ν) through the crystalline medium defined by these values of D and M. For this purpose we shall eliminate first the *magnetic* and then the *electric* forces from our fundamental equations (1) and (2) respectively, and examine the electric and the magnetic oscillations separately. The elimination of α, β, γ from equations (1) and that of P, Q, R from equations (2) evidently give

$$\frac{D_1}{v_0}\frac{d^2P}{dt^2} = \frac{v_0}{M_2}\frac{d}{dz}\left(\frac{dP}{dz} - \frac{dR}{dx}\right) - \frac{v_0}{M_3}\frac{d}{dy}\left(\frac{dQ}{dx} - \frac{dP}{dy}\right)$$

and similar equations in Q and R, and

$$\frac{M_1}{v_0}\frac{d^2\alpha}{dt^2} = \frac{v_0}{D_3}\frac{d}{dy}\left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right) - \frac{v_0}{D_2}\frac{d}{dz}\left(\frac{d\gamma}{dx} - \frac{d\alpha}{dz}\right)$$

and similar equations in β and γ respectively.

Replace here the forces P, Q, R and α, β, γ by their respective moments X, Y, Z and a, b, c, and we have

$$\left.\begin{aligned}
\frac{1}{v_0^2}\frac{d^2X}{dt^2} &= \frac{1}{M_2}\frac{d}{dz}\left(\frac{1}{D_1}\frac{dX}{dz} - \frac{1}{D_3}\frac{dZ}{dx}\right) - \frac{1}{M_3}\frac{d}{dy}\left(\frac{1}{D_2}\frac{dY}{dx} - \frac{1}{D_1}\frac{dX}{dy}\right),\\
\frac{1}{v_0^2}\frac{d^2Y}{dt^2} &= \frac{1}{M_3}\frac{d}{dx}\left(\frac{1}{D_2}\frac{dY}{dx} - \frac{1}{D_1}\frac{dX}{dy}\right) - \frac{1}{M_1}\frac{d}{dz}\left(\frac{1}{D_3}\frac{dZ}{dy} - \frac{1}{D_2}\frac{dY}{dz}\right),\\
\frac{1}{v_0^2}\frac{d^2Z}{dt^2} &= \frac{1}{M_2}\frac{d}{dy}\left(\frac{1}{D_3}\frac{dZ}{dy} - \frac{1}{D_2}\frac{dY}{dz}\right) - \frac{1}{M_2}\frac{d}{dx}\left(\frac{1}{D_1}\frac{dX}{dz} - \frac{1}{D_3}\frac{dZ}{dx}\right)
\end{aligned}\right\} \quad (28)$$

and

$$\left.\begin{aligned}
\frac{1}{v_0^2}\frac{d^2a}{dt^2} &= \frac{1}{D_3}\frac{d}{dy}\left(\frac{1}{M_1}\frac{da}{dy} - \frac{1}{M_2}\frac{db}{dx}\right) - \frac{1}{D_2}\frac{d}{dz}\left(\frac{1}{M_3}\frac{dc}{dx} - \frac{1}{M_1}\frac{da}{dz}\right),\\
\frac{1}{v_0^2}\frac{d^2b}{dt^2} &= \frac{1}{D_1}\frac{d}{dz}\left(\frac{1}{M_2}\frac{db}{dz} - \frac{1}{M_3}\frac{dc}{dy}\right) - \frac{1}{D_3}\frac{d}{dx}\left(\frac{1}{M_1}\frac{da}{dy} - \frac{1}{M_2}\frac{db}{dx}\right),\\
\frac{1}{v_0^2}\frac{d^2c}{dt^2} &= \frac{1}{D_2}\frac{d}{dx}\left(\frac{1}{M_3}\frac{dc}{dx} - \frac{1}{M_1}\frac{da}{dz}\right) - \frac{1}{D_1}\frac{d}{dy}\left(\frac{1}{M_2}\frac{db}{dz} - \frac{1}{M_3}\frac{dc}{dy}\right).
\end{aligned}\right\} \quad \ldots.(29)$$

The Electric Oscillations.—In order that electromagnetic linearly polarized plane-waves may be transmitted through the given medium, the above expressions (6) and (9) for the moments X, Y, Z and a, b, c respectively must evidently satisfy the differential equations (28) and

(29). Let us first examine the electric oscillations; we replace X, Y, Z by their values (6) in formulae (28), and we find the following conditional equations between the direction-cosines λ, μ, ν, which are given, and those ξ, η, ζ and the velocity of propagation v, which are sought:

$$\frac{v^2}{{v_0}^2}\xi = \frac{\nu}{M_2}\left(\frac{\nu\xi}{D_1} - \frac{\lambda\zeta}{D_3}\right) - \frac{\mu}{M_3}\left(\frac{\lambda\eta}{D_2} - \frac{\mu\xi}{D_1}\right),$$

$$\frac{v^2}{{v_0}^2}\eta = \frac{\lambda}{M_3}\left(\frac{\lambda\eta}{D_2} - \frac{\mu\xi}{D_1}\right) - \frac{\nu}{M_1}\left(\frac{\mu\zeta}{D_3} \quad \frac{\nu\eta}{D_2}\right),$$

$$\frac{v^2}{{v_0}^2}\zeta = \frac{\mu}{M_1}\left(\frac{\mu\zeta}{D_3} - \frac{\nu\eta}{D_2}\right) - \frac{\lambda}{M_2}\left(\frac{\nu\xi}{D_1} - \frac{\lambda\zeta}{D_3}\right)$$

(cf. also formulae (B), Ex. 1, at end of chap.), which for $M_1 = M_2 = M_3 = M$ reduce to

$$\left.\begin{aligned} v^2\xi &= A^2(\mu^2+\nu^2)\xi - B^2\lambda\mu\eta - C^2\lambda\nu\zeta, \\ v^2\eta &= B^2(\lambda^2+\nu^2)\eta - C^2\mu\nu\zeta - A^2\lambda\mu\xi, \\ v^2\zeta &= C^2(\lambda^2+\mu^2)\zeta - A^2\lambda\nu\xi - B^2\mu\nu\eta, \end{aligned}\right\} \quad \text{.............(30)}$$

where $$A^2 = \frac{{v_0}^2}{D_1 M}, \quad B^2 = \frac{{v_0}^2}{D_2 M}, \quad C^2 = \frac{{v_0}^2}{D_3 M}. \quad \text{.............(30A)}$$

By the geometrical relation

$$\lambda^2 + \mu^2 + \nu^2 = 1,$$

these conditional equations (30) can be written in the form

$$(A^2 - v^2)\xi = \lambda f, \quad (B^2 - v^2)\eta = \mu f, \quad (C^2 - v^2)\zeta = \nu f, \quad \text{.........(31)}$$

where $$f = A^2\lambda\xi + B^2\mu\eta + C^2\nu\zeta. \quad \text{.........................(31A)}$$

We observe that for $M_1 \quad M_2 \gtrless M_3$ the conditional equations cannot be brought into this simple form (31) (cf. p. 336).

Let us now introduce into the above formulae that direction (of oscillation), which is at right angles both to the normal (λ, μ, ν) to the given wave-front and to the direction of oscillation (ξ, η, ζ) sought; if we denote its direction-cosines by ξ', η', ζ', the following geometrical relations will then evidently hold between these direction-cosines and those λ, μ, ν and ξ, η, ζ:

$$\xi' = \nu\eta - \mu\zeta, \quad \eta' = \lambda\zeta - \nu\xi, \quad \zeta' = \mu\xi - \lambda\eta; \quad \text{...............(32)}$$

these relations are now not independent of one another, but are evidently connected by the three familiar geometrical relations

$$\left.\begin{aligned} \lambda\xi + \mu\eta + \nu\zeta &= 0, \\ \lambda\xi' + \mu\eta' + \nu\zeta' &= 0, \\ \xi\xi' + \eta\eta' + \zeta\zeta' &= 0. \end{aligned}\right\} \quad \text{..........................(33)}$$

Multiplying now the above conditional equations (31), the first by ξ', the second by η' and the third by ζ', add, and we have

$$A^2\xi\xi' + B^2\eta\eta' + C^2\zeta\zeta' - v^2(\xi\xi' + \eta\eta' + \zeta\zeta') = (\lambda\xi' + \mu\eta' + \nu\zeta')f$$

or, by formulae (33),

$$A^2\xi\xi' + B^2\eta\eta' + C^2\zeta\zeta' = 0. \quad\text{.....................}(34)$$

Next, multiply the conditional equations (31), the first by ξ, the second by η and the third by ζ, add, and we have

$$A^2\xi^2 + B^2\eta^2 + C^2\zeta^2 - v^2(\xi^2+\eta^2+\zeta^2) = (\lambda\xi + \mu\eta + \nu\zeta)f$$

or, by formulae (33) and since $\xi^2+\eta^2+\zeta^2 = 1$,

$$v^2 = A^2\xi^2 + B^2\eta^2 + C^2\zeta^2, \quad\text{......................}(35)$$

an equation for the determination of the velocity of propagation v as function of A, B, C and ξ, η, ζ.

Lastly, write the conditional equations (31) in the form

$$\xi = \frac{\lambda}{A^2-v^2}f, \quad \eta = \frac{\mu}{B^2-v^2}f, \quad \zeta = \frac{\nu}{C^2-v^2}f, \quad\text{..............}(36)$$

multiply these, the first by λ, the second by μ and the third by ν, add, and we have

$$\lambda\xi + \mu\eta + \nu\zeta = \left(\frac{\lambda^2}{A^2-v^2} + \frac{\mu^2}{B^2-v^2} + \frac{\nu^2}{C^2-v^2}\right)f$$

or, by formulae (33),

$$0 = \frac{\lambda^2}{A^2-v^2} + \frac{\mu^2}{B^2-v^2} + \frac{\nu^2}{C^2-v^2}, \quad\text{.....................}(37)$$

a quadratic equation for the determination of v^2 as function of the medium constants A, B, C (cf. formulae (30A)) and the direction-cosines λ, μ, ν.

Two Directions of Oscillation with Different Velocities of Propagation; Determination of these Singular Directions and Velocities of Propagation.—The conditional equations (31) and hence the fundamental differential equations (1) and (2) will evidently be satisfied, when the above formulae (34)-(37) hold; these formulae will, therefore, serve, provided they can be satisfied, for the determination of the quantities sought, the possible directions of oscillation and the respective velocities of propagation in any assigned direction. It follows now from formula (37) that there will be two and only two possible velocities of propagation for waves transmitted through the given medium in any assigned direction (λ, μ, ν), and hence from formulae (36) that there will be two and only two possible directions of oscillation for waves propagated in that direction; on the other hand, it is evident from formula (34) that these two directions of oscillation are those whose direction-cosines are ξ, η, ζ and ξ', η', ζ', that is, the two possible directions of oscillation will be at right angles to each other and both at right angles to the normal (λ, μ, ν) to the given wave-front (cf. also pp. 335 and 336). These two possible directions of oscillation in any wave-front are now known as the "singular directions." By formula (34) the two singular directions (ξ, η, ζ) and (ξ', η', ζ') sought are now defined as the directions of conjugate diameters; on the other hand, since they are at right angles to each other (cf. formulae (33)), they will evidently be determined as the principal axes of the ellipse formed by the intersection of the plane

$$\lambda x+\mu y+\nu z=0, \quad \text{……………………………}(38)$$

and the ellipsoid

$$A^2x^2+B^2y^2+C^2z^2=1. \quad \text{……………………………}(39)$$

Lastly, it then follows from formula (35) that the velocity of propagation v of the oscillations parallel to either principal axis of this ellipse will be determined by the reciprocal value of that principal axis.

Fresnel's Construction.—It is evident from the above that electric plane-waves of two and only two directions of oscillation but with different velocities of propagation can be transmitted through a crystalline medium in any assigned direction; these two possible directions of oscillation or singular directions are determined by the principal axes of the ellipse formed by the intersection of the plane (38) parallel to the given wave-fronts and the ellipsoid (39), whereas the velocity of propagation of the oscillations parallel to either principal axis is determined by the reciprocal value of that axis. This method of determination of the singular directions and the corresponding velocities of propagation is known as "Fresnel's construction." It is evident that not only the singular directions but also the respective velocities of propagation will vary according to the direction of propagation chosen.

The Optical Axes: Biaxal and Uniaxal Crystals; the Ordinary and Extraordinary Oscillations.—It follows from Fresnel's construction for the determination of the two singular directions of oscillation and the respective velocities of propagation that there are, in general, two directions of propagation (λ, μ, ν) in a crystalline medium, each of which will give one and the same velocity of propagation for both singular directions; these two directions of propagation are now evidently determined by the normals to the two planes (38) that intersect the ellipsoid (39) in circles. As the circle has no principal axes, it will follow that the oscillations in these two particular wave-fronts may take place in any direction. The two directions of propagation thus characterized are now designated as the "optical axes" of the medium; crystals possessing two such optical axes are, therefore, termed "biaxal crystals. For $A=B$, $A=C$ or $B=C$ the ellipsoid (39) will degenerate to one of revolution, and the two planes (38) that intersect it in circles will evidently coincide with each other. In such crystals there will, therefore, be only one direction, in which both (all) oscillations will be propagated with one and the same velocity, whereby, as above, the oscillations themselves may take place in any direction at right angles to that direction; this direction of propagation evidently coincides with the axis of revolution of the given ellipsoid of revolution, which for $B=C$ is its x-axis. Crystals possessing only one such optical axis are, therefore, known as "uniaxal" crystals. It follows now from Fresnel's construction, modified for uniaxal crystals ($A>B=C$),* that the shorter axes of the ellipses, formed by the intersections of the planes (38) and the ellipsoid (39), will have one and the same length for all directions of propagation (λ, μ, ν), and hence that the oscillations that take place parallel to those axes will all be propagated with one and the same velocity (cf. also below); as this common velocity of propagation is now that of plane-waves transmitted through a similarly constituted isotropic medium, the oscillations propagated with that common velocity are, therefore, designated as the "ordinary" oscillations or waves. The other oscillations, or those that take place parallel to the longer axes of the ellipses, will be propagated with velocities that are proportional to the lengths of those axes (cf. above); since now, by Fresnel's construction, the length of the longer axis of any such ellipse evidently varies with the direction of propagation chosen, these oscillations will be propagated with different velocities, and they are thus known as the "extraordinary" oscillations or waves.

Actual Determination of the Velocities of Propagation of the Ordinary and Extraordinary Waves.—To determine analytically the velocities of propagation of the two possible systems of plane-waves that may be transmitted in any assigned direction through a biaxal crystal, we must solve the above quadratic equation (37) in v^2; multiplying it out, we have

$$\begin{aligned} v^4-[(B^2+C^2)\lambda^2+(A^2+C^2)\mu^2+(A^2+B^2)\nu^2]v^2 \\ +B^2C^2\lambda^2+A^2C^2\mu^2+A^2B^2\nu^2=0\,; \end{aligned}$$

which gives

$$\left.\begin{aligned} v^2=\tfrac{1}{2}\{(B^2+C^2)\lambda^2+(A^2+C^2)\mu^2+(A^2+B^2)\nu^2 \\ \pm\sqrt{[(B^2-C^2)\lambda^2-(A^2-C^2)\mu^2-(A^2-B^2)\nu^2]^2 \\ +4(A^2-C^2)(B^2-C^2)\lambda^2\mu^2}\} \end{aligned}\right\}. \quad (40)$$

Suppose now that $A^2>B^2>C^2$—we can evidently always choose our system of coordinates so that this be the case; the expression under the square-root sign will then be positive and hence its square-root always real, and the given quadratic equation will have two *real* roots. Before we consider further these two roots for biaxal crystals, let us examine the particular form assumed by formula (40) in uniaxal crystals. Take $B^2=C^2$, that is, the x-axis as optical axis; the expression for v^2 will then reduce to

$$v^2=\tfrac{1}{2}\{2C^2\lambda^2+(A^2+C^2)(\mu^2+\nu^2)\pm\sqrt{[-(A^2-C^2)(\mu^2+\nu^2)]^2}\}$$

---

* We are assuming here, as below, that $A>B(\geqq C)$.

or, by the geometrical relation $\lambda^2+\mu^2+\nu^2=1$, to

$$v^2=\tfrac{1}{2}\{2C^2\lambda^2+[(A^2+C^2)\pm(-A^2+C^2)](1-\lambda^2)\};$$

which gives the two simple values for v^2

$$\left.\begin{aligned} &v_1^2=C^2 \\ \text{and}\qquad &v_2^2=A^2-(A^2-C^2)\lambda^2. \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(40\text{A})$$

The former of these values is evidently the velocity of propagation squared of the ordinary waves (cf above); it corresponds to that root of the quadratic equation in v^2, where the positive sign has been chosen before the square-root sign in the general expression (40) for v^2; similarly it is easy to show that for $A^2=B^2$ the positive sign must be chosen before the square-root sign, in order that we may obtain the velocity of propagation of the ordinary waves. If we use this as criterion in discriminating between the two possible systems of waves, the ordinary and extraordinary (refracted) ones, in biaxal crystals $(A>B>C)$, the velocity of propagation squared of the former would be given by the expression

$$\left.\begin{aligned} v_0^2=\tfrac{1}{2}\{&(B^2+C^2)\lambda^2+(A^2+C^2)\mu^2+(A^2+B^2)\nu^2 \\ &+\sqrt{[(B^2-C^2)\lambda^2-(A^2-C^2)\mu^2-(A^2-B^2)\nu^2]^2 +4(A^2-C^2)(B^2-C^2)\lambda^2\mu^2}\} \end{aligned}\right\}\ldots(41)$$

and that of the latter by

$$\left.\begin{aligned} v_e^2=\tfrac{1}{2}\{&(B^2+C^2)\lambda^2+(A^2+C^2)\mu^2+(A^2+B^2)\nu^2 \\ &-\sqrt{[(B^2-C^2)\lambda^2-(A^2-C^2)\mu^2-(A^2-B^2)\nu^2]^2 +4(A^2-C^2)(B^2-C^2)\lambda^2\mu^2}\}. \end{aligned}\right\}\ldots(42)$$

Determination of Position of the Optical Axes.—We have already observed that in biaxal crystals there are two directions $(\lambda,\ \mu,\ \nu)$, known as the optical axes of the crystal, along either of which both (all) systems of waves, the ordinary and extraordinary ones, are propagated with one and the same velocity; this is, here

$$v_e^2=v_0^2,$$

which by formulae (41) and (42) can be written

$$[(B^2-C^2)\lambda^2-(A^2-C^2)\mu^2-(A^2-B^2)\nu^2]^2+4(A^2-C^2)(B^2-C^2)\lambda^2\mu^2=0,$$

where A, B, C are to be regarded as given and λ, μ, ν are sought.

Since now, by assumption, $A^2>B^2>C^2$, both terms of the left-hand member of this conditional equation will be positive; the given equation can, therefore, evidently be satisfied only when

$$(B^2-C^2)\lambda^2-(A^2-C^2)\mu^2-(A^2-B^2)\nu^2=0$$

and

$$4(A^2-C^2)(B^2-C^2)\lambda^2\mu^2=0.$$

That these two conditional equations may be satisfied, we must now evidently put either

$$\lambda_1^2=0 \quad\text{and}\quad (A^2-C^2)\mu_1^2-(A^2-B^2)\nu_1^2=0$$

or

$$\mu_2^2=0 \quad\text{and}\quad (B^2-C^2)\lambda_2^2-(A^2-B^2)\nu_2^2=0.$$

Together with the geometrical relation $\lambda^2+\mu^2+\nu^2=1$, these conditional equations between the direction-cosines λ, μ, ν sought and the medium constants A, B, C would evidently give the following values for the former:

$$\lambda_1^2=0, \qquad \mu_1^2=\frac{B^2-A^2}{B^2-C^2}, \qquad \nu_1^2=\frac{A^2-C^2}{B^2-C^2}$$

or

$$\lambda_2^2=\frac{A^2-B^2}{A^2-C^2}, \qquad \mu_2^2=0, \qquad \nu_2^2=\frac{B^2-C^2}{A^2-C^2}.$$

Since now, by assumption, $A^2>B^2>C^2$, μ_1^2 would be negative and hence μ_1 itself imaginary. The former values λ_1, μ_1, ν_1 for the direction-cosines λ, μ, ν would, therefore, have no physical meaning and must thus be rejected; the latter values are, therefore, the ones sought; we can write them in the form

$$\lambda=\pm\sqrt{\frac{A^2-B^2}{A^2-C^2}}, \qquad \mu=0, \qquad \nu=\pm\sqrt{\frac{B^2-C^2}{A^2-C^2}}, \quad\ldots\ldots\ldots\ldots(43)$$

where we have dropped the index 2. These values correspond to four directions (λ, μ, ν) of normal, two of which are evidently oppositely directed to the other two; they are now familiar to us as the expressions that determine the directions of the normals to the two circular cross-sections of the ellipsoid (39). Of the four directions determined by these values for λ, μ, ν it is customary to choose as optical axes two that make an acute angle with one another; let two such directions be

$$\left.\begin{aligned}\lambda_1&=\sqrt{\frac{A^2-B^2}{A^2-C^2}}, \qquad \mu_1=0, \qquad \nu_1=\sqrt{\frac{B^2-C^2}{A^2-C^2}}\\ \text{and}\qquad \lambda_2&=\sqrt{\frac{A^2-B^2}{A^2-C^2}}, \qquad \mu_2=0, \qquad \nu_2=-\sqrt{\frac{B^2-C^2}{A^2-C^2}}\end{aligned}\right\}\quad\ldots\ldots\ldots(43\text{A})$$

Observe that in uniaxal crystals ($A>B=C$) these values reduce to

$$\lambda=1, \qquad \mu=\nu=0,$$

the (positive) x-axis (cf. p. 344).

Velocity of Propagation along the Optical Axes.—Let us, next, determine the velocity of propagation of waves transmitted along either optical axis. We have just seen that this velocity is characterized by the vanishing of the square-root expression in the general formula (40) for v^2; the velocity sought will thus be given by the expression

$$v^2=\tfrac{1}{2}\left[(B^2+C^2)\lambda^2+(A^2+C^2)\mu^2+(A^2+B^2)\nu^2\right].$$

If we choose as optical axes the two determined by formulae (43A), we find, on replacing here λ, μ, ν by those values,

$$v_1{}^2 = v_2{}^2 = B^2; \qquad \text{..............} \ . \ \text{............}(44)$$

that is, all (both) waves will be propagated along both optical axes with one and the same velocity.

The Angles between Optical Axes and Normal to Wave-Front as Variables.—It is often convenient to refer our formulae to the optical axes instead of to the coordinate axes x, y, z employed above (cf. p. 331). For this purpose we introduce the angles u_1 and u_2, which the normal (λ, μ, ν) to the given wave-fronts makes with the optical axes. These angles are now evidently determined by the following expressions in terms of the direction-cosines λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 of the optical axes and those λ, μ, ν of the normal to wave-front:

$$\cos u_1 = \lambda\lambda_1 + \mu\mu_1 + \nu\nu_1$$

and

$$\cos u_2 = \lambda\lambda_2 + \mu\mu_2 + \nu\nu_2,$$

or, since by formulae (43) $\mu_1 = \mu_2 = 0$ here $(A > B > C)$,

$$\cos u_1 = \lambda\lambda_1 + \nu\nu_1$$

and

$$\cos u_2 = \lambda\lambda_2 + \nu\nu_2.$$

If we choose as optical axes the two determined by formulae (43A), we find, on replacing here λ_1, ν_1 and λ_2, ν_2 by those values,

$$\cos u_1 = \lambda\sqrt{\frac{A^2-B^2}{A^2-C^2}} + \nu\sqrt{\frac{B^2-C^2}{A^2-C^2}}$$

and

$$\cos u_2 = \lambda\sqrt{\frac{A^2-B^2}{A^2-C^2}} - \nu\sqrt{\frac{B^2-C^2}{A^2-C^2}};$$

which give

$$\left.\begin{aligned} \lambda &= \sqrt{\frac{A^2-C^2}{A^2-B^2}}\,\frac{\cos u_1 + \cos u_2}{2}, \\ \nu &= \sqrt{\frac{A^2-C^2}{B^2-C^2}}\,\frac{\cos u_1 - \cos u_2}{2}, \\ \text{hence} \qquad \mu &= \sqrt{1-\lambda^2-\nu^2}, \end{aligned}\right\} \qquad \text{..................}(45)$$

that is, λ, μ, ν expressed in terms of the angles u_1 and u_2 and the medium-constants (cf. formulae (30A)).

Velocity of Propagation expressed in Terms of u_1 and u_2.—To express the velocity of propagation (cf. formula (40)) in terms of the new variables u_1 and u_2, we must determine the two expressions

$$(B^2+C^2)\lambda^2 + (A^2+C^2)\mu^2 + (A^2+B^2)\nu^2$$

and $[(B^2-C^2)\lambda^2 - (A^2-C^2)\mu^2 - (A^2-B^2)\nu^2]^2 + 4(A^2-C^2)(B^2-C^2)\lambda^2\mu^2$

as functions of those variables. For this purpose we replace λ, μ, ν by their values (45) in terms of u_1 and u_2 in these expressions, and we have

$$\begin{aligned}(B^2+C^2)\lambda^2+(A^2+C^2)\mu^2+(A^2+B^2)\nu^2 &= (B^2+C^2)\frac{A^2-C^2}{A^2-B^2}p^2+(A^2+C^2)\\ &\quad\times\left[1-\frac{A^2-C^2}{A^2-B^2}p^2-\frac{A^2-C^2}{B^2-C^2}q^2\right]+(A^2+B^2)\frac{A^2-C^2}{B^2-C^2}q^2\\ &= A^2+C^2-(A^2-C^2)(p^2-q^2)\end{aligned}$$

and

$$\begin{aligned}&[(B^2-C^2)\lambda^2-(A^2-C^2)\mu^2-(A^2-B^2)\nu^2]^2+4(A^2-C^2)(B^2-C^2)\lambda^2\mu^2\\ &= \left\{(B^2-C^2)\frac{A^2-C^2}{A^2-B^2}p^2-(A^2-C^2)\left[1-\frac{A^2-C^2}{A^2-B^2}p^2-\frac{A^2-C^2}{B^2-C^2}q^2\right]\right.\\ &\qquad\qquad\left.-(A^2-B^2)\frac{A^2-C^2}{B^2-C^2}q^2\right\}^2\\ &\quad+4\frac{(A^2-C^2)^2(B^2-C^2)}{A^2-B^2}p^2\left[1-\frac{A^2-C^2}{A^2-B^2}p^2-\frac{A^2-C^2}{B^2-C^2}q^2\right]\\ &= (A^2-C^2)^2\left\{\left[\frac{A^2+B^2-2C^2}{A^2-B^2}p^2+q^2-1\right]^2\right.\\ &\qquad\left.+\frac{4(B^2-C^2)}{A^2-B^2}p^2\left[1-\frac{A^2-C^2}{A^2-B^2}p^2-\frac{A^2-C^2}{B^2-C^2}q^2\right]\right\}\\ &= (A^2-C^2)^2[(p^2-q^2)^2-2(p^2+q^2)+1],\end{aligned}$$

where $p=\dfrac{\cos u_1+\cos u_2}{2}$ and $q=\dfrac{\cos u_1-\cos u_2}{2}$,

or, on replacing p and q by their values,

$$(B^2+C^2)\lambda^2+(A^2+C^2)\mu^2+(A^2+B^2)\nu^2=A^2+C^2-(A^2-C^2)\cos u_1\cos u_2$$

and

$$\begin{aligned}&[(B^2-C^2)\lambda^2-(A^2-C^2)\mu^2-(A^2-B^2)\nu^2]^2+4(A^2-C^2)(B^2-C^2)\lambda^2\mu^2\\ &= (A^2-C^2)^2[\cos^2u_1\cos^2u_2-(\cos^2u_1+\cos^2u_2)+1]\\ &= (A^2-C^2)^2\sin^2u_1\sin^2u_2.\end{aligned}$$

Replace the given expressions by these in formula (40) for v^2, and we have

$$v^2=\frac{A^2+C^2}{2}-\frac{A^2-C^2}{2}\cos(u_1\pm u_2), \quad\ldots\ldots\ldots\ldots\ldots\ldots(46)$$

that is, the velocity of propagation expressed in terms of the angles u_1 and u_2, which the optical axes make with the normal to the given wave-fronts.

Expressed in terms of u_1 and u_2 the velocities of propagation of the ordinary and the extraordinary waves would, therefore, evidently be given by

$$\left.\begin{aligned}v_0^2&=\frac{A^2+C^2}{2}-\frac{A^2-C^2}{2}\cos(u_1+u_2)\\ \text{and}\quad v_e^2&=\frac{A^2+C^2}{2}-\frac{A^2-C^2}{2}\cos(u_1-u_2)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(46\text{A})$$

respectively (cf. p. 345).

The Magnetic Oscillations.—A similar treatment of the magnetic moments of formulae (29) shows that a crystalline medium is also capable of transmitting in any assigned direction (λ, μ, ν) two systems of magnetic waves, whose directions of oscillation are at right angles to the electric oscillations that can be transmitted in that direction and whose velocities of propagation are those of the respective electric waves (cf. Ex.'s 7 and 8 at end of chapter).

The Ray.—The rather abstract conception "ray" plays such an important part in the theory of light, that we should hardly feel justified in making no mention of it here; it is often introduced, because certain formulae, as those of reflection and refraction on the surface of a crystalline medium, assume simpler form, when referred to the "ray" than to the normal to wave-front (cf. pp. 366-367). We can now define the "ray" as the direction determined by the normal (taken in the direction of propagation of the wave) to the plane that passes through the direction of action of the resultant electric force P, Q, R and that of the resultant magnetic force α, β, γ or, if we assume that $M_1 = M_2 = M_3$, that of the resultant magnetic moment a, b, c.

Relative Position of Ray to Forces and Moments.—By formulae (12) and (13) both the electric and the magnetic oscillations X, Y, Z and a, b, c respectively at any point O (cf. Fig. 37) take place at right angles to the normal λ, μ, ν to the wave-front at that point, whereas, by formula (16), the resultant electric moment X, Y, Z and the resultant magnetic force α, β, γ make a right angle with each other, as roughly indicated in the annexed figure. If we assume that $M_1 = M_2 = M_3$, the resultant magnetic moment a, b, c will then be in the direction of action of the resultant magnetic force α, β, γ, that is, the latter will also lie in the given wave-front at right angles to the resultant electric moment. By

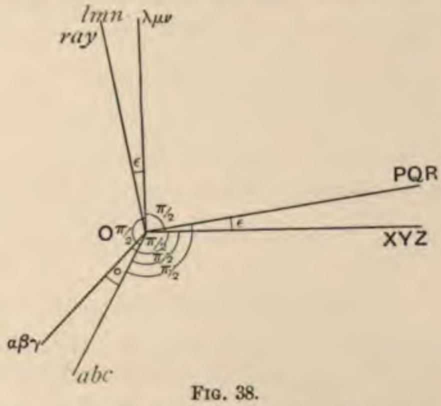

FIG. 38.

formula (17) the resultant electric force P, Q, R acts at right angles to the resultant magnetic moment a, b, c or, if $M_1 = M_2 = M_3$, to the resultant magnetic force α, β, γ; since now the latter (a, b, c) then ($M_1 = M_2 = M_3$) makes a right angle with the resultant electric moment X, Y, Z, it follows that the resultant electric force P, Q, R will lie in the plane that passes through the resultant electric moment X, Y, Z and the normal λ, μ, ν to the given wave-front; let us denote the angle, which the resultant electric force P, Q, R makes with the resultant electric moment X, Y, Z, by ϵ. By our above definition the ray is now determined by the normal to the plane that passes through the direction of action of the resultant electric force P, Q, R and that of the resultant magnetic force α, β, γ or moment a, b, c ($M_1 = M_2 = M_3$); since now the resultant electric force P, Q, R lies in the plane that passes through the resultant electric moment X, Y, Z and the normal λ, μ, ν to the given wave-front, making the angle ϵ with the former (direction), and the resultant magnetic force or moment is normal to this plane, the ray will lie in that same plane, making the same angle ϵ with the normal λ, μ, ν to the given wave-front. The ray evidently stands in the same relation to the electric force as the normal to the wave-front to the electric moment.

Determination of the Angle ϵ between Ray and Normal to Wave-Front.—Let us now determine the angle ϵ between the ray and the normal (λ, μ, ν) to wave-front, which, as we have seen above, is identical to the angle, which the resultant electric force P, Q, R makes with the resultant electric moment X, Y, Z. If we denote the direction-cosines of the resultant electric force referred to the principal axes of the crystalline medium by p, q, r, we evidently have

$$p = \frac{P}{\sqrt{P^2 + Q^2 + R^2}}, \quad q = \frac{Q}{\sqrt{P^2 + Q^2 + R^2}}, \quad r = \frac{R}{\sqrt{P^2 + Q^2 + R^2}}.$$

Replace here the component forces P, Q, R by their values (7) for the plane-waves (6) that can be transmitted in the assigned direction (λ, μ, ν) through the given medium, and we find, by formulae (30A),

$$\left.\begin{aligned} p &= \frac{A^2\xi}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}}, \quad q = \frac{B^2\eta}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}}, \\ r &= \frac{C^2\zeta}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}}, \end{aligned}\right\} \quad \text{.........(47)}$$

that is, p, q, r expressed in terms of ξ, η, ζ, the direction-cosines of one of the two possible (singular) directions of oscillation in the given wave-fronts, and the medium constants A, B, C. These relations between p, q, r, ξ, η, ζ, and A, B, C evidently give

$$p\xi + q\eta + r\zeta = \frac{A^2\xi^2 + B^2\eta^2 + C^2\zeta^2}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}}.$$

The left-hand member of this equation is now the analytic expression for the cosine of the angle between the two directions P, Q, R and X, Y, Z, that is, the above angle ϵ; we thus have

$$\cos\epsilon = \frac{A^2\xi^2 + B^2\eta^2 + C^2\zeta^2}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}}$$

or, by formula (35),

$$\cos\epsilon = \frac{v^2}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}}. \quad \text{......................(48)}$$

If we denote the velocity of propagation of the ray by v_r, we can evidently express $\cos\epsilon$ in the form $\frac{v}{v_r}$ and thus write

$$\frac{v}{v_r} = \frac{v^2}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}},$$

hence

$$v^2 v_r^2 = A^4\xi^2 + B^4\eta^2 + C^4\zeta^2. \quad \text{........................(48A)}$$

Derivation of Formulae for Ray.—To obtain formulae, where the quantities are referred to the ray (l, m, n) instead of to the normal (λ, μ, ν) to wave-front, we must replace ξ, η, ζ in the formulae already found by their values from formulae (47) in terms of p, q, r and the medium constants, whereas in those formulae, where v appears, we must introduce the velocity of propagation v_r of the ray in its place.

Replace, first, ξ, η, ζ by their values from formulae (47) in formula (35), and we have

$$v^2 = \left(\frac{p^2}{A^2} + \frac{q^2}{B^2} + \frac{r^2}{C^2}\right)(A^4\xi^2 + B^4\eta^2 + C^4\zeta^2),$$

which by formula (48A) can be written

$$v^2 = \left(\frac{p^2}{A^2} + \frac{q^2}{B^2} + \frac{r^2}{C^2}\right) v^2 v_r^2,$$

hence

$$\frac{1}{v_r^2} = \frac{p^2}{A^2} + \frac{q^2}{B^2} + \frac{r^2}{C^2}, \quad \text{........ ....(49)}$$

an equation for the determination of v_r in terms of p, q, r and A, B, C.

Next, replace ξ, η, ζ by their values (47) in the geometrical relation

$$\xi\xi' + \eta\eta' + \zeta\zeta' = 0$$

(cf. formulae (33)) and in formulae (34), and we have

$$\left.\begin{aligned} &\frac{p\xi'}{A^2} + \frac{q\eta'}{B^2} + \frac{r\zeta'}{C^2} = 0 \\ \text{and} \quad &p\xi' + q\eta' + r\zeta' = 0\,; \end{aligned}\right\} \quad \text{..........................(50)}$$

the latter states that the resultant electric force acts at right angles to the resultant magnetic moment or force ($M_1 = M_2 = M_3$).

It follows now from formulae (49) and (50) and considerations similar to those on pp. 342-343, that the two (singular) directions p, q, r and $\xi' \eta', \zeta'$ of action of the electric force for any assigned direction (l, m, n) of ray will be determined by the principal axes of the ellipse intersected on the plane

$$lx+my+nz=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51)$$

by the ellipsoid

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}=1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52)$$

whereas the velocity of propagation of the respective ray v_r will be given by the length of one of the principal axes of that ellipse (cf. below).

Determination of (Singular) Directions of Force for any given Ray (l, m, n); **Fresnel's Construction.**—We observe that the ellipsoid (52), like that (39) employed for the determination of the two singular directions of (electric) oscillation, is determined alone by the values of the medium-constants A, B, C; its principal axes evidently coincide in direction with those of the ellipsoid (39), whereas the lengths of these axes are the reciprocals of those of the latter ellipsoid (39); these two ellipsoids are, therefore, known as "reciprocal" ellipsoids. The determination of the two possible directions of action of the electric force in any crystalline medium corresponding to any given ray (l, m, n) is evidently similar to that of the two singular directions of (electric) oscillation in any given wave-front (λ, μ, ν) and is effected by Fresnel's construction (cf. p. 343): we lay namely the plane (51), to which the given ray is normal, through the centre of the ellipsoid (52) and seek the two principal axes of the ellipse intersected on that plane by that ellipsoid; these principal axes then give the two possible directions of action of the electric force for the given ray (l, m, n), whereas the velocity of propagation of the ray corresponding to the direction of action of the force along either principal axis is determined by the length of that axis.

Equation between Velocity of Propagation of Ray, its Direction-Cosines and the Medium Constants.—The derivation of the quadratic equation for the determination of v_r^2 in terms of the direction-cosines l, m, n of the ray and the medium-constants A, B, C corresponding to formula (37), where v^2 is determined as a function of λ, μ, ν and A, B, C, offers certain difficulties, for it involves several purely analytical transformations.

We start from formulae (31) and (31A); by formulae (47) we can now write the latter in the form

$$f=A^2\lambda\xi+B^2\mu\eta+C^2\nu\zeta=(\lambda p+\mu q+\nu r)\sqrt{A^4\xi^2+B^4\eta^2+C^4\zeta^2}$$

or, by figure 38 and formula (48A),

$$f = \cos(90^\circ - \epsilon)\, vv_r = vv_r \sin\epsilon$$

or, since

$$\epsilon = \arccos\frac{v}{v_r},$$

$$f = v\sqrt{v_r^2 - v^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

Replace f by this value in formulae (31), and we have

$$(A^2 - v^2)\,\xi = \lambda v\sqrt{v_r^2 - v^2},$$
$$(B^2 - v^2)\,\eta = \mu v\sqrt{v_r^2 - v^2},$$
$$(C^2 - v^2)\,\zeta = \nu v\sqrt{v_r^2 - v^2}.$$

Since now the ray (l, m, n), the direction of action of the resultant electric force (p, q, r) and the resultant magnetic moment (ξ', η', ζ') all make right angles with one another, the following analytic relations will hold between their direction-cosines:

$$l = r\eta' - q\zeta', \quad m = p\zeta' - r\xi', \quad n = q\xi' - p\eta'$$

(cf. also formulae (32)). Replace here p, q, r by their values from formulae (47), and we have

$$l = \frac{C^2\zeta\eta' - B^2\eta\zeta'}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}}$$

and similar expressions for m and n, hence, by formula (48A),

$$vv_r l = C^2\zeta\eta' - B^2\eta\zeta'$$

and similarly

$$vv_r m = A^2\xi\zeta' - B^2\zeta\xi'$$

and

$$vv_r n = B^2\eta\xi' - A^2\xi\eta'.$$

Next, replace here ξ', η', ζ' by their values (32) in terms of λ, μ, ν and ξ, η, ζ, and we can write these relations in the form

$$\begin{aligned} vv_r l &= C^2\zeta(\lambda\zeta - \nu\xi) - B^2\eta(\mu\xi - \lambda\eta) \\ &= \lambda(A^2\xi^2 + B^2\eta^2 + C^2\zeta^2) - \xi(A^2\lambda\xi + B^2\mu\eta + C^2\nu\zeta) \end{aligned}$$

or, by formulae (35) and (53) (cf. formula (31A)),

$$vv_r l = \lambda v^2 - \xi v\sqrt{v_r^2 - v^2},$$

hence

$$v_r l = \lambda v - \xi\sqrt{v_r^2 - v^2}$$

and similarly

$$v_r m = \mu v - \eta\sqrt{v_r^2 - v^2}$$

and

$$v_r n = \nu v - \zeta\sqrt{v_r^2 - v^2},$$

that is, λ, μ, ν expressed in terms of l, m, n, ξ, η, ζ and the v's.

Replace now λ, μ, ν by their values from these last relations in the above form of formulae (31), and we have

$$(A^2 - v^2)\,\xi = \sqrt{v_r^2 - v^2}\left(v_r l - \xi\sqrt{v_r^2 - v^2}\right),$$

hence

$$(A^2 - v_r^2)\,\xi = v_r l\sqrt{v_r^2 - v^2}$$

and similarly

$$(B^2 - v_r^2)\,\eta = v_r m\sqrt{v_r^2 - v^2}$$

and

$$(C^2 - v_r^2)\,\zeta = v_r n\sqrt{v_r^2 - v^2}$$

or

$$\xi = \frac{lv_r\sqrt{v_r^2 - v^2}}{A^2 - v_r^2}, \quad \eta = \frac{mv_r\sqrt{v_r^2 - v^2}}{B^2 - v_r^2}, \quad \zeta = \frac{nv_r\sqrt{v_r^2 - v^2}}{C^2 - v_r^2}.$$

that is, ξ, η, ζ expressed in terms of l, m, n, the v's and the mediu constants.

Replace ξ, η, ζ by these values in formulae (47) for p, q, r, and have

$$p = \frac{A^2}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}} \frac{lv_r\sqrt{v_r^2 - v^2}}{A^2 - v_r^2},$$

$$q = \frac{B^2}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}} \frac{mv_r\sqrt{v_r^2 - v^2}}{B^2 - v_r^2},$$

$$r = \frac{C^2}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}} \frac{nv_r\sqrt{v_r^2 - v^2}}{C^2 - v_r^2}.$$

Lastly, multiply these equations, the first by l, the second by m a the third by n, add, and we have

$$lp + mq + nr = \left(\frac{A^2l^2}{A^2 - v_r^2} + \frac{B^2m^2}{B^2 - v_r^2} + \frac{C^2n^2}{C^2 - v_r^2}\right) \frac{v_r\sqrt{v_r^2 - v^2}}{\sqrt{A^4\xi^2 + B^4\eta^2 + C^4\zeta^2}}$$

or, since $lp + mq + nr = 0$ (cf. pp. 349–350),

$$0 = \frac{A^2l^2}{A^2 - v_r^2} + \frac{B^2m^2}{B^2 - v_r^2} + \frac{C^2n^2}{C^2 - v_r^2}, \quad \dots\dots\dots\dots\dots\dots (5$$

the quadratic equation in v_r^2 sought; by this equation we can det mine the two possible velocities of propagation of ray for any assign direction (l, m, n) of the same.

Reciprocal Relations between Ray and Normal to Wave-Front. A comparison of the above equations (49)–(54) between the for and the ray with the foregoing ones between the moments and t normal to wave-front reveals a certain reciprocal relation betwe these two sets of quantities. It is evident that to obtain a formula between the forces and the ray, we have only to make t following substitutions in the respective formula between the mome and the normal to wave-front:

$$\begin{array}{lllll} \lambda\mu\nu, & \xi\eta\zeta, & \xi'\eta'\zeta', & v, & A^2B^2C^2, \\ lmn, & pqr, & \xi'\eta'\zeta', & \dfrac{1}{v_r}, & \dfrac{1}{A^2}\dfrac{1}{B^2}\dfrac{1}{C^2}. \end{array}$$

Reflection and Refraction.—The manifold electromagnetic phe mena exhibited by crystals are to be ascribed indirectly alm exclusively to the peculiar behaviour of electromagnetic waves u reflection and refraction on their surface. We shall, therefore, con our further treatment of electromagnetic waves in crystalline me to the examination of the laws of reflection and refraction on the surface of crystals and their subsequent behaviour, that of the reflected and refracted waves, within the same. We shall examine here as above only the non-conducting crystalline media (cf. p. 330). We choose the dividing-surface between any two such media, for example, the cleavage surfaces of two crystals pressed closely together or the cleavage surface of a crystal placed in an isotropic medium (air), as yz-plane of a system of rectangular coordinates x, y, z with origin at that point on the same, where the given electromagnetic waves are incident. The principal axes (D_1, D_2, D_3) of either medium will not, in general, coincide with these axes; henceforth let us, therefore, denote the former axes by x', y', z' (x_1', y_1', z_1') and retain x, y, z for the latter.

Maxwell's Equations for Aeolotropic Insulators.—The electromagnetic state in an aeolotropic insulator or crystal referred to any system of rectangular coordinates x, y, z is now defined by the differential equations

$$\left.\begin{aligned} \frac{4\pi}{v_0}\frac{dX}{dt}=\frac{d\beta}{dz}-\frac{d\gamma}{dy}, \quad \frac{4\pi}{v_0}\frac{dY}{dt}=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}, \\ \frac{4\pi}{v_0}\frac{dZ}{dt}=\frac{d\alpha}{dy}-\frac{d\beta}{dx} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(55)$$

and

$$\left.\begin{aligned} \frac{4\pi}{v_0}\frac{da}{dt}=\frac{dR}{dy}-\frac{dQ}{dz}, \quad \frac{4\pi}{v_0}\frac{db}{dt}=\frac{dP}{dz}-\frac{dR}{dx}, \\ \frac{4\pi}{v_0}\frac{dc}{dt}=\frac{dQ}{dx}-\frac{dP}{dy} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(56)$$

(cf. formulae (1 and 2, I.)), where, however, the electric moments are given by the expressions

$$\left.\begin{aligned} X&=\frac{1}{4\pi}(D_{11}P+D_{12}Q+D_{13}R), \\ Y&=\frac{1}{4\pi}(D_{21}P+D_{22}Q+D_{23}R), \\ Z&=\frac{1}{4\pi}(D_{31}P+D_{32}Q+D_{33}R) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(57)$$

(cf. formulae (9, I.)); these D's are functions of D_1, D_2, D_3, the constants of electric induction along the principal axes x', y', z' of the crystal (cf. p. 331), and the cosines of the angles between these axes and the coordinate-axes x, y, z; D_{11} and D_{12} are, for example, given by the expressions

$$D_{11}=D_1\cos^2(x', x)+D_2\cos^2(x', y)+D_3\cos^2(x', z),$$
$$D_{12}=D_1\cos(x', x)\cos(y', x)+D_2\cos(x', y)\cos(y', y)+D_3\cos(x', z)\cos(y', z)$$

(cf. p. 8). Since we shall assume here as above that the simple relations (4, I.) will hold between the ma and the magnetic forces.

The Surface-Conditions.—Let now the system of coo to which equations (55)–(57) are referred, be that partic above, where the dividing-surface between the two cr as yz-plane, etc.; our differential equations (55) and evidently assume the following familiar form on that su

$$\left.\begin{array}{lll} X_1 = X_0, & Q_1 = Q_0, & R_1 = R_0, \\ \alpha_1 = \alpha_0, & \beta_1 = \beta_0, & \gamma_1 = \gamma_0 \end{array}\right\} \cdots\cdots$$

(cf. formulae (3, VII.)), where the index 0 or 1 de component moment or force to be taken is that or tl acting in the one (0) or the other (1) crystal respectivel

Incident (Electric) Plane Waves and the Reflected Waves.—As above, let us represent the component el or oscillations X, Y, Z in incident plane-waves that can in any assigned direction (λ, μ, ν) through the crystal (cf. Fig. 39 below) by the functions

$$\left.\begin{array}{l} X = a\xi e^{in\left(t+\frac{s}{v}\right)}, \\ Y = a\eta e^{in\left(t+\frac{s}{v}\right)}, \\ Z = a\zeta e^{in\left(t+\frac{s}{v}\right)}, \end{array}\right\} \cdots\cdots\cdots\cdots\cdots$$

where $$s = \lambda x + \mu y + \nu z \quad \text{and} \quad n = \frac{2\pi v}{\lambda} = \frac{2\pi}{T}$$

(cf. p. 267), ξ, η, ζ denoting the direction-cosines of (possible (singular) directions of oscillation in the gi (λ, μ, ν). Not only a and λ, μ, ν, but also ξ, η, ζ and can be determined by Fresnel's construction (cf. p. 343 direction of propagation (λ, μ, ν), are to be regarded The displacement in any wave-front may of course direction, but, if that direction does not coincide w two singular directions peculiar to that wave-front (cl its components parallel to those singular directions will as permanent waves, and, as we have seen above, each velocity. This evidently accounts for the familiar bifur or rays upon entering a crystalline medium. If both tropic, incident waves represented by the functions (59 shall denote the direction-cosines of one of the two sin in the given wave-fronts (λ, μ, ν), will evidently, bifurcated both upon refraction and upon reflection hand, if the medium (0) of the incident waves is isotropic, there will evidently be, in general, two refracted waves and only one reflected one. Let us first examine the most general case, where both media are aeolotropic. Incident waves represented by the functions (59) will then give rise to two reflected and two refracted waves of the following form:

$$\left.\begin{aligned} X_0 &= a_0\xi_0 e^{in\left(t-\frac{s_0}{v_0}\right)}, \\ Y_0 &= a_0\eta_0 e^{in\left(t-\frac{s_0}{v_0}\right)}, \\ Z_0 &= a_0\zeta_0 e^{in\left(t-\frac{s_0}{v_0}\right)}, \end{aligned}\right\} \qquad \text{(60)*}$$

where $$s_0 = \lambda_0 x + \mu_0 y + \nu_0 z, \qquad \text{(60A)}$$

and

$$\left.\begin{aligned} X_e &= a_e\xi_e e^{in\left(t-\frac{s_e}{v_e}\right)}, \\ Y_e &= a_e\eta_e e^{in\left(t-\frac{s_e}{v_e}\right)}, \\ Z_e &= a_e\zeta_e e^{in\left(t-\frac{s_e}{v_e}\right)}, \end{aligned}\right\} \qquad \text{(61)}$$

where $$s_e = \lambda_e x + \mu_e y + \nu_e z, \qquad \text{(61A)}$$

the ordinary and extraordinary reflected waves respectively, and

$$\left.\begin{aligned} X_0' &= a_0'\xi'_0 e^{in\left(t-\frac{s'_0}{v'_0}\right)}, \\ Y_0' &= a_0'\eta_0' e^{in\left(t-\frac{s'_0}{v'_0}\right)}, \\ Z_0' &= a_0'\zeta_0' e^{in\left(t-\frac{s'_0}{v'_0}\right)}, \end{aligned}\right\} \qquad \text{(62)*}$$

where $$s_0' = \lambda_0' x + \mu_0' y + \nu_0' z, \qquad \text{(62A)}$$

and

$$\left.\begin{aligned} X_e' &= a_e'\xi_e' e^{in\left(t-\frac{s'_e}{v'_e}\right)}, \\ Y_e' &= a_e'\eta_e' e^{in\left(t-\frac{s'_e}{v'_e}\right)}, \\ Z_e' &= a_e'\zeta_e' e^{in\left(t-\frac{s'_e}{v'_e}\right)}, \end{aligned}\right\} \qquad \text{(63)}$$

where $$s_e' = \lambda_e' x + \mu_e' y + \nu_e' z, \qquad \text{(63A)}$$

the ordinary and extraordinary refracted waves respectively.

If we can determine the λ, μ, ν's of the above functions (60)–(63A) for the reflected and refracted waves, we can find by Fresnel's construction (cf. p. 343) the respective ξ, η, ζ's and v's; that is, the latter quantities may be regarded here as known, provided the former can be determined.

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* Cf. foot-note, p. 358.

Determination of the λ, μ, ν's and a's of Reflected and Refracted Waves from Surface-Conditions.—To determine the λ, μ, ν's and the a's of formulae (60)-(63A), we make use of our surface-conditions (58), which can be written in the explicit form

$$\left.\begin{aligned} X_0'+X_e' &= X+X_0+X_e \\ Q_0'+Q_e' &= Q+Q_0+Q_e \\ R_0'+R_e' &= R+R_0+R_e \end{aligned}\right\} \ldots\ldots(64)^* \qquad \left.\begin{aligned} a_0'+a_e' &= a+a_0+a_e \\ \beta_0'+\beta_e' &= \beta+\beta_0+\beta_e \\ \gamma_0'+\gamma_e' &= \gamma+\gamma_0+\gamma_e \end{aligned}\right\} \ldots\ldots(65)^*$$

Replace the X's by their values on the given dividing surface, $x=0$, in the first condition (64), and we evidently have

$$\left.\begin{aligned} a_0'\xi_0'e^{-in\left(\frac{\mu_0'y+\nu_0'z}{v_0'}\right)}+a_e'\xi_e'e^{-in\left(\frac{\mu_e'y+\nu_e'z}{v_e'}\right)} \\ =a\xi e^{+in\left(\frac{\mu y+\nu z}{v}\right)}+a_0\xi_0 e^{-in\left(\frac{\mu_0 y+\nu_0 z}{v_0}\right)}+a_e\xi_e e^{-in\left(\frac{\mu_e y+\nu_e z}{v_e}\right)}. \end{aligned}\right\} \ldots\ldots(66)$$

This condition must now hold for all values of y and z, that is, at all points on the given dividing surface; this is evidently only possible, when both

$$-\frac{\mu_0'}{v_0'}=-\frac{\mu_e'}{v_e'}=\frac{\mu}{v}=-\frac{\mu_0}{v_0}=-\frac{\mu_e}{v_e}$$

and

$$-\frac{\nu_0'}{v_0'}=-\frac{\nu_e'}{v_e'}=\frac{\nu}{v}=-\frac{\nu_0}{v_0}=-\frac{\nu_e}{v_e}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(66\text{A})$$

The above formulae will now assume a much simpler form, if we lay the plane of incidence (of the incident waves) in either the xy or the xz-plane of the above system of coordinates x, y, z; let us choose here, as in Chapter VII., the former as plane of incidence. We observe that the generality of the given problem will in no way be affected by this choice of the plane of incidence, for the only restriction put upon the above system of coordinates was that the x-axis be normal to the given dividing surface, whereas the y and z-axes were left entirely arbitrary in that surface.

***xy*-Plane as Incidence-Plane.**—For the xy-plane as plane of incidence $\nu=0$, and the latter of the two conditional relations (66A) will thus assume the form

$$\nu_0'=\nu_e'=\nu_0=\nu_e=0\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(67)$$

that is, both the reflected and the refracted waves will also be propagated in the plane of incidence, the xy-plane.

If we now denote the angle, which the normal to the wave-front of any incident wave makes with the x-axis, here the angle of incidence, by ϕ, and the angles, which the normals to the wave-fronts of the two reflected and the two refracted waves make with the positive and negative x-axis respectively, here the angles of reflection and refraction, by ϕ_0, ϕ_e and ϕ_0', ϕ_e' respectively, as indicated in the annexed figure, we can then write the direction-cosines λ μ, λ_0 μ_0, etc. in the form

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* Here the index o (*o* in Fig. 39) referring to the *ordinary* waves, although similarly written, is not to be confounded with that zero (0) employed above (cf. formulae (58)); as it is always evident which index is referred to, we shall attempt no further discrimination in orthography.

$$\left.\begin{array}{llll} \lambda=\cos\phi, & \mu=-\sin\phi, & & \\ \lambda_0=\cos\phi_0, & \mu_0=\sin\phi_0, & \lambda_e=\cos\phi_e, & \mu_e=\sin\phi_e, \\ \lambda_0'=-\cos\phi_0', & \mu_0'=\sin\phi_0', & \lambda_e'=-\cos\phi_e', & \mu_e'=\sin\phi_e' \end{array}\right\} \quad \ldots\ldots(68)$$

(cf. the annexed figure).

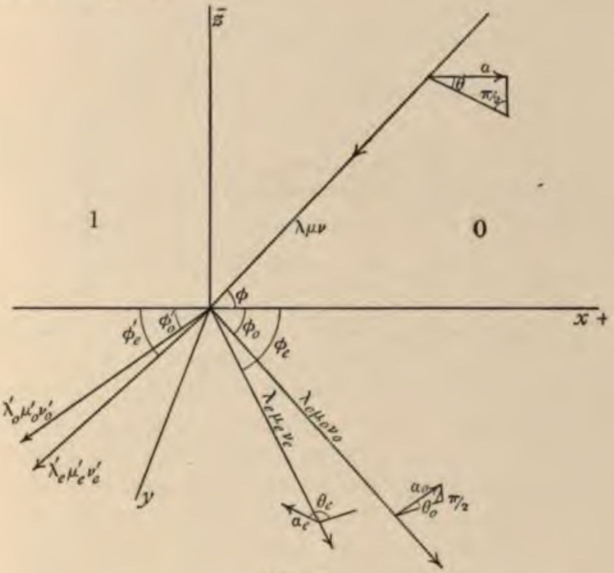

FIG. 39.

Laws of Reflection and Refraction.—On replacing the μ's by their values (68) in the former conditional relation (66A), we have

$$\frac{\sin\phi_0'}{v_0'}=\frac{\sin\phi_e'}{v_e'}=\frac{\sin\phi}{v}=\frac{\sin\phi_0}{v_0}=\frac{\sin\phi_e}{v_e}, \ldots\ldots\ldots\ldots\ldots\ldots \quad (69)$$

the familiar relation (laws) between the angles of incidence, reflection and refraction and the velocities of propagation of the incident, reflected and refracted waves.

Oscillations in Incident, Reflected and Refracted Waves referred to Incidence-Plane; the Azimuth of Oscillation.—Since the above incident, reflected and refracted waves all lie in one and the same plane, the plane of incidence, we can refer their directions or planes of oscillation to that plane. The angle, which the direction or plane of oscillation in any wave makes with its incidence plane, is now known as its "azimuth" of oscillation. If we denote the azimuths of the above incident, the two reflected and the two refracted waves by θ, θ_0 θ_e and θ_0' θ_e' respectively, we can evidently write their component moments in the form

$$\left.\begin{array}{l}
X = a\cos\theta\sin\phi\, e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)},\\
Y = a\cos\theta\cos\phi\, e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)},\\
Z = a\sin\theta\, e^{in\left(t - \frac{y\sin\phi - x\cos\phi}{v}\right)},\\
X_0 = a_0\cos\theta_0\sin\phi_0\, e^{in\left(t - \frac{y\sin\phi_0 + x\cos\phi_0}{v_0}\right)},\\
Y_0 = -a_0\cos\theta_0\cos\phi_0\, e^{in\left(t - \frac{y\sin\phi_0 + x\cos\phi_0}{v_0}\right)},\\
Z_0 = a_0\sin\theta_0\, e^{in\left(t - \frac{y\sin\phi_0 + x\cos\phi_0}{v_0}\right)},\\
\text{with similar expressions for the component moments}\\
\text{of the extraordinary reflected wave, and}\\
X_0' = a_0'\cos\theta_0'\sin\phi_0'\, e^{in\left(t - \frac{y\sin\phi'_0 - x\cos\phi'_0}{v'_0}\right)},\\
Y_0' = a_0'\cos\theta_0'\cos\phi_0'\, e^{in\left(t - \frac{y\sin\phi'_0 - x\cos\phi'_0}{v'_0}\right)},\\
Z_0' = a_0'\sin\theta_0'\, e^{in\left(t - \frac{y\sin\phi'_0 - x\cos\phi'_0}{v'_0}\right)},
\end{array}\right\}\quad\dots\dots\dots\dots(70)$$

with similar expressions for the component moments of the extraordinary refracted wave (cf. Fig. 39 and formulae (60)-(63A) and (68)).

The Incident Magnetic Waves; their Amplitude of Oscillation.—Since, by assumption, $M_1 = M_2 = M_3$ in either crystal, the above electric and the accompanying magnetic oscillations will take place at right angles to each other (cf. formulae (20)); we can, therefore, represent the component moments of the magnetic oscillations that accompany the incident electric ones (59) by the functions

$$\left.\begin{array}{l}
a = A\xi' e^{in\left(t + \frac{s}{v}\right)},\\
b = A\eta' e^{in\left(t + \frac{s}{v}\right)},\\
c = A\zeta' e^{in\left(t + \frac{s}{v}\right)},
\end{array}\right\}\quad\dots\dots\dots\dots\dots\dots(71)$$

where A denotes their amplitude of oscillation, and ξ', η', ζ', their direction-cosines of oscillation, are connected with the direction-cosines ξ, η, ζ of electric oscillation and those λ, μ, ν of normal to wave-front by the relations

$$\xi' = \nu\eta - \mu\zeta,\quad \eta' = \lambda\zeta - \nu\xi,\quad \zeta' = \mu\xi - \lambda\eta$$

(cf. formulae (32)). Replace in these relations λ, μ,' ν and ξ, η, ζ by their values (67) and (68) and the following respectively referred to the above system of coordinates (xy-plane as incidence-plane):

$$\xi=\cos\theta\sin\phi,\quad \eta=\cos\theta\cos\phi,\quad \zeta=\sin\theta$$

(cf. Fig. 39), and we have

$$\xi'=\sin\theta\sin\phi,\quad \eta'=\sin\theta\cos\phi,\quad \zeta'=-\cos\theta.$$

Replace ξ', η', ζ' by these values in formulae (71), and we can write the component magnetic moments in the form

$$\left.\begin{aligned} a&=A\sin\theta\sin\phi\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},\\ b&=A\sin\theta\cos\phi\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},\\ c&=-A\cos\theta\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)} \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(72)$$

(cf. formulae (70)).

To determine the amplitude A of the magnetic oscillations (72), we make use of the first equation of formulae (55), which evidently reduces here (xy-plane as incidence plane) to

$$\frac{4\pi}{\bar{v}}\frac{dX}{dt}=-\frac{d\gamma}{dy},$$

where we are writing $\bar{v}$ for the velocity of propagation of electromagnetic waves in the standard medium (vacuum) instead of v_0, which we have been employing above for the velocity of propagation of the ordinary reflected wave (cf. formulae (60)*). Replace here X by its value from formulae (70), and we have

$$\frac{4\pi}{\bar{v}}ain\cos\theta\sin\phi\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}=-\frac{d\gamma}{dy};$$

which integrated gives

$$\gamma=\frac{4\pi v}{\bar{v}}a\cos\theta\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}.$$

By formulae (4) ($M_1=M_2=M_3=M$), we thus find the following expression for c:

$$c=\frac{Mv}{\bar{v}}a\cos\theta\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}.$$

A comparison of this expression for c with the above (cf. formulae (72)) shows that the amplitude A of the magnetic oscillations (72) that accompany the incident electric ones (59) must be

$$A=-\frac{Mv}{\bar{v}}a,$$

where a denotes the amplitude of the latter.

---

* Cf. foot-note, p. 358.

The expressions (72) for the component magnetic moments a, b, c can thus be written:

$$\left.\begin{aligned}
\bar{a}^{*} &= -\frac{Mv}{\bar{v}}a\sin\theta\sin\phi\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},\\
b &= -\frac{Mv}{\bar{v}}a\sin\theta\cos\phi\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)},\\
c &= \frac{Mv}{\bar{v}}a\cos\theta\, e^{in\left(t-\frac{y\sin\phi-x\cos\phi}{v}\right)}.
\end{aligned}\right\}\quad\text{..........(72A)}$$

The Reflected and Refracted Magnetic Waves.—Similarly, we can write the component moments of the reflected and refracted magnetic waves that accompany the reflected and refracted electric ones of formulae (70) as follows:

$$\left.\begin{aligned}
\bar{a}_0^{*} &= -\frac{Mv_0}{\bar{v}}a_0\sin\theta_0\sin\phi_0\, e^{in\left(t-\frac{y\sin\phi_0+x\cos\phi_0}{r_0}\right)},\\
b_0 &= \frac{Mv_0}{\bar{v}}a_0\sin\theta_0\cos\phi_0\, e^{in\left(t-\frac{y\sin\phi_0+x\cos\phi_0}{r_0}\right)},\\
c_0 &= \frac{Mv_0}{\bar{v}}a_0\cos\theta_0\, e^{in\left(t-\frac{y\sin\phi_0+x\cos\phi_0}{v_0}\right)},
\end{aligned}\right\}\quad\text{.........(73)}$$

with similar expressions for the component moments of the extraordinary reflected (magnetic) wave, and

$$\left.\begin{aligned}
\bar{a}_0'^{*} &= -\frac{M'v_0'}{\bar{v}}a_0'\sin\theta_0'\sin\phi_0' e^{in\left(t-\frac{y\sin\phi'_0-x\cos\phi'_0}{v'_0}\right)},\\
b_0' &= -\frac{M'v_0'}{\bar{v}}a_0'\sin\theta_0'\cos\phi_0' e^{in\left(t-\frac{y\sin\phi'_0-x\cos\phi'_0}{v'_0}\right)},\\
c_0' &= \frac{M'v_0'}{\bar{v}}a_0'\cos\theta_0' e^{in\left(t-\frac{y\sin\phi'_0-x\cos\phi'_0}{v'_0}\right)},
\end{aligned}\right\}\quad\text{.........(74)}$$

with similar expressions for the component moments of the extraordinary refracted (magnetic) wave.

Since, by assumption, there is no variation in the constant of magnetic induction in either crystal, the component *forces* acting in the above magnetic waves will be proportional to the respective component *moments*, the expressions (72)–(74).

The Amplitudes of the Reflected and Refracted Waves and the Surface-Conditions.—We have seen above that the two conditional relations (66A), or, if referred to the incidence-plane, (67) and (69), must hold, in order that the first surface-condition (64) may be satisfied; the latter will now evidently be satisfied only, when the following conditional equation holds between the a's and the ξ's:

$$a_0'\xi_0' + a_e'\xi_e' = a\xi + a_0\xi_0 + a_e\xi_e,$$

---

* We write $\bar{a}$, $\bar{a}_0$, $\bar{a}'_0$ for the component magnetic moments a, a_0, a'_0 to distinguish them from the amplitudes a, a_0, a'_0.

or, if referred to the incidence-plane, the following equation between the a's and the θ's and ϕ's:

$$\left.\begin{aligned} &a_0' \cos\theta_0' \sin\phi_0' + a_e' \cos\theta_e' \sin\phi_e' \\ &\quad = a\cos\theta\sin\phi + a_0\cos\theta_0\sin\phi_0 + a_e\cos\theta_e\sin\phi_e; \end{aligned}\right\} \quad \text{......(75)}$$

we obtain the former equation, on replacing the μ's and ν's by their values (66A) in the surface condition (66), and the latter on replacing the ξ's by their values in terms of the θ's and ϕ's (cf. formulae (70)) (the xy-plane as incidence-plane) in the former.

The Surface-Conditions (65).—On the assumption of the validity of the conditional equation (75), we have one equation for the determination of the a's. To obtain other equations for the determination of these four unknown amplitudes a_0, a_e, a'_0 and a'_e, we must make use of our other surface-conditions (64) and (65). Let us, first, examine the latter surface-conditions and of these the second one; we replace there the β's by their values (cf. formulae (72A)–(74)) on the given dividing surface, $x=0$, and we have

$$\begin{aligned} &-v_0'a_0' \sin\theta_0' \cos\phi_0' e^{in\left(t-\frac{y\sin\phi'_0}{v'_0}\right)} - v_e'a_e' \sin\theta_e' \cos\phi_e' e^{in\left(t-\frac{y\sin\phi'_e}{v'_e}\right)} \\ &= -va\sin\theta\cos\phi\, e^{in\left(t-\frac{y\sin\phi}{v}\right)} + v_0a_0\sin\theta_0\cos\phi_0\, e^{in\left(t-\frac{y\sin\phi_0}{v_0}\right)} \\ &\quad + v_ea_e\sin\theta_e\cos\phi_e\, e^{in\left(t-\frac{y\sin\phi_e}{v_e}\right)}. \end{aligned}$$

By formula (69), which will evidently hold here, since the surface-conditions must hold for all values of y (z) and t, this surface-condition leads to the conditional equation

$$\begin{aligned} &-v_0'a_0' \sin\theta_0' \cos\phi_0' - v_e'a_e' \sin\theta_e' \cos\phi_e' \\ &\quad = -va\sin\theta\cos\phi + v_0a_0\sin\theta_0\cos\phi_0 + v_ea_e\sin\theta_e\cos\phi_e. \end{aligned}$$

By the same formula (69), we can now express the v's as functions of the ϕ's, and thus write this conditional equation in the form

$$\left.\begin{aligned} &-a_0' \sin\theta_0' \sin\phi_0' \cos\phi_0' - a_e' \sin\theta_e' \sin\phi_e' \cos\phi_e' \\ &= -a\sin\theta\sin\phi\cos\phi + a_0\sin\theta_0\sin\phi_0\cos\phi_0 + a_e\sin\theta_e\sin\phi_e\cos\phi_e. \end{aligned}\right\} \quad (76)$$

Similarly treated, the other two surface-conditions (65) lead to the conditional equations

$$\left.\begin{aligned} &M'(a_0' \sin\theta_0' \sin^2\phi_0' + a_e' \sin\theta_e' \sin^2\phi_e') \\ &\quad = M(a\sin\theta\sin^2\phi + a_0\sin\theta_0\sin^2\phi_0 + a_e\sin\theta_e\sin^2\phi_e) \end{aligned}\right\} \quad \text{......(77)}$$

and

$$\begin{aligned} &a_0' \cos\theta_0' \sin\phi_0' + a_e' \cos\theta_e' \sin\phi_e' \\ &\quad = a\cos\theta\sin\phi + a_0\cos\theta_0\sin\phi_0 + a_e\cos\theta_e\sin\phi_e; \end{aligned}$$

the latter is the conditional equation (75) already derived from the first surface-condition (64)

The Surface-Conditions (64) between the Q's and the R's; the Expression for Q.—The four surface-conditions just examined give only three independent conditional equations (75)–(77) for the determination of the four amplitudes a_0, a_e, a'_0 and a'_e sought. To obtain a fourth equation, we must have recourse to the two remaining surface-conditions (64); let us examine here the second one. We must now express the Q's of this surface-condition in terms of known quantities, aside from the four unknown amplitudes sought; let us first seek that expression for Q. For this purpose we introduce a third system of rectangular coordinates x'', y'', z'' with origin at O, the common origin of the two systems x, y, z and x', y', z' already employed;

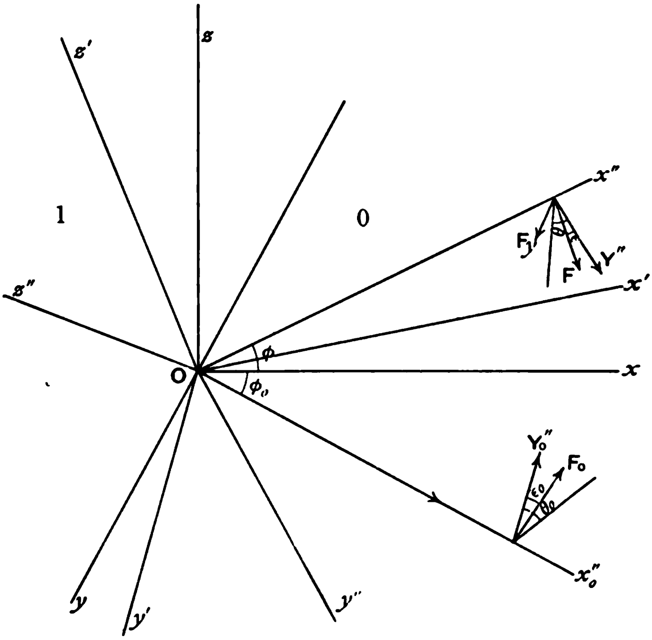

FIG. 40.

the x''-axis of this new system shall coincide with the normal from O to the wave-fronts of the incident wave and the y''-axis shall be taken parallel to the common direction of oscillation in those wave-fronts, as indicated in the annexed figure. Referred to this new system of coordinates the component electric moments of the incident wave (59) can evidently be written

$$X'' = Z'' = 0,$$

$$Y'' = ae^{in\left(t+\frac{x''}{v}\right)}.$$

The components of this resultant moment Y'' parallel to the principal axes x', y', z' of the crystal 0 (of the incident waves) are evidently

$$\left.\begin{aligned} X' &= Y''\cos(x', y'') = a\cos(x', y'')\,e^{in\left(t+\frac{x''}{v}\right)}, \\ Y' &= Y''\cos(y', y'') = a\cos(y', y'')\,e^{in\left(t+\frac{x''}{v}\right)}, \\ Z' &= Y''\cos(z', y'') = a\cos(z', y'')\,e^{in\left(t+\frac{x''}{v}\right)}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(78)$$

The component electric force Q acting parallel to the y-axis of the system of coordinates x, y, z can now evidently be written in the form

$$Q = P'\cos(x', y) + Q'\cos(y', y) + R'\cos(z', y),$$

where P', Q', R' denote the component electric forces acting parallel to the principal axes of the crystal (cf. Fig. 40); since P', Q', R' are the component forces acting parallel to the principal axes x', y', z' of the crystal, we can put

$$P' = \frac{4\pi}{D_1}X', \quad Q' = \frac{4\pi}{D_2}Y', \quad R' = \frac{4\pi}{D_3}Z'$$

(cf. formulae (3)) and thus write the expression for Q in the form

$$Q = 4\pi\left[\frac{X'}{D_1}\cos(x', y) + \frac{Y'}{D_2}\cos(y', y) + \frac{Z'}{D_3}\cos(z', y)\right],$$

or, on replacing here X', Y', Z' by their values (78),

$$\left.\begin{aligned} Q = 4\pi a\Bigg[&\frac{\cos(x', y'')\cos(x', y)}{D_1} + \frac{\cos(y', y'')\cos(y', y)}{D_2} \\ &+ \frac{\cos(z', y'')\cos(z', y)}{D_3}\Bigg]e^{in\left(t+\frac{x''}{v}\right)}. \end{aligned}\right\} \quad \ldots\ldots\ldots(79)$$

$\cos(x', y'')$, $\cos(y', y'')$ and $\cos(z', y'')$ are the direction-cosines of the resultant electric moment Y'' with respect to the principal axes x', y', z' of the crystal, that is, the direction-cosines ξ, η, ζ of formulae (6); these direction-cosines are now related to the direction-cosines p', q', r'—the p, q, r of formulae (47)—of the resultant electric force F with respect to the principal axes, by the formulae

$$\cos(x', y'') = \xi' = \frac{\sqrt{A^4\xi'^2 + B^4\eta'^2 + C^4\zeta'^2}}{A^2}\,p',$$

$$\cos(y', y'') = \eta' = \frac{\sqrt{A^4\xi'^2 + B^4\eta'^2 + C^4\zeta'^2}}{B^2}\,q',$$

$$\cos(z', y'') = \zeta' = \frac{\sqrt{A^4\xi'^2 + B^4\eta'^2 + C^4\zeta'^2}}{C^2}\,r$$

(cf. formulae (47)), where, according to the present notation, we are denoting the direction-cosines of the resultant electric moment Y'' and the resultant electric force F with respect to the principal axes x', y', z' of the crystal by ξ', η', ζ' and p', q', r' respectively. By formulae (30A) and (48A) these expressions for $\cos(x', y'')$, $\cos(y', y'')$ and $\cos(z', y'')$ can be written

$$\cos(x', y'') = \frac{vv_r}{v^2} D_1 M p', \qquad \cos(y', y'') = \frac{vv_r}{v^2} D_2 M q',$$

$$\cos(z', y'') = \frac{vv_r}{v^2} D_3 M r',$$

where v_r denotes the velocity of propagation of the ray.

By these values for the direction-cosines we can now write the above expression (79) for Q in the form

$$Q = \frac{4\pi a v v_r M}{v^2}\left[p' \cos(x', y) + q' \cos(y', y) + r' \cos(z', y)\right] e^{in\left(t + \frac{x''}{v}\right)}$$

$$= \frac{4\pi a v v_r M}{v^2} \cos(F, y) e^{in\left(t + \frac{x''}{v}\right)}, \qquad \ldots\ldots(80)$$

where (F, y) denotes the angle between the direction of action of the resultant electric force F and the y-axis (cf. Fig. 40). This cosine can evidently be replaced by the quotient $\frac{F_y}{F}$, where F_y denotes the component of the resultant force F parallel to the y-axis.

The component of the resultant electric force F in the direction of the resultant electric moment Y'' is now

$$F \cos \epsilon$$

and its other component in the direction of the (negative) normal to wave-front

$$F \sin \epsilon$$

(cf. Figs. 38 and 40), where ϵ denotes the angle between the resultant electric force F and the resultant electric moment Y'' (cf. p. 350).

The components of $F \cos \epsilon$ and $F \sin \epsilon$ in the plane of incidence, the xy-plane, are evidently

$$F \cos \epsilon \cos \theta \quad \text{and} \quad F \sin \epsilon$$

respectively, where θ denotes the azimuth of the given oscillations, and hence the components of these two component forces parallel to the y-axis

$$F \cos \epsilon \cos \theta \cos \phi \quad \text{and} \quad F \sin \epsilon \sin \phi$$

respectively (cf. Fig. 40).

The total component electric force acting parallel to the y-axis, F_y, will thus be $\quad F_y = (\cos \epsilon \cos \theta \cos \phi + \sin \epsilon \sin \phi) F$,

hence $\quad F_y/F = \cos \epsilon \cos \theta \cos \phi + \sin \epsilon \sin \phi$.

Replace $\cos(F, y)=F_y/F$ by this value in the above expression (80) for Q, and we have

$$Q=\frac{4\pi a v v_r M}{\bar{v}^2}(\cos\epsilon\cos\theta\cos\phi+\sin\epsilon\sin\phi)e^{in\left(t+\frac{x''}{v}\right)},$$

or, since by formulae (48) and (48A)

$$\cos\epsilon=\frac{v}{v_r},$$

$$Q=\frac{4\pi a v^2 M}{\bar{v}^2}(\cos\theta\cos\phi+\tan\epsilon\sin\phi)e^{in\left(t+\frac{x''}{v}\right)}. \quad\ldots\ldots\ldots\ldots(81)$$

The Expressions for Q_0 and Q_e.—The determination of the expressions for Q_0 and Q_e is similar to that for Q. The position of the auxiliary system of coordinates x_0'', y_0'', z_0'' for Q_0 is also roughly indicated in Fig. 40. We find, as above,

$$Q_0=\frac{4\pi a_0 v_0 v_{0r} M}{\bar{v}^2}\cos(F_0, y)e^{in\left(t-\frac{x_0''}{v_0}\right)}$$

and a similar expression for Q_e.

On evaluating $\cos(F_0, y)$ we observe, however, that the components of $F_0\cos\epsilon_0\cos\theta_0$ and $F_0\sin\epsilon_0$ parallel to the y-axis must evidently be written

$$-F_0\cos\epsilon_0\cos\theta_0\cos\phi_0 \quad\text{and}\quad F_0\sin\epsilon_0\sin\phi_0$$

respectively (cf. Fig. 40), and hence the final expression for Q_0 in the form

$$Q_0=-\frac{4\pi a_0 v_0^2 M}{\bar{v}^2}(\cos\theta_0\cos\phi_0-\tan\epsilon_0\sin\phi_0)e^{in\left(t-\frac{x_0''}{v_0}\right)};\ldots\ldots(82)$$

similarly, the evaluation of $\cos(F_e, y)$, etc., and the final expression for Q_e.

The Expressions for Q_0' and Q_e'.—The expressions for Q_0' and Q_e' will evidently be similar to that (81) for Q; we find

$$Q_0'=\frac{4\pi a_0' v_0'^2 M'}{\bar{v}^2}(\cos\theta_0'\cos\phi_0'+\tan\epsilon_0'\sin\phi_0')e^{in\left(t-\frac{x_0'''}{v_0'}\right)} \quad\ldots\ldots(83)$$

and a similar expression for Q_e'.

Derivation of the two remaining Conditional Equations from the Surface-Conditions in Q and R.—Replace the Q's by their values on the given dividing-surface, $x=0$, in the second surface-condition (64), and we find, by formula (69),

$$\begin{aligned}
&a_0'v_0'^2M'(\cos\theta_0'\cos\phi_0'+\tan\epsilon_0'\sin\phi_0')\\
&\qquad\qquad+a_e'v_e'^2M'(\cos\theta_e'\cos\phi_e'+\tan\epsilon_e'\sin\phi_e')\\
&=av^2M(\cos\theta\cos\phi+\tan\epsilon\sin\phi)-a_0v_0^2M(\cos\theta_0\cos\phi_0-\tan\epsilon_0\sin\phi_0)\\
&\qquad\qquad-a_ev_e^2M(\cos\theta_e\cos\phi_e-\tan\epsilon_e\sin\phi_e),
\end{aligned}$$

or, on expressing the v's as functions of the ϕ's,

$$\left.\begin{aligned}&a_0' \sin^2\phi_0' (\cos\theta_0' \cos\phi_0' + \tan\epsilon_0' \sin\phi_0')\\ &\quad + a_e' \sin^2\phi_e' (\cos\theta_e' \cos\phi_e' + \tan\epsilon_e' \sin\phi_e')\\ &= a \sin^2\phi (\cos\theta \cos\phi + \tan\epsilon \sin\phi)\\ &\quad - a_0 \sin^2\phi_0 (\cos\theta_0 \cos\phi_0 - \tan\epsilon_0 \sin\phi_0)\\ &- a_e \sin^2\phi_e (\cos\theta_e \cos\phi_e - \tan\epsilon_e \sin\phi_e),\end{aligned}\right\} \ldots\ldots\ldots\ldots(84)$$

where we have put $M' = M$ (cf. foot-note p. 298), the fourth conditional equation for the determination of the four amplitudes a_0, a_e, a_0' and a_e'. The last surface-condition (64) similarly treated leads to a fifth conditional equation, which is similar to (84) but is not independent of those already found.

Summary.—The conditional equations (75)–(77) and (84) evidently suffice for the determination of the four unknown amplitudes a_0, a_e, a_0' and a_e'. These amplitudes are, strictly speaking, the only remaining unknowns in the above equations; we observe, however, that the relation (69) between the ϕ's and the v's gives only the ratios between the sines of the former and the latter. To find the ϕ's and v's of the reflected and refracted waves, for example the ϕ_0 and v_0 of the ordinary reflected wave, to which any incident wave of angle of incidence ϕ and velocity of propagation v gives rise, we first determine the direction-cosines $\bar{\lambda}_0$, $\bar{\mu}_0$, $\bar{\nu}_0$ (λ, μ, ν) of the normal to the wave-fronts of that wave with respect to the principal axes x', y', z' of the crystal 0 as functions of ϕ_0 and the cosines between those principal axes x', y', z' and the coordinate-axes x, y, z, to which the given dividing-surface is referred (cf. pp. 355 and 358); these direction-cosines are evidently given by the expressions

$$\begin{aligned}\bar{\lambda}_0 &= \cos\phi_0 \cos(x', x) + \sin\phi_0 \cos(x', y),\\ \bar{\mu}_0 &= \cos\phi_0 \cos(y', x) + \sin\phi_0 \cos(y', y),\\ \bar{\nu}_0 &= \cos\phi_0 \cos(z', x) + \sin\phi_0 \cos(z', y);\end{aligned}$$

we then replace $\bar{\lambda}_0$, $\bar{\mu}_0$, $\bar{\nu}_0$ (λ, μ, ν) by these values in formula (40), and we thus obtain an equation between ϕ_0 and $v_0(v)$; by this equation and the relation (69) between ϕ, ϕ_0, v and v_0 we can then determine ϕ_0 and v_0 uniquely as functions of ϕ, v, the medium constants A, B, C and the cosines between the coordinate-axes x', y', z' and x, y, z, all of which are given.

The General Problem and its Solution.—The actual solution of the conditional equations (75)–(77) and (84) with respect to the four unknown amplitudes offers no material difficulties. We observe, however, that the further examination of the resulting expressions for this most general case, where both media are aeolotropic, is of little interest, since an empirical verification of the results could be obtained only with difficulty, whereas, on the other hand, quite similar results can be deduced more readily from the simpler equations that hold for the particular case, where only one of the adjacent media is aeolotropic. The particular case generally investigated by experimenters, and thus of special interest here, is now that, where the waves pass from an isotropic into an aeolotropic medium. We shall, therefore, confine our ensuing investigations to this particular case.

The Medium 0 of the Incident Waves Isotropic.—Here $v_0 = v_e = v$, so that there will be only one system of reflected waves instead of two, and the four unknown quantities will evidently be the two amplitudes a_0' and a_e' and the resultant amplitude a_1 and azimuth θ_1 of the reflected waves of that single system. The conditional equations (75)–(77) and (84) will then, by formula (69), evidently assume the simpler form

$$\left.\begin{aligned} & a_0' \cos\theta_0' \sin\phi_0' + a_e' \cos\theta_e' \sin\phi_e' = (a\cos\theta + a_1\cos\theta_1)\sin\phi, \\ & \quad a_0' \sin\theta_0' \sin\phi_0' \cos\phi_0' + a_e' \sin\theta_e' \sin\phi_e' \cos\phi_e' \\ & \qquad = (a\sin\theta - a_1\sin\theta_1)\sin\phi\cos\phi, \\ & a_0' \sin\theta_0' \sin^2\phi_0' + a_e' \sin\theta_e' \sin^2\phi_e' = (a\sin\theta + a_1\sin\theta_1)\sin^2\phi \\ \text{and} \quad & \quad a_0' \sin^2\phi_0' (\cos\theta_0' \cos\phi_0' + \tan\epsilon_0' \sin\phi_0') \\ & \qquad + a_e' \sin^2\phi_e' (\cos\theta_e' \cos\phi_e' + \tan\epsilon_e' \sin\phi_e') \\ & \qquad = (a\cos\theta - a_1\cos\theta_1)\sin^2\phi\cos\phi. \end{aligned}\right\} \ldots(85)$$

The Uniradial Azimuths.—The examination of the conditional equations (85) can now be greatly simplified by the introduction of the so-called "uniradial" azimuths employed by MacCullagh;* these azimuths are those two particular ones θ of the incident waves, which bring about the extinction of either the ordinary or the extraordinary refracted waves; let us denote them by Θ_e and Θ_0 respectively. Such particular values of θ are consistent with our conditional equations (85), for put there, for example, $a_e' = 0$, and these equations will reduce to the following, which can evidently always be satisfied:

$$\left.\begin{aligned} & A_0' \cos\theta_0' \sin\phi_0' = (a\cos\Theta_0 + A_{1.0}\cos\Theta_{1.0})\sin\phi, \\ & A_0' \sin\theta_0' \sin\phi_0' \cos\phi_0' = (a\sin\Theta_0 - A_{1.0}\sin\Theta_{1.0})\sin\phi\cos\phi, \\ & A_0' \sin\theta_0' \sin^2\phi_0' = (a\sin\Theta_0 + A_{1.0}\sin\Theta_{1.0})\sin^2\phi \\ \text{and} \quad & A_0' \sin^2\phi_0' (\cos\theta_0' \cos\phi_0' + \tan\epsilon_0' \sin\phi_0') \\ & \qquad\qquad = (a\cos\Theta_0 - A_{1.0}\cos\Theta_{1.0})\sin^2\phi\cos\phi, \end{aligned}\right\} \quad (85\text{A})$$

where $\Theta_{1.0}$, $A_{1.0}$ and A_0' denote the particular values assumed by θ_1, a_1 and a_0' respectively for that value Θ_0 of θ of the incident oscillations, which brings about the extinction of the extraordinary refracted waves; here Θ_0, $\Theta_{1.0}$, $A_{1.0}$ and A_0' are evidently the four unknown quantities, whereas a, ϕ, ϕ_0', θ_0' and ϵ_0' are either given or can be determined as functions of given quantities (cf. above), being entirely independent of the value of the azimuth of the incident oscillations. Since now the equations (85A) can always be satisfied, the four quantities Θ_0, $\Theta_{1.0}$, $A_{1.0}$ and A_0' being determined uniquely thereby, it follows that there will always be a particular value Θ_0 of θ for each and every angle of incidence ϕ, which will bring about the extinction of the extraordinary refracted waves. Similarly, it is evident that there will always be a particular value Θ_e of θ for every value of ϕ, for which the ordinary refracted waves will be extinguished; this value Θ_e and the corresponding values of the three other unknown quantities $\Theta_{1.e}$, $A_{1.e}$ and A_e' will evidently be determined by similar equations to the above (85A); we obtain these equations on replacing there the index 0 (o)* by e. For brevity we shall henceforth drop these indices, and the equations in question will then hold for either uniradial azimuth Θ_0 or Θ_e.

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* *Transactions of the Irish Academy*, vol. xxi.

It follows from the above that for any given angle of incidence ϕ there are always two uniradial azimuths Θ_0 and Θ_e, that is, two values of θ, for which either the extraordinary or the ordinary refracted waves respectively will be extinguished; these uniradial azimuths will evidently differ for different values of ϕ and, in general, from one another for any given ϕ.

Determination of the Uniradial Azimuths.—The actual determination of Θ, Θ_1, A_1 and A' from equations (85A), (after we have dropped the index 0 (o)*) offers no difficulties; solved with regard to $a \sin\Theta$, $A_1 \sin\Theta_1$, $a\cos\Theta$ and $A_1\cos\Theta_1$, they evidently give

$$a \sin\Theta = A' \sin\theta' \frac{\sin\phi'}{\sin\phi} \frac{\sin(\phi+\phi')}{\sin 2\phi},$$

$$A_1 \sin\Theta_1 = -A' \sin\theta' \frac{\sin\phi'}{\sin\phi} \frac{\sin(\phi-\phi')}{\sin 2\phi},$$

$$\begin{aligned} a\cos\Theta &= A'\cos\theta' \frac{\sin\phi'}{\sin\phi} \frac{\cos\theta'(\sin 2\phi + \sin 2\phi') + 2\tan\epsilon' \sin^2\phi'}{2\cos\theta' \sin 2\phi} \\ &= A'\cos\theta' \frac{\sin\phi'}{\sin\phi} \frac{\cos\theta' \sin(\phi+\phi')\cos(\phi-\phi') + \tan\epsilon' \sin^2\phi'}{\cos\theta' \sin 2\phi} \end{aligned}$$

and

$$\begin{aligned} A_1\cos\Theta_1 &= A'\cos\theta' \frac{\sin\phi'}{\sin\phi} \frac{\cos\theta'(\sin 2\phi - \sin 2\phi') - 2\tan\epsilon' \sin^2\phi'}{2\cos\theta' \sin 2\phi} \\ &= A'\cos\theta' \frac{\sin\phi'}{\sin\phi} \frac{\cos\theta' \sin(\phi-\phi')\cos(\phi+\phi') - \tan\epsilon' \sin^2\phi'}{\cos\theta' \sin 2\phi}; \end{aligned}$$

hence

$$\left.\begin{aligned} \tan\Theta &= \frac{\sin\theta' \sin(\phi+\phi')}{\cos\theta' \sin(\phi+\phi')\cos(\phi-\phi') + \tan\epsilon' \sin^2\phi'}, \\ \tan\Theta_1 &= -\frac{\sin\theta' \sin(\phi-\phi')}{\cos\theta' \sin(\phi-\phi')\cos(\phi+\phi') - \tan\epsilon' \sin^2\phi'} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (86)$$

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* Cf. foot-note, p. 358.

and

$$\left.\begin{aligned}
A_1\sin\Theta_1 &= -a\sin\Theta\frac{\sin(\phi-\phi')}{\sin(\phi+\phi')},\\
A_1\cos\Theta_1 &= a\cos\Theta\frac{\cos\theta'\sin(\phi-\phi')\cos(\phi+\phi')-\tan\epsilon'\sin^2\phi'}{\cos\theta'\sin(\phi+\phi')\cos(\phi-\phi')+\tan\epsilon'\sin^2\phi'},\\
A'\sin\theta' &= a\sin\Theta\frac{\sin\phi}{\sin\phi'}\,\frac{\sin 2\phi}{\sin(\phi+\phi')},\\
A'\cos\theta' &= a\cos\Theta\frac{\sin\phi}{\sin\phi'}\,\frac{\cos\theta'\sin 2\phi}{\cos\theta'\sin(\phi+\phi')\cos(\phi-\phi')+\tan\epsilon'\sin^2\phi'},
\end{aligned}\right\}\quad(87)$$

from which the values for A_1 and A' follow directly.

If both media are isotropic, $\epsilon'=0$ and formulae (86) reduce to

$$\tan\Theta = \frac{\tan\theta'}{\cos(\phi-\phi')},$$

$$\tan\Theta_1 = -\frac{\tan\theta'}{\cos(\phi+\phi')},$$

hence

$$\left.\begin{aligned}
\tan\Theta_1 &= -\tan\Theta\frac{\cos(\phi-\phi')}{\cos(\phi+\phi')},\\
\tan\theta' &= \tan\Theta\cos(\phi-\phi').
\end{aligned}\right\}\quad\text{..................}(86\text{A})$$

Observe that these expressions for the azimuths are identical in form to those (41) found in Chapter VII. for two isotropic media.

Apparent Similarity between Expressions for Component-Amplitudes at ⊥ to Incidence-Plane of Reflected and Refracted Oscillations along Uniradial Azimuths and those for same Component Amplitudes in adjacent Isotropic Media; Similarity only for Perpendicular Incidence. On comparing formulae (87) with those (18 and 19, VII.) (cf. formulae (34A, VII.)) for two isotropic media, we observe that the component amplitudes of the reflected and the refracted oscillations at right angles to the plane of incidence are given by the same expressions in both cases, whereas those in the plane of incidence undergo changes, when the isotropic medium 1 (of the refracted waves) is replaced by an aeolotropic one. Waves incident on the surface of a crystalline medium would, therefore, be reflected and refracted *apparently* according to the same laws as on the surface of an isotropic medium, when their oscillations were taking place at right angles to their plane of incidence; for put $\Theta=\pi/2$ in formulae (87), and we have

$$A_1\sin\Theta_1 = -a\frac{\sin(\phi-\phi')}{\sin(\phi+\phi')},$$

$$A_1\cos\Theta_1 = 0,$$

$$A'\sin\theta' = a\frac{\sin\phi}{\sin\phi'}\,\frac{\sin 2\phi}{\sin(\phi+\phi')},$$

$$A'\cos\theta' = 0;$$

hence $$\tan\Theta_1 = \tan\theta' = \infty,$$
or $$\Theta_1 = \theta' = \pi/2, \quad \ldots\ldots(88)$$

$$\left.\begin{aligned} A_1 &= -a\frac{\sin(\phi-\phi')}{\sin(\phi+\phi')},\\ A' &= a\frac{\sin\phi}{\sin\phi'}\frac{\sin 2\phi}{\sin(\phi+\phi')}, \end{aligned}\right\} \quad \ldots\ldots(89)$$

the expressions (18 and 19, VII.) (cf. formulae (34A, VII.)) already found for adjacent isotropic media, when the incident oscillations were taking place at right angles to the plane of incidence. These formulae (89), although identical to formulae (18 and 19, VII.) in form, differ from them materially in the following respect: The angle of incidence ϕ was entirely arbitrary in the latter, whereas only those two (one) values of ϕ are compatible with the former, for which $\Theta = \pi/2$ is an uniradial azimuth. To determine those values of ϕ, put $\Theta = \Theta_1 = \theta' = \pi/2$ (cf. formulae (88)) in formulae (86) for Θ and Θ_1, and we have

$$\infty = \frac{\sin(\phi+\phi')}{\tan\epsilon'\sin^2\phi'} = \frac{\sin(\phi-\phi')}{\tan\epsilon'\sin^2\phi'},$$

hence $$\phi = \phi' = 0;$$

that is, the two values of ϕ sought evidently coincide here.

For $\phi = \phi' = 0$ the expressions (89) for A_1 and A' become now indeterminate; to find their real values, we write them in the form

$$A_1 = -a\frac{\frac{\sin\phi}{\sin\phi'}\cos\phi' - \cos\phi}{\frac{\sin\phi}{\sin\phi'}\cos\phi' + \cos\phi}$$

and $$A' = 2a\frac{\sin\phi}{\sin\phi'}\frac{\frac{\sin\phi}{\sin\phi'}\cos\phi}{\frac{\sin\phi}{\sin\phi'}\cos\phi' + \cos\phi},$$

replace here $\frac{\sin\phi}{\sin\phi'}$ by its value $\frac{v}{v'}$ (cf. formula (69)), put then $\phi = \phi' = 0$,

and we have $$\left.A_1 = -a\frac{v-v'}{v+v'}, \quad A' = 2a\frac{v^2}{v'(v+v')};\right\} \quad \ldots\ldots(90)$$

these expressions are now identical to those for the component amplitudes of the reflected and refracted waves, to which waves striking the surface of an isotropic insulator at perpendicular incidence (cf. formulae (36, VII.)) give rise. For waves incident on the surface of a crystalline medium (and whose oscillations are taking place at right angles to their plane of incidence) we must, therefore, replace formulae (89) by these particular ones (90) and modify our above statement as follows: The only angle of incidence, at which waves (whose oscillations are taking place at right angles to their plane of incidence) will be reflected and refracted on the surface of a crystalline medium according to the same laws as on the surface of an isotropic insulator, is perpendicular incidence, $\phi = 0$. The validity of the following more general statement then follows directly from the above development (cf. formulae (87) and (88)): The only angle of incidence, at which waves will be reflected and refracted on the surfaces of crystalline and isotropic media according to the same laws, is perpendicular incidence.

The General Problem; Azimuth of Incident Oscillations Arbitrary.—Formulae (86) and (87) evidently hold only for the two particular cases, where the azimuth of the incident oscillations is one of the two uniradial azimuths. The general case, where the azimuth θ of the incident oscillations is entirely arbitrary, can be treated as follows: We determine as above the two uniradial azimuths that correspond to the given angle of incidence ϕ and resolve the incident oscillations of arbitrary azimuth θ along those two azimuths; each component will then give rise to only one refracted wave, the one to an ordinary and the other to an extraordinary wave. Since now these component oscillations take place along uniradial azimuths, each can be treated singly as above, the azimuths and amplitudes sought being determined by formulae (86) and (87) (cf. also below).

Determination of Component Amplitudes of Incident Oscillations along Uniradial Azimuths.—To determine the component-amplitudes a_0* and a_e along the uniradial azimuths Θ_0* and Θ_e respectively of oscillations of azimuth θ and amplitude a, we represent these amplitudes and azimuths in any wave-front graphically as in the annexed figure, where a is the diagonal of the parallelogram, whose sides a_0

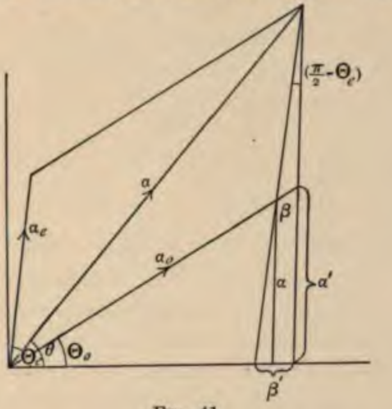

FIG. 41.

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* Cf. foot-note, p. 358.

and a_e are sought The figure evidently gives the following geometrical relations between these quantities and the auxiliary ones α, β, α', β':

$$\frac{\alpha}{a_0} = \sin\Theta_0, \qquad \frac{\beta}{a_e} = \cos\Theta_e,$$

$$\frac{\alpha'}{a\cos\theta} = \tan\Theta_0, \qquad \frac{\beta'}{a\sin\theta} = \frac{1}{\tan\Theta_e},$$

$$\frac{\alpha'-\alpha}{\beta} = \tan\Theta_0, \qquad \frac{\beta'-\beta}{\alpha} = \frac{1}{\tan\Theta_e}.$$

On eliminating α' and β' from the last four relations, we have

$$a\cos\theta\tan\Theta_0 - \alpha = \beta\tan\Theta_0$$

and

$$a\sin\theta - \beta\tan\Theta_e = \alpha;$$

which give the following values for α and β:

$$\alpha = a\frac{(\sin\theta - \cos\theta\tan\Theta_e)\tan\Theta_0}{\tan\Theta_0 - \tan\Theta_e}$$

and

$$\beta = a\frac{\cos\theta\tan\Theta_0 - \sin\theta}{\tan\Theta_0 - \tan\Theta_e}.$$

Replace α and β by these values in the first two relations, and we find the following values for a_0 and a_e, the component amplitudes along the uniradial azimuths Θ_0 and Θ_e respectively:

$$\left.\begin{aligned} a_0 &= a\frac{\sin\theta - \cos\theta\tan\Theta_e}{\sin\Theta_0 - \cos\Theta_0\tan\Theta_e} \\ \text{and} \qquad a_e &= a\frac{\sin\theta - \cos\theta\tan\Theta_0}{\sin\Theta_e - \cos\Theta_e\tan\Theta_0}, \end{aligned}\right\} \quad \text{.....................(91)}$$

where Θ_0 and Θ_e are to be replaced by their values from formulae (86).

The Amplitudes of the Two Refracted Waves.—The amplitudes A_0' and A_e' of the two refracted waves will evidently be determined by the formulae

$$\left.\begin{aligned} &A_\kappa'\sin\theta_\kappa' = a_\kappa\sin\Theta_\kappa\frac{\sin\phi}{\sin\phi_\kappa'}\frac{\sin 2\phi}{\sin(\phi+\phi_\kappa')} \\ &\text{and} \\ &A_\kappa'\cos\theta_\kappa' = a_\kappa\cos\Theta_\kappa\frac{\sin\phi}{\sin\phi_\kappa'}\frac{\cos\theta_\kappa'\sin 2\phi}{\cos\theta_\kappa'\sin(\phi+\phi_\kappa')\cos(\phi-\phi_\kappa') + \tan\epsilon_\kappa'\sin^2\phi_\kappa'} \end{aligned}\right\} (92)$$

(cf. formulae (87)), where κ is to be replaced by 0 (o)* for the ordinary and by e for the extraordinary wave and the component amplitude a_κ by its respective value from formulae (91).

The Resultant Amplitude and Azimuth of the Reflected Oscillations.—The amplitude a_1 of the reflected oscillations will evidently be the resultant of the two component amplitudes $A_{1,0}$ and $A_{1,e}$, whose values are determined by the formulae

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* Cf. foot-note, p. 358.

$$\left.\begin{aligned} A_{1,\kappa}\sin\Theta_{1,\kappa} &= -a_\kappa\sin\Theta_\kappa\frac{\sin(\phi-\phi_\kappa')}{\sin(\phi+\phi_\kappa')},\\ A_{1,\kappa}\cos\Theta_{1,\kappa} &= a_\kappa\cos\Theta_\kappa\frac{\cos\theta_\kappa'\sin(\phi-\phi_\kappa')\cos(\phi+\phi_\kappa')-\tan\epsilon_\kappa'\sin^2\phi_\kappa'}{\cos\theta_\kappa'\sin(\phi+\phi_\kappa')\cos(\phi-\phi_\kappa')+\tan\epsilon_\kappa'\sin^2\phi_\kappa'}\end{aligned}\right\} \quad (93)$$

(cf. formulae (87)), where κ is to be replaced by 0 (o) and e respectively and a_κ by its respective value from formulae (91).

The resultant component of the two component amplitudes $A_{1,0}$ and $A_{1,e}$ at right angles to the plane of incidence is evidently

$$A_{1,0}\sin\Theta_{1,0}+A_{1,e}\sin\Theta_{1,e} = -\sum_{\kappa=0,e}a_\kappa\sin\Theta_\kappa\frac{\sin(\phi-\phi_\kappa')}{\sin(\phi+\phi_\kappa')}, \quad \ldots(93\text{A})$$

and their resultant component in that plane

$$\begin{aligned} &A_{1,0}\cos\Theta_{1,0}+A_{1,e}\cos\Theta_{1,e}\\ &\quad=\sum_{\kappa=0,e}a_\kappa\cos\Theta_\kappa\frac{\cos\theta_\kappa'\sin(\phi-\phi_\kappa')\cos(\phi+\phi_\kappa')-\tan\epsilon_\kappa\sin^2\phi_\kappa'}{\cos\theta_\kappa'\sin(\phi+\phi_\kappa')\cos(\phi-\phi_\kappa')+\tan\epsilon_\kappa\sin^2\phi_\kappa'}. \quad (93\text{B})\end{aligned}$$

The amplitude a_1 of the reflected oscillations will, therefore, be given by the expression

$$\begin{aligned} a_1^2 = &\left[\sum_{\kappa=0,e}a_\kappa\sin\Theta_\kappa\frac{\sin(\phi-\phi_\kappa')}{\sin(\phi+\phi_\kappa')}\right]^2\\ &+\left[\sum_{\kappa=0,e}a_\kappa\cos\Theta_\kappa\frac{\cos\theta_\kappa'\sin(\phi-\phi_\kappa')\cos(\phi+\phi_\kappa')-\tan\epsilon_\kappa\sin^2\phi_\kappa'}{\cos\theta_\kappa'\sin(\phi+\phi_\kappa')\cos(\phi-\phi_\kappa')+\tan\epsilon_\kappa\sin^2\phi_\kappa'}\right]^2 \quad (94)\end{aligned}$$

and their azimuth θ_1 by

$$\tan\theta_1 = -\frac{\displaystyle\sum_{\kappa=0,e}a_\kappa\sin\Theta_\kappa\frac{\sin(\phi-\phi_\kappa')}{\sin(\phi+\phi_\kappa')}}{\displaystyle\sum_{\kappa=0,e}a_\kappa\cos\Theta_\kappa\frac{\cos\theta_\kappa'\sin(\phi-\phi_\kappa')\cos(\phi+\phi_\kappa')-\tan\epsilon_\kappa'\sin^2\phi_\kappa'}{\cos\theta_\kappa'\sin(\phi+\phi_\kappa')\cos(\phi-\phi_\kappa')+\tan\epsilon_\kappa'\sin^2\phi_\kappa'}}. \quad (95)$$

The general problem can thus be treated as follows: We determine first a_0 and a_e as functions of a, θ, Θ_0 and Θ_e by formulae (91), then the Θ_κ's as functions of ϕ, ϕ_κ', θ_κ' and ϵ_κ' by formulae (86) and lastly ϕ_κ', θ_κ' and ϵ_κ' as functions of ϕ, v, etc. by Fresnel's construction (cf. also p. 368); the amplitudes A_0' and A_e' of the two refracted waves then follow directly from formulae (92) and the amplitude a_1 and azimuth θ_1 of the reflected wave from formulae (94) and (95) respectively.

Coincidence of the Uniradial Azimuths of the Reflected Oscillations; the Angle of Polarization.—It is evident from formulae (93) that there is at least one angle of incidence ϕ, for which the uniradial azimuths $\Theta_{1,0}$ and $\Theta_{1,e}$ of the two component reflected oscillations along those azimuths will coincide with one another. The condition that these two uniradial azimuths coincide is

$$\tan\Theta_{1,0} = \tan\Theta_{1,e},$$

which, by formulae (93), can evidently be written in the form

$$\left.\begin{aligned}&\tan\Theta_0\frac{\sin(\Phi-\phi_0')}{\sin(\Phi+\phi_0')}\cdot\frac{\cos\theta_0'\sin(\Phi+\phi_0')\cos(\Phi-\phi_0')+\tan\epsilon_0'\sin^2\phi_0'}{\cos\theta_0'\sin(\Phi-\phi_0')\cos(\Phi+\phi_0')-\tan\epsilon_0'\sin^2\phi_0'}\\&=\tan\Theta_e\frac{\sin(\Phi-\phi_e')}{\sin(\Phi+\phi_e')}\cdot\frac{\cos\theta_e'\sin(\Phi+\phi_e')\cos(\Phi-\phi_e')+\tan\epsilon_e'\sin^2\phi_e'}{\cos\theta_e'\sin(\Phi-\phi_e')\cos(\Phi+\phi_e')-\tan\epsilon_e'\sin^2\phi_e'},\end{aligned}\right\}(96)$$

where Φ denotes the angle of incidence sought. It is evident from this conditional equation for the determination of Φ that Φ will be a function of (Θ_κ), θ_κ', ϵ_κ', etc., but not of the azimuth θ of the incident oscillations; we observe that the component amplitudes a_0 and a_e along the uniradial azimuths Θ_0 and Θ_e respectively are alone functions of this azimuth θ (cf. below). For any given case (crystal and reflecting surface given) Φ will, therefore, be determined by one and the same value for all values of θ. On the other hand, since for any given Φ there are only two uniradial azimuths Θ_0 and Θ_e, waves incident at that angle Φ will all be reflected in one and the same plane (θ_1), whatever be their azimuths of oscillation. If we let ordinary or non-polarized waves fall on the surface of a crystal at this particular angle of incidence Φ, they will, therefore, be reflected as linearly polarized ones; this angle Φ is thus known as the angle of polarization (cf. also p. 279). We have now found in Chapter VII., when waves were incident on the surface of an isotropic insulator at the angle of polarization, that only the component oscillations at right angles to the plane of incidence were reflected; we observe that this is not the case here, where the reflecting surface employed is that of a crystal, for, by formula (95), the azimuth θ_1 of the reflected oscillations will, in general, be quite arbitrary, and not $\pi/2$ (cf. Exs. 23 and 34 at end of chapter). The conception that only the component oscillations at right angles to the plane of incidence are reflected for the angle of polarization is not, therefore, in general, identical to that of polarization.

The Resultant Amplitude of the Reflected Oscillations for Angle of Polarization.—Lastly, we observe, although the angle of polarization Φ and the two respective uniradial azimuths Θ_{10} and Θ_{1e} of the reflected oscillations, which are then equal, are entirely independent of the azimuth θ of the incident oscillations, that the resultant amplitude a_1 of the reflected oscillations is then a function of that azimuth, for a_1 is, by formula (94), a function of a_0 and a_e and the latter are functions of θ (cf. formulae (91)).

Total Reflection. Azimuth of Incident Oscillations an Uniradial one.—The above formulae for reflection and refraction evidently hold only for partial reflection, that is, for the two cases, where

$$v > r_\kappa' \text{ and } r < r_\kappa', \text{ but } \sin\phi < \frac{v}{r_\kappa'}.$$

The following cases thus remain to be examined:

$$v < v_\kappa', \text{ but } \sin\phi > \frac{v}{v_\kappa'},$$

and $$v < v_\kappa', \text{ but either } \frac{v}{v_e'} > \sin\phi > \frac{v}{v_0'} \text{ or } \frac{v}{v_0'} > \sin\phi > \frac{v}{v_e'}.$$

Only the former of these cases is, strictly speaking, one of total reflection, since either the ordinary or the extraordinary refracted wave only is totally extinguished in the latter. Let us examine here the case of total reflection. The treatment of this case is evidently similar to that of total reflection (from the formulae for partial reflection) on the surface of an isotropic insulator: we obtain namely the formulae sought, on replacing in the formulae for partial reflection on the surface of a crystalline medium the real angles of refraction ϕ_0' and ϕ_e' by the complex ones $\left(\frac{\pi}{2}+i\phi_0'\right)$ and $\left(\frac{\pi}{2}+i\phi_e'\right)$ respectively (cf. pp. 285, 287, and 290). Since the two refracted waves are extinguished almost immediately upon entering the crystal (cf. pp. 287 and 291), only the reflected oscillations will be of interest here. Let us now examine the reflected oscillations first for the particular case, where the azimuth of the incident oscillations is one of the two uniradial azimuths Θ_0 or Θ_e; for partial reflection the resultant amplitude and azimuth of the reflected oscillations were then determined by the formulae

$$A_1 \sin\Theta_1 = -a \sin\Theta \frac{\sin(\phi-\phi')}{\sin(\phi+\phi')},$$

their component amplitude at right angles to the plane of incidence, and

$$A_1 \cos\Theta_1 = a \cos\Theta \frac{\cos\theta' \sin(\phi-\phi')\cos(\phi+\phi') - \tan\epsilon' \sin^2\phi'}{\cos\theta' \sin(\phi+\phi')\cos(\phi-\phi') + \tan\epsilon' \sin^2\phi'}, \qquad \ldots(97)$$

their component amplitude in the plane of incidence (cf. formulae (87)), where the index 0 (*o*) or *e* to Θ, Θ_1, ϕ' and ϵ' corresponding to the uniradial azimuth Θ_0 or Θ_e respectively has been dropped (cf. p. 370).

For total reflection we must now replace the real angle ϕ' by the complex one $\pi/2+i\phi'$ in formulae (97) for the component amplitudes sought, that is, we must put there $\sin\phi' = \sin(\pi/2+i\phi')$

and $$\cos\phi' = -i\sqrt{\sin^2(\pi/2+i\phi')-1},$$

(cf. p. 286). The expression for the component amplitude at right angles to the plane of incidence will then assume the form

$$A_1 \sin\Theta_1 = -a\sin\Theta\, \frac{-i\sin\phi\sqrt{\sin^2(\pi/2+i\phi')-1} - \cos\phi\sin(\pi/2+i\phi')}{-i\sin\phi\sqrt{\sin^2(\pi/2+i\phi')-1} + \cos\phi\sin(\pi/2+i\phi')}$$

$$= -\frac{a\sin\Theta}{\sin^2\phi - \sin^2(\pi/2+i\phi')}\left[\sin^2\phi + \cos 2\phi \sin^2(\pi/2+i\phi')\right.$$

$$\left. + i\sin 2\phi \sin(\pi/2+i\phi')\sqrt{\sin^2(\pi/2+i\phi')-1}\right].$$

To determine the real amplitude and change in phase of the respective component oscillations, we must now bring this expression into the normal form

$$A_1 \sin \Theta_1 = -\frac{a \sin \Theta}{\sin^2\phi - \sin^2(\pi/2 + i\phi')} N e^{i\omega_1}$$
$$= -\frac{a \sin \Theta}{\sin^2\phi - \sin^2(\pi/2 + i\phi')} N(\cos \omega_1 + i \sin \omega_1),$$

where N and ω_1 are sought in terms of ϕ and $(\pi/2 + i\phi')$ (cf. Chapter VII., Total Reflection). On comparing the real and the imaginary parts of the above expression with the respective ones of this expression, we evidently have the two following equations for the determination of N and ω_1:

$$\sin^2\phi + \cos 2\phi \sin^2(\pi/2 + i\phi') = N \cos \omega_1$$

and $$\sin 2\phi \sin(\pi/2 + i\phi')\sqrt{\sin^2(\pi/2 + i\phi') - 1} = N \sin \omega_1,$$

which give $$N = \sin^2\phi - \sin^2(\pi/2 + i\phi')$$

and $$\tan \omega_1 = \frac{\sin 2\phi \sin(\pi/2 + i\phi')\sqrt{\sin^2(\pi/2 + i\phi') - 1}}{\sin^2\phi + \cos 2\phi \sin^2(\pi/2 + i\phi')}.$$

We can thus write the above expression for the component amplitude at right angles to the plane of incidence in the form

$$A_1 \sin \Theta_1 = -a \sin \Theta_1 e^{i \arctan \frac{\sin 2\phi \sin(\pi/2 + i\phi')\sqrt{\sin^2(\pi/2 + i\phi') - 1}}{\sin^2\phi + \cos 2\phi \sin^2(\pi/2 + i\phi')}} \quad \text{.............(98)}$$

It follows from this expression that the given component oscillations at right angles to the plane of incidence will undergo a change only in phase upon total reflection, the same result, which we found, when oscillations taking place at right angles to the plane of incidence were totally reflected on the surface of an isotropic insulator (cf. p 285); we observe, moreover, that this expression for the change in phase is identical to that found in Chapter VII, where the reflecting surface was that of an isotropic insulator (cf. formulae (49, VII.)). It thus follows, when the incident oscillations are taking place along either uniradial azimuth, that their component oscillations at right angles to the plane of incidence will be reflected in the same manner as on the surface of an isotropic insulator.

Let us next examine the component amplitude of the given reflected oscillations in the plane of incidence; for total reflection the expression for this amplitude will evidently assume the form

$$A_1 \cos \Theta_1 = a \cos \Theta \frac{\cos\theta'[\sin\phi \sin(\pi/2 + i\phi') + i \cos\phi\sqrt{\sin^2(\pi/2 + i\phi') - 1}] - \tan\epsilon' \sin^2(\pi/2 + i\phi')}{\cos\theta'[\sin\phi \sin(\pi/2 + i\phi') - i \cos\phi\sqrt{\sin^2(\pi/2 + i\phi') - 1}] + \tan\epsilon' \sin^2(\pi/2 + i\phi')}, \quad (99)$$

(cf. formulae (97)), where, however, the azimuth θ' and the angle ϵ', both of which are functions of ϕ' (cf. above), are to be expressed in terms of $\pi/2+i\phi'$ in place of ϕ'; as the actual determination of these quantities would now demand rather dilate geometrical investigations, which for the general case are of no special interest, we shall imagine the same as already determined and in the desired form

$$\cos\theta' = f_1 + if_2 \quad\text{and}\quad \tan\epsilon' = h_1 + ih_2, \quad\ldots\ldots\ldots\ldots\ldots(100)$$

where f_1, f_2, h_1 and h_2 shall contain $(\pi/2+i\phi')$ only in the form

$$\sin(\pi/2+i\phi') \quad\text{and}\quad \sqrt{\sin^2(\pi/2+i\phi')-1}$$

(cf. pp. 283 and 286).

Replace $\cos\theta'$ and $\tan\epsilon'$ by their values (100) in formula (99), and we can write the component amplitude sought in the form

$$A_1\cos\Theta_1 = a\cos\Theta\frac{\begin{gathered}(f_1+if_2)[\sin\phi\sin(\pi/2+i\phi')+i\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}]\\ -(h_1+ih_2)\sin^2(\pi/2+i\phi')\end{gathered}}{\begin{gathered}(f_1+if_2)[\sin\phi\sin(\pi/2+i\phi')-i\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}]\\ +(h_1+ih_2)\sin^2(\pi/2+i\phi')\end{gathered}},$$

$$=\frac{\begin{aligned}&f_1\sin\phi\sin(\pi/2+i\phi')-f_2\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}\\ &\quad-h_1\sin^2(\pi/2+i\phi')+i[f_1\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}\\ &\quad+f_2\sin\phi\sin(\pi/2+i\phi')-h_2\sin^2(\pi/2+i\phi')]\end{aligned}}{\begin{aligned}&f_1\sin\phi\sin(\pi/2+i\phi')+f_2\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}\\ &\quad+h_1\sin^2(\pi/2+i\phi')+i[-f_1\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}\\ &\quad+f_2\sin\phi\sin(\pi/2+i\phi')+h_2\sin^2(\pi/2+i\phi')]\end{aligned}};$$

multiply both numerator and denominator of this expression by

$$\begin{aligned}&f_1\sin\phi\sin(\pi/2+i\phi')+f_2\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}\\ &\quad+h_1\sin^2(\pi/2+i\phi')-i[-f_1\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}\\ &\quad+f_2\sin\phi\sin(\pi/2+i\phi')+h_2\sin^2(\pi/2+i\phi')]\end{aligned}$$

and we find, on separating it into its real and imaginary parts,

$$A_1\cos\Theta_1 = a\cos\Theta\frac{\begin{aligned}&(f_1^2+f_2^2)[\cos^2\phi-\cos2\phi\sin^2(\pi/2+i\phi')]-(h_1^2+h_2^2)\\ &\quad\times\sin^4(\pi/2+i\phi')+2(f_1h_2-f_2h_1)\cos\phi\sin^2(\pi/2+i\phi')\\ &\quad\times\sqrt{\sin^2(\pi/2+i\phi')-1}+2i[(f_1^2+f_2^2)\sin\phi\cos\phi\\ &\quad\times\sin(\pi/2+i\phi')\sqrt{\sin^2(\pi/2+i\phi')-1}\\ &\quad-(f_1h_2-f_2h_1)\sin\phi\sin^3(\pi/2+i\phi')]\end{aligned}}{\begin{aligned}&(f_1^2+f_2^2)[\sin^2(\pi/2+i\phi')-\cos^2\phi]-2(f_1h_2-f_2h_1)\\ &\quad\times\cos\phi\sin^2(\pi/2+i\phi')\sqrt{\sin^2(\pi/2+i\phi')-1}\\ &\quad+2(f_1h_1+f_2h_2)\sin\phi\sin^3(\pi/2+i\phi')\\ &\quad+(h_1^2+h_2^2)\sin^4(\pi/2+i\phi')\end{aligned}},$$

which can be written in the normal form

$$A_1\cos\Theta_1 = a\cos\Theta Ne^{i\omega_1}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(101)$$

where N and ω_1 are evidently determined by the expressions

$$N^{2\prime}=\frac{\begin{array}{l}(f_1^2+f_2^2)^2[\cos^2\phi-\sin^2(\pi/2+i\phi')]^2+4(f_1h_2-f_2h_1)\\ \quad\times\sin^2(\pi/2+i\phi')[\cos^2\phi-\sin^2(\pi/2+i\phi')]\\ \quad\times[(f_1^2+f_2^2)\cos\phi\sqrt{\sin^2(\pi/2+i\phi')-1}-(f_1h_2-f_2h_1)\\ \quad\times\sin^2(\pi/2+i\phi')]-(h_1^2+h_2^2)\sin^4(\pi/2+i\phi')\\ \quad\times\{2(f_1^2+f_2^2)[\cos^2\phi-\cos 2\phi\sin^2(\pi/2+i\phi')+4(f_1h_2-f_2h_1)\\ \quad\times\cos\phi\sin^2(\pi/2+i\phi')\sqrt{\sin^2(\pi/2+i\phi')-1}-(h_1^2+h_2^2)\\ \quad\times\sin^4(\pi/2+i\phi')\}\end{array}}{\begin{array}{l}\{(f_1^2+f_2^2)[\cos^2\phi-\sin^2(\pi/2+i\phi')]+2(f_1h_2-f_2h_1)\\ \quad\times\cos\phi\sin^2(\pi/2+i\phi')\sqrt{\sin^2(\pi/2+i\phi')-1}-2(f_1h_1+f_2h_2)\\ \quad\times\sin\phi\sin^3(\pi/2+i\phi')-(h_1^2+h_2^2)\sin^4(\pi/2+i\phi')\}^2\end{array}}\qquad(102)$$

and

$$\tan\omega_1=\frac{\begin{array}{l}(f_1^2+f_2^2)\sin 2\phi\sin(\pi/2+i\phi')\sqrt{\sin^2(\pi/2+i\phi')-1}\\ \qquad\qquad-2(f_1h_2-f_2h_1)\sin\phi\sin^3(\pi/2+i\phi')\end{array}}{\begin{array}{l}(f_1^2+f_2^2)[\cos^2\phi-\cos 2\phi\sin^2(\pi/2+i\phi')]-(h_1^2+h_2^2)\\ \quad\times\sin^4(\pi/2+i\phi')+2(f_1h_2-f_2h_1)\cos\phi\sin^2(\pi/2+i\phi')\\ \quad\times\sqrt{\sin^2(\pi/2+i\phi')-1}\end{array}}.\qquad(103)$$

Actual Determination of θ' and ϵ'.—We have observed that the actual determination of θ' and ϵ' or f_1, f_2 and h_1, h_2 respectively (cf. formulae (100)) would demand dilate geometrical investigations; on the other hand, the determination of ϵ' or h_1, h_2 as functions of θ' or f_1, f_2 and the other variables offers no material difficulties, at least for the particular case, where the principal axes x', y', z' of the crystal coincide with those x, y, z, to which the reflecting surface is referred. Let us examine here briefly this particular case. By formula (F), Ex. 14, at end of chapter, the angle ϵ', which the ray makes with the normal to wave-front, is now determined by the following expression in terms of v, v', ϕ, ϕ', θ', ω' and the medium constants A' B' C', when the principal axis x' of the crystal coincides with the normal (x) to the reflecting surface:

$$\cos\epsilon'=\frac{v'^2}{\sqrt{\begin{array}{l}A'^4\cos^2\theta'\sin^2\phi'+B'^4(\cos\theta'\cos\phi'\cos\omega'-\sin\theta'\sin\omega')^2\\ \qquad+C'^4(\cos\theta'\cos\phi'\sin\omega'+\sin\theta'\cos\omega')^2\end{array}}},$$

where ω' denotes the angle, which the principal axis y' or z' of the crystal makes with the coordinate axis y or z respectively, to which the reflecting surface is referred. For the given case, $\omega'=0$, this expression for $\cos\epsilon'$ reduces to

$$\cos\epsilon'=\frac{v'^2}{\sqrt{A'^4\cos^2\theta'\sin^2\phi'+B'^4\cos^2\theta'\cos^2\phi'+C'^4\sin^2\theta'}},$$

or, if we replace here v' by its value in terms of v, ϕ and ϕ' from formula (69), to

$$\cos\epsilon' = \frac{v^2 \sin^2\phi'}{\sin^2\phi\sqrt{A'^4\cos^2\theta'\sin^2\phi' + B'^4\cos^2\theta'\cos^2\phi' + C'^4\sin^2\theta'}}.$$

To obtain the expression for $\cos\epsilon'$ for total reflection, we must now replace the real angle ϕ' in this expression for $\cos\epsilon'$ for partial reflection by the complex one $\pi/2 + i\phi'$; we then have

$$\cos\epsilon' = \frac{v^2 \sin^2(\pi/2 + i\phi')}{\sin^2\phi\sqrt{[(A'^4 - B'^4)\sin^2(\pi/2 + i\phi') + B'^4]\cos^2\theta' + C'^4\sin^2\theta'}}, \quad (104)$$

or, on writing $\cos\theta'$ here in the complex form $f_1 + if_2$ (cf. formulae (100)),

$$\cos\epsilon' = \frac{v^2 \sin^2(\pi/2 + i\phi')}{\sin^2\phi\sqrt{[(A'^4 - B'^4)\sin^2(\pi/2 + i\phi') + (B'^4 - C'^4)](f_1 - if_2)^2 + C'^4}},$$

which gives

$$\tan\epsilon' = \frac{\sqrt{\begin{array}{r}[(A'^4 - B'^4)\sin^2(\pi/2 + i\phi') + (B'^4 - C'^4)] \\ \times \sin^4\phi(f_1 + if_2)^2 + C'^4\sin^4\phi - v^4\sin^4(\pi/2 + i\phi')\end{array}}}{v^2\sin^2(\pi/2 + i\phi')}. \quad (104\text{A})$$

To bring this expression for $\tan\epsilon'$ into the desired form $h_1 + ih_2$ (cf. formulae (100)), we square it and separate it into its real and imaginary parts; we then find the two following equations for the determination of the two real functions h_1 and h_2 in terms of f_1, f_2, ϕ, v, $\sin(\pi/2 + i\phi')$ and the medium constants A', B', C':

$$\left.\begin{array}{l} \dfrac{\begin{array}{c}(f_1^2 - f_2^2)[(A'^4 - B'^4)\sin^2(\pi/2 + i\phi') + (B'^4 - C'^4)]\sin^4\phi \\ + C'^4\sin^4\phi - v^4\sin^4(\pi/2 + i\phi')\end{array}}{v^4\sin^4(\pi/2 + i\phi')} = h_1^2 - h_2^2 \\ \text{and} \quad \dfrac{f_1 f_2[(A'^4 - B'^4)\sin^2(\pi/2 + i\phi') + (B'^4 - C'^4)]\sin^4\phi}{v^4\sin^4(\pi/2 + i\phi')} = h_1 h_2. \end{array}\right\} \quad (105)$$

Upon the determination of h_1 and h_2 from these two equations we could then express $\tan\epsilon'$ in the desired form $h_1 + ih_2$.

Lastly, we observe that for the given particular case the azimuths θ_0' and θ_e' would be determined in complex form by the directions of the principal axes of the ellipse intersected on the plane

$$\left.\begin{array}{r} i\sqrt{\sin^2(\pi/2 + i\phi') - 1}\; x + \sin(\pi/2 + i\phi')y = 0 \\ \text{by the ellipsoid} \qquad A'^2x^2 + B'^2y^2 + C'^2z^2 = 1 \end{array}\right\} \quad \ldots\ldots\ldots\ldots(106)$$

(cf. formula (I), Ex. 14, at end of chapter).

The Resultant Totally Reflected Oscillations Elliptically Polarized.— It follows from formulae (101)-(103) that both the amplitude and phase of the above component oscillations in the plane of incidence will undergo changes upon total reflection, and, moreover, from the complicated form of the expressions for those quantities, that these changes will not be the same as those suffered by oscillations taking place in the plane of incidence on the surface of an isotropic insulator (cf. formulae (62)-(67) VII.); in this respect the given component oscillations in the plane of incidence will differ from those at right angles to that plane, the latter alone being totally reflected as on the surface of an isotropic insulator. It is thus evident that the resultant totally reflected oscillations, to which incident oscillations taking place along either uniradial azimuth Θ_0 or Θ_e give rise, will, in general, be elliptically polarized. We determine the two uniradial azimuths, on replacing in the expression for $\tan\Theta$ for partial reflection (cf. formulae (86)) the real angles of refraction ϕ_0' and ϕ_e' by the complex ones $\pi/2+i\phi_0'$ and $\pi/2+i\phi_e$ respectively and also the θ's and ϵ's by their values, determined as formulated above (cf. formulae (102)-(106)), in terms of ϕ, v, $\pi/2+i\phi'$, A', B', C' and the direction-cosines between the two systems of coordinates x', y', z' and x, y, z; it is evident that the two uniradial azimuths will also be given here by complex quantities.

The General Problem: Azimuth of Incident Oscillations Arbitrary; Totally Reflected Waves Elliptically Polarized.—The formulae for the general problem on total reflection, where namely the azimuth of oscillation of the incident waves is arbitrary, can be deduced from those above for the two particular cases, where the azimuths of oscillation are the two uniradial ones, and in a similar manner to that, in which the general formulae for partial reflection were obtained from those for the two particular cases (cf. pp. 373-375). We observe that the two component oscillations at right angles to the plane of incidence of the component oscillations taking place along the two uniradial azimuths undergo each a change in phase, different for each component, but none in amplitude, whereas the two component oscillations in the plane of incidence of the component oscillations along the two uniradial azimuths undergo changes both in amplitude and in phase, and each component different ones. The resultant totally reflected oscillations will, therefore, be elliptically polarized, but evidently not in the same elliptic paths as when the incident waves are reflected from the surface of an isotropic insulator.

Extinction of one of the Refracted Waves; the Resultant Reflected Waves also Elliptically Polarized.—The treatment of the two particular cases

$$v<v_e', \text{ but } \frac{v}{v_e'}>\sin\phi>\frac{v}{v_0'} \text{ and } \frac{v}{v_0'}>\sin\phi>\frac{v}{v_e'}$$

mentioned above evidently presents no further difficulties. Here either only the ordinary or the extraordinary refracted wave is extinguished;

in the former case ϕ_0' only is, therefore, to be replaced by the complex angle $\pi/2+i\phi_0'$, whereas ϕ_e' remains real; in the latter ϕ_0' remains real and ϕ_e' is to be replaced by $\pi/2+i\phi_e'$. In the determination of the uniradial azimuths and the other variables a similar distinction is also to be observed. It is evident that the resultant reflected oscillations will also be elliptically polarized here, but in the two cases differently and also differently from waves that are *totally* reflected either from the surface of that crystal or from that of an isotropic insulator.

EXAMPLES.

1. Show that formulae (10) are satisfied by the particular values (7) and (8) for the component electric and magnetic forces P, Q, R and a, β, γ respectively, provided the given waves are propagated with the velocity v, where

$$\left.\begin{aligned} v^2=v_0^2\Big[&-\frac{\lambda\mu\xi\eta}{M_3D_2}+\frac{\mu^2\xi^2}{M_3D_1}+\frac{\nu^2\xi^2}{M_2D_1}-\frac{\lambda\nu\xi\zeta}{M_2D_3}\\ &-\frac{\mu\nu\eta\zeta}{M_1D_3}+\frac{\nu^2\eta^2}{M_1D_2}+\frac{\lambda^2\eta^2}{M_3D_2}-\frac{\lambda\mu\xi\eta}{M_3D_1}\\ &-\frac{\lambda\nu\xi\zeta}{M_2D_1}+\frac{\lambda^2\zeta^2}{M_2D_3}+\frac{\mu^2\zeta^2}{M_1D_3}-\frac{\mu\nu\eta\zeta}{M_1D_2}\Big].\end{aligned}\right\}\quad\text{(A)}$$

Replace P, Q, R and a, β, γ by their values (7) and (8) respectively in formulae (10), and we have

$$\left.\begin{aligned} \frac{v^2}{v_0^2}\xi&=-\frac{\mu}{M_3}\left(\frac{\lambda\eta}{D_2}-\frac{\mu\xi}{D_1}\right)+\frac{\nu}{M_2}\left(\frac{\nu\xi}{D_1}-\frac{\lambda\zeta}{D_3}\right)\\ \frac{v^2}{v_0^2}\eta&=-\frac{\nu}{M_1}\left(\frac{\mu\zeta}{D_3}-\frac{\nu\eta}{D_2}\right)+\frac{\lambda}{M_3}\left(\frac{\lambda\eta}{D_2}-\frac{\mu\xi}{D_1}\right)\\ \frac{v^2}{v_0^2}\zeta&=-\frac{\lambda}{M_2}\left(\frac{\nu\xi}{D_1}-\frac{\lambda\zeta}{D_3}\right)+\frac{\mu}{M_1}\left(\frac{\mu\zeta}{D_3}-\frac{\nu\eta}{D_2}\right);\end{aligned}\right\}\quad\text{(B)}$$

on multiplying these equations, the first by ξ, the second by η and the third by ζ, and adding, we evidently find, since $\xi^2+\eta^2+\zeta^2=1$, the above expression for the velocity of propagation v. Q.E.D.

2. Show, for the particular case, where $D_1 \gtrless D_2=D_3=D$, $M_1 \gtrless M_2=M_3=M$ and the magnetic oscillations (9) are taking place in planes parallel to the yz-plane (cf. Fig. 37), that the general expression (A), Ex. 1, for v^2 reduces to that (24) found in text.

Here $$a=0,$$
that is, by formulae (9),

$$\mu\zeta-\nu\eta=0.\quad (D_2=D_3).\quad\text{(A)}$$

In addition to this relation between the direction-cosines λ, μ, ν and ξ, η, ζ, we obtain another relation between these quantities, on differentiating formulae (6), the general expressions for the component electric moments X, Y, Z for plane-wave-motion, the first with regard to x, the second to y and the third to z, and adding; we have then

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=-\frac{ain}{v}(\lambda\xi+\mu\eta+\nu\zeta)\,e^{in\left(t-\frac{s}{v}\right)}\quad\text{(B)}$$

(cf. formula (6A)).

Formulae (1), which can, by formulae (3), be written in the form

$$\frac{4\pi}{v_0}\frac{dX}{dt}=\frac{d\beta}{dz}-\frac{d\gamma}{dy}$$
$$\frac{4\pi}{v_0}\frac{dY}{dt}=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}$$
$$\frac{4\pi}{v_0}\frac{dZ}{dt}=\frac{d\alpha}{dy}-\frac{d\beta}{dx},$$

give now, when differentiated, the first with regard to x, the second to y and the third to z, and added

$$\frac{4\pi}{v_0}\frac{d}{dt}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)=0,$$

hence for wave-motion (cf. p. 10)

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=0.$$

Formula (B) can, therefore, evidently be satisfied only when

$$\lambda\xi+\mu\eta+\nu\zeta=0.\ \ldots\ldots\ldots\ldots\ldots\ldots(\text{C})$$

Beside this and the above relation (A) between the direction-cosines λ, μ, ν and ξ, η, ζ, we have the following familiar geometrical ones for the direction-cosines themselves:

$$\xi^2+\eta^2+\zeta^2=1\ \ldots\ldots\ldots\ldots\ldots\ldots(\text{D})$$

and

$$\lambda^2+\mu^2+\nu^2=1.\ \ldots\ldots\ldots\ldots\ldots\ldots(\text{E})$$

Put now $D_2=D_3=D$ and $M_2=M_3=M$ in the general expression (A), Ex. 1, for v^2, and we have

$$\begin{aligned}v^2=v_0^2\Big[&-\frac{\lambda\mu\xi\eta}{MD}+\frac{\mu^2\xi^2}{MD_1}+\frac{\nu^2\xi^2}{MD_1}-\frac{\lambda\nu\xi\zeta}{MD}\\&-\frac{\mu\nu\eta\zeta}{M_1D}+\frac{\nu^2\eta^2}{M_1D}+\frac{\lambda^2\eta^2}{MD}-\frac{\lambda\mu\xi\eta}{MD_1}\\&-\frac{\lambda\nu\xi\zeta}{MD_1}+\frac{\lambda^2\zeta^2}{MD}+\frac{\mu^2\zeta^2}{M_1D}-\frac{\mu\nu\eta\zeta}{M_1D}\Big]\\=\frac{v_0^2}{MD}&[\lambda^2\zeta^2+\lambda^2\eta^2-\lambda\mu\xi\eta-\lambda\nu\xi\zeta]\\&+\frac{v_0^2}{MD_1}[\mu^2\xi^2+\nu^2\xi^2-\lambda\mu\xi\eta-\lambda\nu\xi\zeta]\\&+\frac{v_0^2}{M_1D}[\nu^2\eta^2+\mu^2\zeta^2-\mu\nu\eta\zeta-\mu\nu\eta\zeta]\end{aligned}$$

$$\left.\begin{aligned}=\frac{v_0^2}{MD}&[\lambda^2(\xi^2+\eta^2+\zeta^2)-\lambda\xi(\lambda\xi+\mu\eta+\nu\zeta)]\\&+\frac{v_0^2}{MD_1}[(\lambda^2+\mu^2+\nu^2)\xi^2-\lambda\xi(\lambda\xi+\mu\eta+\nu\zeta)]\\&+\frac{v_0^2}{M_1D}[\nu\eta(\nu\eta-\mu\zeta)+\mu\zeta(\mu\zeta-\nu\eta)],\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(\text{F})$$

which by the above relations (A), (C), (D) and (E) reduces to

$$v^2=\frac{v_0^2}{MD}\lambda^2+\frac{v_0^2}{MD_1}\xi^2.$$

Lastly, the elimination of η and ζ from the three relations (A), (C) and (D) gives the following value for ξ^2 in terms of λ, μ, ν:

$$\xi^2=1-\lambda^2.$$

Replace ξ^2 by this value in the above expression for v^2, and we have

$$v^2 = \frac{v_0^2}{M}\left(\frac{\lambda^2}{D} + \frac{1-\lambda^2}{D_1}\right). \qquad \text{Q.E.D.}$$

3. Show for the particular case, where $D_1 \gtrless D_2 = D_3 = D$, $M_1 \gtrless M_2 = M_3 = M$ and the electric oscillations (6) are taking place in planes parallel to the yz-plane (cf. Fig. 37), that the general expression (A), Ex. 1, for v^2 reduces to that (27) found in text.

Here $X=0$, hence, by formulae (6),

$$\xi = 0. \qquad \text{(A)}$$

As in the preceding example, the same relations (C), (D) and (E), Ex. 2, evidently hold here.

On putting $D_2 = D_3 = D$ and $M_2 = M_3 = M$ in formula (A), Ex. 1, we find the same general expression (F), Ex. 2, for v^2. By the above relations this expression reduces now here to

$$v^2 = \frac{v_0^2}{MD}\lambda^2 + \frac{v_0^2}{M_1 D}[\nu\eta(\nu\eta - \mu\zeta) + \mu\zeta(\mu\zeta - \nu\eta)],$$

where η and ζ are to be replaced by their values in terms of λ, μ, ν. The given relations evidently give the following values for η and ζ:

$$\eta = \pm\frac{\nu}{\sqrt{\mu^2+\nu^2}} \text{ and } \zeta = \mp\frac{\mu}{\sqrt{\mu^2+\nu^2}}$$

(cf. formulae (C), Ex. 2, and (A) for choice of signs).

On replacing η and ζ by these values in the above expression for v^2, we find

$$v^2 = \frac{v_0^2}{MD}\lambda^2 + \frac{v_0^2}{M_1 D}(\mu^2+\nu^2),$$

or, since $\lambda^2 + \mu^2 + \nu^2 = 1$,

$$v^2 = \frac{v_0^2}{D}\left(\frac{\lambda^2}{M} + \frac{1-\lambda^2}{M_1}\right). \qquad \text{Q.E.D.}$$

4. Compare the results obtained on pp. 336-339 for the case, where $D_1 \gtrless D_2 = D_3$ and $M_1 \gtrless M_2 = M_3$, with those found for the most general empirical case, where $D_1 \gtrless D_2 \gtrless D_3$ but $M_1 = M_2 = M_3$.

5. Show that formulae (46A) reduce to

$$v_0^2 = v_e^2 = B^2$$

(cf. formula (44)) for waves propagated along the optical axes.

6. Show that formulae (46A) reduce to the following in the uniaxal crystals $B = C$:

$$v_0^2 = C^2$$

and

$$v_e^2 = A^2 - (A^2 - C^2)\cos^2 u$$

(cf. formulae (40A)), where $u_1 = u_2 = u$.

7. Determine the form (plane of oscillation, etc.) of plane magnetic wave that can be transmitted in any assigned direction (λ, μ, ν) through a crystalline medium.

We represent the component magnetic moments a, b, c acting in any plane-wave in the form

$$\left.\begin{aligned} a &= b\xi_1' e^{in\left(t - \frac{s}{v_1}\right)} \\ b &= b\eta_1' e^{in\left(t - \frac{s}{v_1}\right)} \\ c &= b\zeta_1' e^{in\left(t - \frac{s}{v_1}\right)} \end{aligned}\right\} \qquad \text{(A)}$$

where ξ_1', η_1', ζ_1' denote its direction-cosines of oscillation in any wave-front at the distance

$$s = \lambda x + \mu y + \nu z$$

from the origin of the system of coordinates x, y, z (cf. Fig. 37), b its amplitude of oscillation and v_1 its velocity of propagation; here λ, μ, ν and b are given and ξ_1', η_1', ζ_1' sought.

Replace the moments a, b, c by their values (A) in formulae (29), and we find on putting $M_1 = M_2 = M_3$ the following conditional equations between the given quantities, λ, μ, ν, b and A, B, C, and those sought, ξ_1', η_1', ζ_1':

$$\begin{aligned} v_1^2\xi_1' &= C^2\mu(\mu\xi_1' - \lambda\eta_1') - B^2\nu(\lambda\zeta_1' - \nu\xi_1'), \\ v_1^2\eta_1' &= A^2\nu(\nu\eta_1' - \mu\zeta_1') - C^2\lambda(\mu\xi_1' - \lambda\eta_1'), \\ v_1^2\zeta_1' &= B^2\lambda(\lambda\zeta_1' - \nu\xi_1') - A^2\mu(\nu\eta_1' - \mu\zeta_1'). \end{aligned}$$

On introducing here that direction, which is at right angles to the normal (λ, μ, ν) to the given wave-front and to the direction of oscillation (ξ_1', η_1', ζ_1') sought, we can write these conditional equations in the form

$$\left.\begin{aligned} v_1^2\xi_1' &= C^2\mu\zeta_1 - B^2\nu\eta_1, \\ v_1^2\eta_1' &= A^2\nu\xi_1 - C^2\lambda\zeta_1, \\ v_1^2\zeta_1' &= B^2\lambda\eta_1 - A^2\mu\xi_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

(cf. formulae (32)), where ξ_1, η_1, ζ_1 denote the direction-cosines of that direction.

Multiply these conditional equations (B), the second by ν and the third by μ, subtract, and we have

$$\begin{aligned} v_1^2(\nu\eta_1' - \mu\zeta_1') &= A^2(\nu^2 + \mu^2)\xi_1 - B^2\lambda\mu\eta_1 - C^2\lambda\nu\zeta_1 \\ &= A^2(1 - \lambda^2)\xi_1 - \lambda(B^2\mu\eta_1 + C^2\nu\zeta_1); \end{aligned}$$

or, since

$$\xi_1 = \nu\eta_1' - \mu\zeta_1'$$

(cf. formulae (32)),

$$\left.\begin{aligned} &(A^2 - v_1^2)\xi_1 = \lambda f_1, \\ &\text{where} \quad f_1 = A^2\lambda\xi_1 + B^2\mu\eta_1 + C^2\nu\zeta_1; \\ &\text{similarly, we find} \\ &(B^2 - v_1^2)\eta_1 = \mu f_1 \quad\text{and}\quad (C^2 - v_1^2)\zeta_1 = \nu f_1. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}$$

These equations are similar in form to those (31) and (31A) found in text for plane electric waves; the following formulae will, therefore, hold here for the magnetic waves:

$$A^2\xi_1\xi_1' + B^2\eta_1\eta_1' + C^2\zeta_1\zeta_1' = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(D)}$$

$$v_1^2 = A^2\xi_1^2 + B^2\eta_1^2 + C^2\zeta_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(E)}$$

and

$$\frac{\lambda^2}{A^2 - v_1^2} + \frac{\mu^2}{B^2 - v_1^2} + \frac{\nu^2}{C^2 - v_1^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(F)}$$

(cf. pp. 341, 342). Observe the similarity between these formulae and those (34)-(37) for the electric waves: v and ξ, η, ζ of the latter are replaced here by v_1 and ξ_1, η_1, ζ_1 respectively, whereas ξ', η', ζ' are identical to ξ_1', η_1', ζ_1'. It thus follows that the magnetic oscillations will take place at right angles to the electric ones that can be propagated through the given medium, whereas the velocity of propagation of either of the two possible systems of plane magnetic waves that can be transmitted through the medium will evidently be that of the electric waves, whose oscillations are taking place at right angles to the magnetic oscillations (cf. formulae (E) and (F)), that is, v_1 may be replaced here by v.

8. Determine the amplitude of the magnetic oscillations that accompany the electric oscillations (6) of amplitude a in any assigned direction (λ, μ, ν) through a crystalline medium.

We represent the component moments a, b, c of the magnetic oscillations in question in the form (A) employed in the preceding example; here however their resultant amplitude b is not given but is sought. By formulae (4) the component forces α, β, γ acting in these magnetic oscillations can then be written in the form

$$\left.\begin{aligned}\alpha&=\frac{4\pi}{M}b\xi' e^{in\left(t-\frac{s}{v}\right)},\\ \beta&=\frac{4\pi}{M}b\eta' e^{in\left(t-\frac{s}{v}\right)},\\ \gamma&=\frac{4\pi}{M}b\zeta' e^{in\left(t-\frac{s}{v}\right)},\end{aligned}\right\}\quad\text{..............................(A)}$$

where we have put $M_1=M_2=M_3$ and replaced ξ_1', η_1', ζ_1' and v_1 by ξ', η', ζ' and v respectively (cf. Ex. 7).

Replace now P, Q, R by their values (7) and α, β, γ by the above values (A) in formulae (11), which always hold between the component electric and magnetic forces acting in electromagnetic plane-waves, and we find the following conditional equations for the determination of the amplitude b in terms of known quantities:

$$b\xi'=\frac{v_0}{v}a\left(\frac{\nu\eta}{D_2}-\frac{\mu\zeta}{D_3}\right),\quad b\eta'=\frac{v_0}{v}a\left(\frac{\lambda\zeta}{D_3}-\frac{\nu\xi}{D_1}\right),\quad b\zeta'=\frac{v_0}{v}a\left(\frac{\mu\xi}{D_1}-\frac{\lambda\eta}{D_2}\right).$$

Squared and added, these equations evidently give

$$b^2=\frac{v_0^2}{v^2}a^2\left[\left(\frac{\nu\eta}{D_2}-\frac{\mu\zeta}{D_3}\right)^2+\left(\frac{\lambda\zeta}{D_3}-\frac{\nu\xi}{D_1}\right)^2+\left(\frac{\mu\xi}{D_1}-\frac{\lambda\eta}{D_2}\right)^2\right].$$

Observe that b is here a function of the direction of propagation (λ, μ, ν).

9. Show that the following relation holds between the resultant electric force F and the resultant electric moment Y'' prevailing in electromagnetic waves transmitted through a crystalline medium:

$$F=\frac{4\pi v v_r M}{v^2}Y''=\frac{4\pi v^2 M}{v^2\cos\epsilon}Y''$$

(cf. formula (80), where $Q=F_y$).

10. Examine, as in Chapter VII., the rotation of the plane of polarization upon the reflection on the surface of a crystal of waves incident at different angles ϕ and of different azimuths of oscillation θ.

11. Examine the general problem on reflection and refraction on the surface of a crystal for perpendicular incidence.

Here, $\phi=0$, hence $\phi_1=\phi_\kappa'=0$ (cf. formula (69)), and the expressions (93A) and (93B) for the component-amplitudes at right angles to and in the plane of incidence respectively of the reflected oscillations become indeterminate; to find their real values, we write the general expressions in the form

$$\sum_{\kappa=0,e}A_{1,\kappa}\sin\Theta_{1,\kappa}=-\sum_{\kappa=0,e}a_\kappa\sin\Theta_\kappa\frac{\dfrac{\sin\phi}{\sin\phi_\kappa'}\cos\phi_\kappa'-\cos\phi}{\dfrac{\sin\phi}{\sin\phi_\kappa'}\cos\phi_\kappa'+\cos\phi}$$

and

$$\sum_{\kappa=0,e}A_{1,\kappa}\cos\Theta_{1,\kappa}=\sum_{\kappa=0,e}a_\kappa\cos\Theta_\kappa\frac{\cos\theta_\kappa'\left(\dfrac{\sin\phi}{\sin\phi_\kappa'}\cos\phi-\cos\phi_\kappa'\right)-\tan\epsilon_\kappa'\sin\phi_\kappa'}{\cos\theta_\kappa'\left(\dfrac{\sin\phi}{\sin\phi_\kappa'}\cos\phi+\cos\phi_\kappa'\right)+\tan\epsilon_\kappa'\sin\phi_\kappa'},$$

replace here $\frac{\sin\phi}{\sin\phi_\kappa'}$ by its value $\frac{v}{v_\kappa'}$ from formula (69), then put $\phi=\phi_\kappa'=0$, and we have

$$\left.\begin{aligned}\sum_{\kappa=0,e} A_{1,\kappa}\sin\Theta_{1,\kappa} &= -\sum_{\kappa=0,e} a_\kappa\sin\Theta_\kappa\frac{v-v_\kappa'}{v+v_\kappa'}\\ \text{and}\qquad \sum_{\kappa=0,e} A_{1,\kappa}\cos\Theta_{1,\kappa} &= \sum_{\kappa=0,e} a_\kappa\cos\Theta_\kappa\frac{v-v_\kappa'}{v+v_\kappa'}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(\text{A})$$

The expressions (92) for the component-amplitudes of the (two) refracted waves also become indeterminate for $\phi=\phi_\kappa'=0$; similarly, we find the following real values for those expressions:

$$\left.\begin{aligned}A_\kappa'\sin\theta_\kappa' &= 2a_\kappa\sin\Theta_\kappa\frac{v^2}{(v+v_\kappa')v_\kappa'},\\ \text{and}\qquad A_\kappa'\cos\theta_\kappa' &= 2a_\kappa\cos\Theta_\kappa\frac{v^2}{(v+v_\kappa')v_\kappa'}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(\text{B})$$

where $\kappa=0$ and e. Observe that both these (B) and the above formulae (A) no longer contain the angle ϵ_κ'.

The expressions (86) for the uniradial azimuths Θ_κ and $\Theta_{1,\kappa}$ also become indeterminate for $\phi=0$; similarly, their real values are found to be

$$\Theta_\kappa=\theta_\kappa' \text{ and } \Theta_{1,\kappa}=-\theta_\kappa';\quad\ldots\ldots\ldots\ldots(\text{C})$$

the uniradial azimuths of the incident oscillations will, therefore, coincide here with those of the two refracted oscillations (in the crystal), which can be determined by Fresnel's construction (cf. p. 343).

Since now for $\phi=0$, $\phi_0'=\phi_e'$, there will be no bifurcation of the incident waves upon their passage into the crystal; the azimuths θ_0' and θ_e' of the refracted waves will, therefore, evidently coincide with the two singular directions (cf. p. 342) at right angles to their common direction of propagation $\phi_0'=\phi_e'=0$; we have now seen on p. 343 that the singular directions are always at right angles, that is,

$$\theta_e'-\theta_0'=\pm\pi/2.\quad\ldots\ldots\ldots\ldots(\text{D})$$

By formulae (C), the expressions (91) for the component-amplitudes a_0 and a_e along the uniradial azimuths Θ_0 and Θ_e respectively will assume here the form

$$\left.\begin{aligned}a_0 &= a\frac{\sin\theta-\cos\theta\tan\theta_e'}{\sin\theta_0'-\cos\theta_0'\tan\theta_e'},\\ a_e &= a\frac{\sin\theta-\cos\theta\tan\theta_0'}{\sin\theta_e'-\cos\theta_e'\tan\theta_0'},\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(\text{E})$$

or, by the relation (D) between the two azimuths θ_0' and θ_e', upon the elimination of the latter, the simple form

$$\begin{aligned}a_0 &= a\cos(\theta-\theta_0'),\\ a_e &= \pm a\sin(\theta-\theta_0').\end{aligned}$$

Replace a_0 and a_e by these and Θ_κ and $\Theta_{1,\kappa}$ by their values in terms of θ' and θ_0' from formulae (C) and (D) in formulae (A) and (B), and we have

$$\left.\begin{aligned}\sum_{\kappa=0,e}A_{1,\kappa}\sin\Theta_{1,\kappa} &= -a\left[\sin\theta_0'\cos(\theta-\theta_0')\frac{v-v_0'}{v+v_0'}+\cos\theta_0'\sin(\theta-\theta_0')\frac{v-v_e'}{v+v_e'}\right],\\ \sum_{\kappa=0,e}A_{1,\kappa}\cos\Theta_{1,\kappa} &= a\left[\cos\theta_0'\cos(\theta-\theta_0')\frac{v-v_0'}{v+v_0'}-\sin\theta_0'\sin(\theta-\theta_0')\frac{v-v_e'}{v+v_e'}\right]\end{aligned}\right\}$$

$$\left.\begin{aligned}\text{and}\qquad A_0' &= 2a\cos(\theta-\theta_0')\frac{v^2}{(v+v_0')v_0'},\\ A_e' &= \pm 2a\sin(\theta-\theta_0')\frac{v^2}{(v+v_e')v_e'}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(\text{F})$$

The former expressions evidently give the following values for the resultant amplitude and azimuth of the reflected oscillations:

$$\left.\begin{aligned} &a_1^2=a^2\left[\cos^2(\theta-\theta_0')\left(\frac{v-v_0'}{v+v_0'}\right)^2+\sin^2(\theta-\theta_0')\left(\frac{v-v_e'}{v+v_e'}\right)^2\right] \\ &\text{and} \\ &\tan\theta_1=-\frac{\sin\theta_0'\cos(\theta-\theta_0')(v-v_0')(v+v_e')+\cos\theta_0'\sin(\theta-\theta_0')(v+v_0')(v-v_e')}{\cos\theta_0'\cos(\theta-\theta_0')(v-v_0')(v+v_e')-\sin\theta_0'\sin(\theta-\theta_0')(v+v_0')(v-v_e')}. \end{aligned}\right\}\ldots(\text{G})$$

On comparing these formulae with those (36, VII.) that hold on the surface of an isotropic insulator, we observe that, aside from the (two) refracted waves with common direction of propagation but of different azimuths of oscillation and velocities of propagation in place of a single refracted wave, the reflected oscillations are quite differently constituted in the two cases.

12. Show that for $\theta=\theta_0'$ formulae (F) and (G), Ex. 11, reduce to

$$A_0'=\frac{2av^2}{(v+v_0')v_0'}, \qquad A_e'=0$$

and

$$a_1=a\,\frac{v-v_0'}{v+v_0'}, \qquad \tan\theta_1=-\tan\theta_0'=-\tan\theta.$$

Observe the similarity in form between these formulae and those (36, VII.) (cf. also formulae (34) and (34A), Chapter VII.) for two adjacent isotropic insulators.

13. Examine the problem on reflection and refraction on the surface of a crystal for the particular case, where $\phi+\phi_\kappa'=\pi/2$.

There are evidently two angles of incidence ϕ, for the one of which $\phi+\phi_0'=\pi/2$ and for the other $\phi+\phi_e'=\pi/2$; let us examine here the former case.

For $\kappa=0$ formulae (87) assume here the form

$$A_{1,0}\sin\Theta_{1,0}=a_0\sin\Theta_0\cos 2\phi,$$

$$A_{1,0}\cos\Theta_{1,0}=-a_0\cos\Theta_0\frac{\tan\epsilon_0'}{2\cos\theta_0'\tan\phi+\tan\epsilon_0'},$$

$$A_0'\sin\theta_0'=2a_0\sin\Theta_0\sin^2\phi,$$

$$A_0'\cos\theta_0'=2a_0\cos\Theta_0\frac{\cos\theta_0'\tan\phi}{2\cos\theta_0'+\tan\epsilon_0'\cot\phi},$$

and formulae (86) the form

$$\tan\Theta_0=\frac{\sin\theta_0'}{\cos\theta_0'\sin 2\phi+\tan\epsilon_0'\cos^2\phi},$$

$$\tan\Theta_{1,0}=-\frac{2\sin\theta_0'\tan\phi}{\tan\epsilon_0'}.$$

Θ_0 is to be replaced by this particular value in formulae (91) for the component-amplitudes a_0 and a_e along the two uniradial azimuths Θ_0 and Θ_e respectively. Observe that these amplitudes remain here functions of the azimuth θ of the incident oscillations; it thus follows from formulae (95) that the resultant azimuth θ_1 of the reflected oscillations will vary for different values of θ. We have now seen in Chapter VII. on isotropic insulators that for $\phi+\phi'=\pi/2$ the resultant azimuth of the reflected oscillations was not a function of the azimuth of the incident ones; it thus follows that ordinary electromagnetic (light) waves incident at the angle $\phi+\phi_\kappa'=\pi/2$ will be reflected as linearly polarized waves only when the reflecting surface is that of an isotropic insulator (cf. also p. 376).

14. Examine the problem on reflection and refraction for the particular case, where the waves pass from an isotropic into a crystalline medium and the normal to the reflecting surface employed is parallel to one of the principal axes of the crystal.

Let us take here that cross-section of the crystal as reflecting-surface, whose normal is parallel to its principal axis x' (D_1'). The coordinate-axes x and x' in Figure 40 will then coincide, whereas the angles between the other two pairs of axes, y, y' and z, z', will become equal; let us denote this common angle by ω', as indicated in Fig. 42 below.

For the given problem Maxwell's equations for the electric and magnetic moments will retain their general form (55) and (56), only the relations (57) between the electric moments and the electric forces in the crystal will assume simpler form; we evidently have here

$$\left.\begin{array}{lll} \cos(x', x) = 1, & \cos(x', y) = 0, & \cos(x', z) = 0, \\ \cos(y', x) = 0, & \cos(y', y) = \cos\omega', & \cos(y', z) = \cos(90^\circ + \omega') = -\sin\omega', \\ \cos(z', x) = 0, & \cos(z', y) = \cos(90^\circ - \omega') = \sin\omega', & \cos(z', z) = \cos\omega' \end{array}\right\} \quad (\text{A})$$

(cf. Figure 42), and hence

$$\left.\begin{array}{lll} D'_{11} = D_1', & D'_{12} = 0, & D'_{13} = 0, \\ D'_{21} = 0, & D'_{22} = D_2'\cos^2\omega' + D_3'\sin^2\omega', & D'_{23} = (D_2' - D_3')\sin\omega'\cos\omega', \\ D'_{31} = 0, & D'_{32} = (D_2' - D_3')\sin\omega'\cos\omega', & D'_{33} = D_2'\sin^2\omega' + D_3'\cos^2\omega' \end{array}\right\} \ldots(\text{B})$$

(cf. p. 355).

Replace the D''s by these values in formulae (57), and we have

$$\left.\begin{array}{l} X' = \dfrac{D_1'}{4\pi} P', \\ Y' = \dfrac{1}{4\pi}[(D_2'\cos^2\omega' + D_3'\sin^2\omega')Q' + (D_2' - D_3')\sin\omega'\cos\omega' R'], \\ Z' = \dfrac{1}{4\pi}[(D_2' - D_3')\sin\omega'\cos\omega' Q' + (D_2'\sin^2\omega' + D_3'\cos^2\omega')R'] \end{array}\right\} \ldots\ldots(\text{C})$$

within the crystal, where all quantities shall be characterized by the insertion of the $'$.

The angles ϵ_0' and ϵ_e' are determined by the formula

$$\cos\epsilon_\kappa' = \frac{v_\kappa'^2}{\sqrt{A'^4\bar{\xi}_\kappa'^2 + B'^4\bar{\eta}_\kappa'^2 + C'^4\bar{\zeta}_\kappa'^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{D})$$

(cf. formula (48)), where $\bar{\xi}_\kappa'$, $\bar{\eta}_\kappa'$, $\bar{\zeta}_\kappa'$ denote the direction-cosines of oscillation in the refracted waves κ ($\kappa = 0$ and e) referred to the principal axes x', y', z' of the crystal, v_κ' their velocity of propagation and A', B', C' the constants of the crystal (cf. formulae (30A)). The variables $\bar{\xi}_\kappa'$, $\bar{\eta}_\kappa'$, $\bar{\zeta}_\kappa'$ can now be expressed as functions of θ_κ', ϕ_κ', the variables introduced on pp. 358-360, and the direction cosines between the two systems of coordinates x, y, z and x', y', z' (cf. Fig. 40); for the given case, which is represented graphically in Fig. 42 below, the former can now readily be expressed in terms of θ_κ', ϕ_κ', and the above angle ω', as follows:

The component of the resultant amplitude a_κ' along the xy-plane is

$$a_\kappa'\cos\theta_\kappa'$$

and its component parallel to the z-axis $a_\kappa'\sin\theta_\kappa'$;

	the component of $a_\kappa'\cos\theta_\kappa'$	parallel to the	$x = x'$-axis $a_\kappa'\cos\theta_\kappa'\sin\phi_\kappa'$,
	,, $a_\kappa'\cos\theta_\kappa'$	,,	y-axis $a_\kappa'\cos\theta_\kappa'\cos\phi_\kappa'$,
	,, $a_\kappa'\sin\theta_\kappa'$	,,	y'-axis $-a_\kappa'\sin\theta_\kappa'\sin\omega'$,
	,, $a_\kappa'\sin\theta_\kappa'$	,,	z'-axis $a_\kappa'\sin\theta_\kappa'\cos\omega'$,
	,, $a_\kappa'\cos\theta_\kappa'\cos\phi_\kappa'$	,,	y'-axis $a_\kappa'\cos\theta_\kappa'\cos\phi_\kappa'\cos\omega'$
and	,, $a_\kappa'\cos\theta_\kappa'\cos\phi_\kappa'$	,,	z'-axis $a_\kappa'\cos\theta_\kappa'\cos\phi_\kappa'\sin\omega'$.

The resultant component amplitudes will, therefore, be

$$a_\kappa' \cos\theta_\kappa' \sin\phi_\kappa' \text{ parallel to the } x'\text{-axis},$$

$$a_\kappa' (\cos\theta_\kappa' \cos\phi_\kappa' \cos\omega' - \sin\theta_\kappa' \sin\omega') \text{ parallel to the } y'\text{-axis}$$

and $$a_\kappa' (\cos\theta_\kappa' \cos\phi_\kappa' \sin\omega' + \sin\theta_\kappa' \cos\omega') \text{ parallel to the } z'\text{-axis};$$

hence
$$\left.\begin{aligned} \bar{\xi}_\kappa' &= \cos\theta_\kappa' \sin\phi_\kappa', \\ \bar{\eta}_\kappa' &= \cos\theta_\kappa' \cos\phi_\kappa' \cos\omega' - \sin\theta_\kappa' \sin\omega', \\ \bar{\zeta}_\kappa' &= \cos\theta_\kappa' \cos\phi_\kappa' \sin\omega' + \sin\theta_\kappa' \cos\omega'. \end{aligned}\right\} \quad \text{......................(E)}$$

Formula (D) will thus assume here the following particular form in terms of v_κ', θ_κ', ϕ_κ' and ω':

$$\cos\epsilon_\kappa' = \frac{v_\kappa'^2}{\sqrt{\begin{aligned} A'^4 \cos^2\theta_\kappa' \sin^2\phi_\kappa' &+ B'^4 (\cos\theta_\kappa' \cos\phi_\kappa' \cos\omega' - \sin\theta_\kappa' \sin\omega')^2 \\ &+ C'^4 (\cos\theta_\kappa' \cos\phi_\kappa' \sin\omega' + \sin\theta_\kappa' \cos\omega')^2 \end{aligned}}} \quad \text{.......(F)}$$

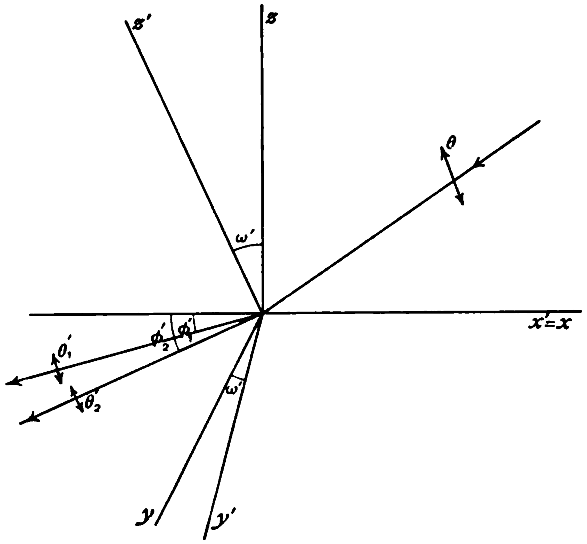

FIG. 42.

The direction-cosines $\bar{\lambda}_\kappa'$, $\bar{\mu}_\kappa'$, $\bar{\nu}_\kappa'$ of the normal n_κ to the wave-fronts of the refracted waves κ (referred to the principal axes x', y', z' of the crystal) can evidently be expressed here as follows in terms of ϕ_κ' and ω':

$$\left.\begin{aligned} &\bar{\lambda}_\kappa' = \cos(n_\kappa', x') = \cos(n_\kappa', x) = \cos(\phi_\kappa' + \pi) = -\cos\phi_\kappa', \\ &\bar{\mu}_\kappa' = \cos(n_\kappa', y') = \cos(n_\kappa', x)\cos(y', x) + \cos(n_\kappa', y)\cos(y', y) + \cos(n_\kappa', z)\cos(y', z), \\ &\text{or, since here } \cos(y', x) = 0 \text{ and } \cos(n_\kappa', z) = 0 \text{ (cf. Fig. 42)}, \\ &\bar{\mu}_\kappa' = \cos(n_\kappa', y)\cos(y', y) = \sin\phi_\kappa' \cos\omega', \end{aligned}\right\} \text{(G)}$$

and similarly $$\bar{\nu}_\kappa' = \sin\phi_\kappa' \sin\omega'$$

(cf. Fig. 42).

By formulae (38) and (39) the azimuth θ_κ' of the refracted oscillations κ is determined by one of the principal axes of the ellipse intersected on the plane

$$\left.\begin{aligned} \bar{\lambda}_\kappa' x' + \bar{\mu}_\kappa' y' + \bar{\nu}_\kappa' z' &= 0 \\ \text{by the ellipsoid} \qquad A'^2 x'^2 + B'^2 y'^2 + C'^2 z'^2 &= 1. \end{aligned}\right\} \quad \text{...................................(H)}$$

To express these equations in terms of the variables ϕ_κ', ω', etc. (referred to the coordinates x, y, z), we observe that the following relations hold here between the two systems of coordinates x, y, z and x', y', z':

$$\begin{aligned} x' &= x, \\ y' &= y \cos\omega' - z \sin\omega', \\ z' &= y \sin\omega' + z\cos\omega' \end{aligned}$$

(cf. Fig. 42). Replace x', y', z' by these and $\bar{\lambda}_\kappa'$, $\bar{\mu}_\kappa'$, $\bar{\nu}_\kappa'$ by the above values (G) in formulae (H), and we have

$$\left.\begin{aligned} &-\cos\phi_\kappa' x + \sin\phi_\kappa' y = 0 \\ \text{and}\qquad &A'^2x^2 + B'^2(y\cos\omega' - z\sin\omega')^2 + C'^2(y\sin\omega' + z\cos\omega')^2 = 1\,; \end{aligned}\right\} \quad\text{............(I)}$$

the azimuth θ_κ' determined by one of the principal axes of the ellipse intersected by these two surfaces (I) will thus be a function of ϕ_κ', ω' and the medium constants A', B', C'.

By formula (40) and the above values (G) for $\bar{\lambda}_\kappa'$, $\bar{\mu}_\kappa'$, $\bar{\nu}_\kappa'$ we can write the velocity of propagation of the refracted waves κ here in the form

$$\left.\begin{aligned} v_\kappa'^2 = \tfrac{1}{2}\Big\{&(B'^2 + C'^2)\cos^2\phi_\kappa' + (A'^2 + C'^2)\sin^2\phi_\kappa'\cos^2\omega' + (A'^2 + B'^2)\sin^2\phi_\kappa'\sin^2\omega' \\ &\pm\sqrt{\begin{aligned}&[(B'^2 - C'^2)\cos^2\phi_\kappa' - (A'^2 - C'^2)\sin^2\phi_\kappa'\cos^2\omega' - (A'^2 - B'^2)\sin^2\phi_\kappa'\sin^2\omega']^2 \\ &\qquad + 4(A'^2 - C'^2)(B'^2 - C'^2)\sin^2\phi_\kappa'\cos^2\phi_\kappa'\cos^2\omega'\end{aligned}}\Big\} \\ = \tfrac{1}{2}\Big\{&A'^2\sin^2\phi_\kappa' + B'^2(1 - \sin^2\phi_\kappa'\cos^2\omega') + C'^2(1 - \sin^2\phi_\kappa'\sin^2\omega') \\ &\pm\sqrt{\begin{aligned}&[-A'^2\sin^2\phi_\kappa' + B'^2(1 - \sin^2\phi_\kappa'\cos^2\omega') - C'^2(\cos^2\phi_\kappa' - \sin^2\phi_\kappa'\cos^2\omega')]^2 \\ &\qquad + 4(A'^2 - C'^2)(B'^2 - C'^2)\sin^2\phi_\kappa'\cos^2\phi_\kappa'\cos^2\omega'\end{aligned}}\Big\} \end{aligned}\right\} \text{(J)}$$

(for the choice of square root-sign see p. 345). On replacing here v_κ' by its value from formula (69), we obtain an equation for the determination of ϕ_κ' as function of ϕ, v, ω' and the medium constants A', B', C', all of which are given; for the actual determination of ϕ_κ' see the ensuing particular cases of the given general one, for which the given equation will assume simpler forms. After having thus determined ϕ_κ', we could then express v_κ' by formula (J) in terms of ϕ, v, ω' and A', B', C'.

Upon the determination of ϕ_κ' and v_κ' in terms of ϕ, v, ω' and A, B', C', we could then express ϵ_κ' by formula (F) in terms of the latter and θ_κ', which (θ_κ') could be determined by formulae (I), as formulated above, as function of ϕ_κ', ω' and A', B', C', hence ϕ, v, ω' and A', B', C' (cf. above).

For the given case, where the medium O or that of the incident waves is isotropic, we determine first by formulae (86) the uniradial azimuths Θ_0 and Θ_e as functions of ϕ, ϕ_κ', θ_κ', and ϵ_κ', then by formulae (91) the component amplitudes a_0 and a_e along those azimuths as functions of a, θ and Θ_κ, hence a, θ, ϕ, ϕ_κ', θ_κ' and ϵ_κ', and lastly the amplitudes A_0' and A_e' of the two refracted waves by formulae (92) and the resultant amplitude a_1 and azimuth θ_1 of the reflected wave by formulae (94) and (95) respectively as functions of a_κ, Θ_κ, ϕ, ϕ_κ', θ_κ' and ϵ_κ', hence a, θ, ϕ, ϕ_κ', θ_κ' and ϵ_κ', and we then replace in the formulae found the quantities ϕ_κ', θ_κ' and ϵ_κ' by the above values in terms of v, ϕ, ω' and A', B', C'.

15. Examine, as in Ex. 14, the problem on reflection and refraction for the particular case, where the waves pass from an isotropic into a crystalline medium and either the principal axis y' (D_2') or z' (D_3') of the crystal coincides with the y or z-axis respectively of the system of coordinates x, y, z, to which the reflecting surface is referred (cf. also Fig. 40).

16. Show for the particular case of Ex. 14, where $\omega'=0$, that is, where the principal axes x', y', z' of the crystal coincide with the coordinate-axes x, y, z (cf. Figs. 40 and 42), that formula (J), Ex. 14, assumes the simple form

$$v_\kappa'^2=\tfrac{1}{2}\left[A'^2\sin^2\phi_\kappa'+B'^2\cos^2\phi_\kappa'+C'^2\pm\sqrt{(-A'^2\sin^2\phi_\kappa'-B'^2\cos^2\phi_\kappa'+C'^2)^2}\right],$$

hence $$v_0'^2=C'^2 \text{ and } v_e'^2=A'^2\sin^2\phi_e'+B'^2\cos^2\phi_e' \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

(cf. p. 345), and then determine ϕ_κ', v_κ', θ_κ' and ϵ_κ' in terms of ϕ, v and A', B', C', and a_κ, Θ_κ, A_κ', a_1 and θ_1 in terms of a, θ, ϕ, v and A', B', C' (cf. Ex. 14).

Replace v_0' and v_e' by their values from formula (69) in formulae (A), and we have

$$\sin^2\phi_0'=\frac{C'^2\sin^2\phi}{v^2}$$

and $$\sin^2\phi_e'=\frac{A'^2\sin^2\phi_e'+B'^2\cos^2\phi_e'}{v^2}\sin^2\phi,$$

hence $$\sin^2\phi_e'=\frac{B'^2\sin^2\phi}{v^2-(A'^2-B'^2)\sin^2\phi},$$

that is, ϕ_0' and ϕ_e' determined as functions of ϕ, v and A', B', C'. By these expressions for ϕ_0' and ϕ_e' and by formula (69) we then find the following values for v_0' and v_e' in terms of ϕ, v and A', B', C':

$$v_0'=C' \text{ and } v_e'=\frac{vB'}{\sqrt{v^2-(A'^2-B'^2)\sin^2\phi}}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

Since now by assumption $A'^2>B'^2>C'^2$ (cf. p. 344), it follows from Fresnel's construction for the determination of the two singular directions of oscillation in a crystalline medium that for all values of ϕ_κ' the one or longer principal axis of the ellipse formed by the intersection of the plane

$$-\cos\phi_\kappa' x+\sin\phi_\kappa' y=0$$

and the ellipsoid $$A'^2x^2+B'^2y^2+C'^2z^2=1,$$

the particular form assumed by formulae (I), Ex. 14, for $\omega'=0$, will coincide here with the z-axis of our coordinates x, y, z, whereas the other principal axis will evidently lie in the xy-plane, the plane of incidence; that is, $\theta_\kappa'=\pi/2$ or 0 ($\kappa=0$ and e). By Fresnel's construction all oscillations parallel to the z-axis will now be propagated here with one and the same velocity (cf. p. 343); by formulae (B) the oscillations that travel with one and the same velocity of propagation for all values of ϕ_κ' are those of the ordinary refracted wave; it thus follows that

$$\theta_0'=\pi/2 \text{ and } \theta_e'=0.$$

Replace ϕ_κ', v_κ', and θ_κ' by the above values in formula (F), Ex. 14, for ϵ_κ', and we have

$$\cos\epsilon_0'=1$$

and
$$\begin{aligned}\cos\epsilon_e'&=\frac{v^2B'^2}{[v^2-(A'^2-B'^2)\sin^2\phi]\sqrt{A'^4\sin^2\phi_e'+B'^4\cos^2\phi_e'}}\\&=\frac{v^2B'}{\sqrt{v^2-(A'^2-B'^2)\sin^2\phi}\sqrt{v^2B'^2+A'^2(A'^2-B'^2)\sin^2\phi}}\\&=\frac{v^2B'}{\sqrt{v^4B'^4+v^2(A'^2-B'^2)^2\sin^2\phi-A'^2(A'^2-B'^2)^2\sin^4\phi}},\end{aligned}$$

hence $$\epsilon_0'=0$$

and $$\tan\epsilon_e'=\frac{(A'^2-B'^2)\sin\phi\sqrt{v^2-A'^2\sin^2\phi}}{v^2B'},$$

that is, ϵ_0' and ϵ_e' determined as functions of ϕ, v and A', B', C' (cf. Ex. 14).

By the above values for θ_κ' and ϵ_κ' formulae (86) evidently give

$$\tan\Theta_0=\infty \quad \text{and} \quad \tan\Theta_e=0$$

or

$$\Theta_0=\pi/2 \quad \text{and} \quad \Theta_e=0,$$

and hence formulae (91)

$$a_0=a\sin\theta \quad \text{and} \quad a_e=a\cos\theta.$$

Next, replace ϕ_κ', θ_κ', ϵ_κ', Θ_κ and a_κ by the above values in formulae (92) for the amplitudes A_0' and A_e' of the two refracted waves, and we have

$$\begin{aligned} A_0' &= a\sin\theta\,\frac{\sin\phi}{\sin\phi_0'}\,\frac{\sin 2\phi}{\sin(\phi+\phi_0')} \\ &= a\sin\theta\,\frac{2v^2\cos\phi}{C'(\sqrt{v^2-C'^2\sin^2\phi}+C'\cos\phi)} \end{aligned}$$

$$\text{and } A_e'=a\cos\theta\,\frac{\sin\phi}{\sin\phi_e'}\,\frac{\sin 2\phi}{\sin(\phi+\phi_e')\cos(\phi-\phi_e')+\dfrac{A'^2-B'^2}{v^2B'}\sin\phi\sqrt{v^2-A'^2\sin^2\phi}\,\sin^2\phi_e'}$$

$$=a\cos\theta\,\frac{2v^2\cos\phi\sqrt{v^2-(A'^2-B'^2)\sin^2\phi}}{B'\left[v^2\cos\phi+B'\sqrt{v^2-A'^2\sin^2\phi}\,.\,\dfrac{v^2+(A'^2-B'^2)\sin^2\phi}{v^2-(A'^2-B'^2)\sin^2\phi}\right]},$$

that is, A_0' and A_e' determined as functions of a, θ, ϕ, v and A', B', C' (cf. Ex. 14).

Lastly, replace ϕ_κ', θ_κ', ϵ_κ', Θ_κ and a_κ by the above values in formulae (94) and (95) for the resultant amplitude a_1 and azimuth θ_1 of the reflected wave, and we find

$$\begin{aligned} a_1^2 = a^2\Bigg\{ & \sin^2\theta\,\frac{\sin^2(\phi-\phi_0')}{\sin^2(\phi+\phi_0')} \\ & +\cos^2\theta\left[\frac{\sin(\phi-\phi_e')\cos(\phi+\phi_e')-\dfrac{A'^2-B'^2}{v^2B'}\sin\phi\sqrt{v^2-A'^2\sin^2\phi}\,\sin^2\phi_e'}{\sin(\phi+\phi_e')\cos(\phi-\phi_e')+\dfrac{A'^2-B'^2}{v^2B'}\sin\phi\sqrt{v^2-A'^2\sin^2\phi}\,\sin^2\phi_e'}\right]^2\Bigg\} \end{aligned}$$

$$\begin{aligned} =a^2\Bigg\{ & \sin^2\theta\left[\frac{\sqrt{v^2-C'^2\sin^2\phi}-C'\cos\phi}{\sqrt{v^2-C'^2\sin^2\phi}+C'\cos\phi}\right]^2+\cos^2\theta \\ & \times\left[\frac{v^2[v^2-(A'^2-B'^2)\sin^2\phi]\cos\phi-B'\sqrt{v^2-A'^2\sin^2\phi}\,[v^2+(A'^2-B'^2)\sin^2\phi]}{v^2[v^2-(A'^2-B'^2)\sin^2\phi]\cos\phi+B'\sqrt{v^2-A'^2\sin^2\phi}\,[v^2+(A'^2-B'^2)\sin^2\phi]}\right]^2\Bigg\} \end{aligned}$$

and

$$\begin{aligned} \tan\theta_1 &= \tan\theta\,\frac{\sin(\phi-\phi_0')}{\sin(\phi+\phi_0')} \\ &\quad\times\frac{\sin(\phi+\phi_e')\cos(\phi-\phi_e')+\dfrac{A'^2-B'^2}{v^2B'}\sin\phi\sqrt{v^2-A'^2\sin^2\phi}\,\sin^2\phi_e'}{\sin(\phi-\phi_e')\cos(\phi+\phi_e')-\dfrac{A'^2-B'^2}{v^2B'}\sin\phi\sqrt{v^2-A'^2\sin^2\phi}\,\sin^2\phi_e'} \\ &= -\tan\theta\,\frac{\sqrt{v^2-C'^2\sin^2\phi}-C'\cos\phi}{\sqrt{v^2-C'^2\sin^2\phi}+C'\cos\phi} \\ &\quad\times\frac{v^2[v^2-(A'^2-B'^2)\sin^2\phi]\cos\phi+B'\sqrt{v^2-A'^2\sin^2\phi}\,[v^2+(A'^2-B'^2)\sin^2\phi]}{v^2[v^2-(A'^2-B'^2)\sin^2\phi]\cos\phi-B'\sqrt{v^2-A'^2\sin^2\phi}\,[v^2+(A'^2-B'^2)\sin^2\phi]}, \end{aligned}$$

that is, a_1 and θ_1 determined as functions of a, θ, ϕ, v and A', B', C' (cf. Ex. 14).

17. Show for the particular case of perpendicular incidence on the surface of a crystal (Ex. 11), where the normal to the reflecting surface coincides with one of the principal axis, the x'-axis (cf. Ex. 14), of the crystal, that the four unknown quantities A_0', A_e', a_1 and θ_1 are determined by the formulae

$$A_0' = 2a \sin(\theta+\omega')\frac{v^2}{(v+C')C'},$$

$$A_e' = 2a \cos(\theta+\omega')\frac{v^2}{(v+B')B'},$$

$$a_1^2 = a^2\left[\sin^2(\theta+\omega')\left(\frac{v-C'}{v+C'}\right)^2 + \cos^2(\theta+\omega')\left(\frac{v-B'}{v+B'}\right)^2\right]$$

and $$\tan\theta_1 = -\frac{\cos\omega'\sin(\theta+\omega')(v-C')(v+B') - \sin\omega'\cos(\theta+\omega')(v+C')(v-B')}{\sin\omega'\sin(\theta+\omega')(v-C')(v+B') + \cos\omega'\cos(\theta+\omega')(v+C')(v-B')}.$$

It is evident from Fresnel's construction that the two azimuths θ_0' and θ_e' of oscillation in the crystal will be here

$$\theta_0' = \pi/2 - \omega' \text{ and } \theta_e' = -\omega' \quad \text{......(A)}$$

(cf. Ex. 16), where the relative position of the axes x, y, z and x', y', z' is that represented in Figure 42.

For $\phi=0$, hence $\phi_\kappa'=0$, formulae (A), Ex. 16, evidently give

$$v_0' = C' \text{ and } v_e' = B'. \quad \text{......(B)}$$

[For perpendicular incidence there will be, strictly speaking, two singular directions of oscillation with one common direction of propagation within the crystal, that is, there will be no bifurcation of the incident waves into ordinary and extraordinary (refracted) ones. We cannot, therefore, well discriminate here between the velocities of propagation of these two systems of oscillation, whether the one or the other value correspond to the velocity of the ordinary or to that of the extraordinary wave; for this reason these values for the velocities of propagation can easily get interchanged according to the method of treatment of the problems in question (cf. following examples)].

By formulae (A), formulae (E), Ex. 11, evidently assume here the simple form

$$\left.\begin{aligned} a_0 &= a\sin(\theta+\omega'),\\ a_e &= a\cos(\theta+\omega'). \end{aligned}\right\} \quad \text{......(C)}$$

Replace θ_κ', v_κ', and a_κ by the above values in formulae (F) and (G), Ex. 11, and we find the formulae sought.

18. Show for $\phi=0$ and $\omega'=0$ that the formulae of both Ex.'s 16 and 17 for the four unknown quantities A_0', A_e', a_1 and θ_1 reduce to

$$A_0' = 2a\sin\theta\frac{v^2}{(v+C')C'},$$

$$A_e' = 2a\cos\theta\frac{v^2}{(v+B')B'},$$

$$a_1^2 = a^2\left[\sin^2\theta\left(\frac{v-C'}{v+C'}\right)^2 + \cos^2\theta\left(\frac{v-B'}{v+B'}\right)^2\right]$$

and $$\tan\theta_1 = -\tan\theta\frac{(v-C')(v+B')}{(v+C')(v-B')}.$$

19. Examine the particular case of perpendicular incidence on the surface of a crystal (Ex. 11), where either the principal axis y' (D_2') or z' (D_3') of the crystal coincides with the y or z-axis respectively of the system of coordinates x, y, z, to which the reflecting surface is referred (cf. Fig. 40).

20. Examine the problem on reflection and refraction on the surface of a crystal for the particular case, where the principal axes of the crystal coincide with the coordinate-axes x, y, z (Ex. 16) and the angle of incidence ϕ is so chosen that $\phi + \phi_0' = \pi/2$.

By Ex. 16, where ϕ was arbitrary, we had

$$\sin^2\phi_0' = \frac{C'^2 \sin^2\phi}{v^2}$$

and

$$\sin^2\phi_e' = \frac{B'^2 \sin^2\phi}{v^2 - (A'^2 - B'^2)\sin^2\phi}.$$

Replace here ϕ_0' by its present value $\left(\frac{\pi}{2} - \phi\right)$, and we find the following value for ϕ:

$$\sin^2\phi = \frac{v^2}{v^2 + C'^2},$$

and hence the following values for ϕ_0' and ϕ_e':

$$\sin^2\phi_0' = \frac{C'^2}{v^2 + C'^2}$$

and

$$\sin^2\phi_e' = \frac{B'^2}{v^2 - A'^2 + B'^2 + C'^2}.$$

Replace ϕ, ϕ_0' and ϕ_e' by these particular values in the more general formulae of Ex. 16, and we find

$$\epsilon_0' = 0, \quad \tan\epsilon_e' = \frac{(A'^2 - B'^2)\sqrt{v^2 - (A'^2 - C'^2)}}{B'(v^2 + C'^2)},$$

$$A_0' = a \sin\theta \frac{2v^2}{v^2 + C'^2},$$

$$A_e' = a\cos\theta \frac{2v^2 C'(v^2 - A'^2 + B'^2 + C'^2)^{\frac{3}{2}}}{B'\sqrt{v^2 + C'^2}\left[vC'(v^2 - A'^2 + B'^2 + C'^2) + B'\sqrt{v^2 - A'^2 + C'^2}\,(v^2 + A'^2 - B'^2 + C'^2)\right]},$$

$$a_1^2 = a^2\left\{\sin^2\theta\left(\frac{v^2 - C'^2}{v^2 + C'^2}\right)^2 + \cos^2\theta\left[\frac{vC'(v^2 - A'^2 + B'^2 + C'^2) - B'\sqrt{v^2 - A'^2 + C'^2}\,(v^2 + A'^2 - B'^2 + C'^2)}{vC'(v^2 - A'^2 + B'^2 + C'^2) + B'\sqrt{v^2 - A'^2 + C'^2}\,(v^2 + A'^2 - B'^2 + C'^2)}\right]^2\right\}$$

and

$$\tan\theta_1 = -\tan\theta\,\frac{v^2 - C'^2}{v^2 + C'^2}\cdot\frac{vC'(v^2 - A'^2 + B'^2 + C'^2) + B'\sqrt{v^2 - A'^2 + C'^2}\,(v^2 + A'^2 - B'^2 + C'^2)}{vC'(v^2 - A'^2 + B'^2 + C'^2) - B'\sqrt{v^2 - A'^2 + C'^2}\,(v^2 + A'^2 - B'^2 + C'^2)}.$$

21. Show for the particular case of reflection and refraction on the surface of a crystal, where the principal axes of the crystal coincide with the coordinate-axes x, y, z (Ex. 16) and the angle of incidence ϕ is so chosen that $\phi + \phi_e' = \pi/2$, that ϕ is determined by the equation

$$(A'^2 - B'^2)\sin^4\phi - (v^2 + A'^2)\sin^2\phi + v^2 = 0 \text{.........................(A)}$$

22. Examine the particular case of reflection and refraction on the surface of a crystal, where the principal axes of the crystal coincide with the coordinate-axes x, y, z (Ex. 16) and the angle of incidence ϕ is so chosen that the two uniradial azimuths $\Theta_{1,0}$ and $\Theta_{1,e}$ of the reflected oscillations coincide.

Replace Θ_{10} and Θ_{1e} by their values in terms of ϕ, ϕ_κ', θ_κ' and ϵ_κ' (cf. formulae (86)) in the conditional equation $\Theta_{10}=\Theta_{1e}$ for the determination of ϕ, and we have

$$\frac{\sin\theta_0'\sin(\phi-\phi_0')}{\cos\theta_0'\sin(\phi-\phi_0')\cos(\phi+\phi_0')-\tan\epsilon_0'\sin^2\phi_0'}$$
$$=\frac{\sin\theta_e'\sin(\phi-\phi_e')}{\cos\theta_e'\sin(\phi-\phi_e')\cos(\phi+\phi_e')-\tan\epsilon_e'\sin^2\phi_e'}$$

(the particular angle of incidence ϕ determined by this equation was denoted by Φ in text).

Next, replace in this conditional equation for ϕ θ_0' and θ_e' by their values $\pi/2$ and 0 respectively for the given particular case $\omega'=0$ (cf. Ex. 16), and we have

$$\frac{\sin(\phi-\phi_0')}{0}=\frac{0}{\sin(\phi-\phi_e')\cos(\phi+\phi_e')-\tan\epsilon_e'\sin^2\phi_e'}, \quad \ldots\ldots\ldots\ldots\ldots(\text{A})$$

hence
$$\sin(\phi-\phi_0')=0$$
or
$$\sin(\phi-\phi_e')\cos(\phi+\phi_e')-\tan\epsilon_e'\sin^2\phi_e'=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{B})$$

The former of these two conditional equations evidently gives either $C'=v$, which corresponds to no dividing surface (with respect to the $z=z'$ axis), or $\phi=\phi_0'=0$, hence $\phi_e'=0$, for which the azimuths Θ_{10} and Θ_{1e} do not coincide, as we have seen in Ex. 11—the left hand member of the conditional equation (A) then becomes indeterminate ($\gtrless 0$).

On replacing ϕ_e' and ϵ_e' by their values from Ex. 16 for the given case, $\omega'=0$, in the latter conditional equation (B), we can write it in the form

$$v^2[v^2-(A'^2-B'^2)\sin^2\phi]\cos\phi=B'\sqrt{v^2-A'^2\sin^2\phi}\,[v^2+(A'^2-B'^2)\sin^2\phi], \;\ldots\ldots(\text{C})$$

the equation sought for the determination of ϕ in terms of v and A', B', C'; observe that this equation does not contain the azimuth of oscillation θ of the incident waves.

Equation (C) gives

$$(A'^2-B'^2)(A'^2B'^2-v^2)\sin^6\phi+(A'^2-B'^2)[2v^4+(A'^2-B'^2)v^2+(A'^2+B'^2)B'^2]v^2\sin^4\phi$$
$$-[v^4+2(A'^2-B'^2)v^2+(A'^2-2B'^2)B'^2]v^4\sin^2\phi+(v^2-B'^2)v^6=0, \;(\text{D})$$

an equation of the third degree in $\sin^2\phi$. The actual solution of this equation and the examination of its roots is of particular interest (cf. C. Curry: "On the Electromagnetic Theory of Reflection and Refraction on the Surface of Crystals." *Report of British Association*, Bristol, 1897).

23. Examine the problem on reflection and refraction on the surface of the uniaxal crystal $A'=B'$ for perpendicular incidence.

The velocity of propagation v_κ' within the crystal will be given here by the expression

$$v_\kappa'^2=\tfrac{1}{2}\{(B'^2+C'^2)[\cos^2(x',x)+\cos^2(y',x)]+2B'^2\cos^2(z',x)$$
$$\pm(B'^2-C'^2)[\cos^2(x',x)+\cos^2(y',x)]\}$$

(cf. formula (40) and the expressions on p. 368 for $\bar{\lambda}_0$, $\bar{\mu}_0$, $\bar{\nu}_0$), hence

$$v_0'^2=B'^2[\cos^2(x',x)+\cos^2(y',x)+\cos^2(z',x)]$$
and
$$v_e'^2=C'^2[\cos^2(x',x)+\cos^2(y',x)]+B'^2\cos^2(z',x),$$
or, since
$$\cos^2(x',x)+\cos^2(y',x)+\cos^2(z',x)=1,$$
$$v_0'^2=B'^2$$
and
$$v_e'^2=C'^2\sin^2(z',x)+B'^2\cos^2(z',x)$$

(cf. text to formulae (B), Ex. 17).

Replace v_0' and v_e' by these values in formulae (F) and (G), Ex. 11, which hold for the corresponding case in biaxal crystals, and we obtain the formulae sought for the determination of the four unknown quantities A_0', A_e', a_1 and θ_1.

24. Show for perpendicular incidence on the surface of the uniaxal crystal $B'=C'$ that the velocities of propagation of the refracted waves are given by the expressions

$$v_0'^2 = C'^2,$$

and

$$v_e'^2 = A'^2\sin^2(x',\ x) + C'^2\cos^2(x',\ x).$$

We obtain the formulae for the four unknown quantities A_0', A_e', a_1 and θ_1, on replacing v_0' and v_e' by these values in formulae (F) and (G), Ex. 11.

25. Determine for the particular case, where the normal to the reflecting surface coincides with the principal axis x' (D_1') (cf. Fig. 40) of the uniaxal crystal $A'=B'$, the velocities of propagation and the angles of refraction of the two refracted waves.

For $A'=B'$ formula (J), Ex. 14, which holds for the corresponding case in biaxal crystals, assumes the form

$$v_\kappa'^2 = \tfrac{1}{2}[B'^2(1+\sin^2\phi_\kappa'\sin^2\omega') + C'^2(1-\sin^2\phi_\kappa'\sin^2\omega') \pm (B'^2 - C'^2)(\cos^2\phi_\kappa' + \sin^2\phi_\kappa'\cos^2\omega')],$$

hence

$$v_0'^2 = B'^2$$

and

$$v_e'^2 = (B'^2 - C'^2)\sin^2\phi_e'\sin^2\omega' + C'^2.$$

Replace v_0' and v_e' by these values in the relation (69) between the ϕ's and the v's, and we find the following values for ϕ_0' and ϕ_e' in terms of ϕ, v, ω' and A', B', C':

$$\sin^2\phi_0' = \frac{B'^2\sin^2\phi}{v^2}$$

and

$$\sin^2\phi_e' = \frac{C'^2\sin^2\phi}{v^2 - (B'^2 - C'^2)\sin^2\phi\sin^2\omega'},$$

and hence the following values for v_0' and v_e' in terms of ϕ, v, ω' and A', B', C':

$$v_0'^2 = B'^2,$$

and

$$v_e'^2 = \frac{v^2C'^2}{v^2 - (B'^2 - C'^2)\sin^2\phi\sin^2\omega'}.$$

26. Determine the four unknown quantities A_0', A_e', a_1 and θ_1 in terms of a, θ, ϕ, v and A', B', C' (cf. Ex. 14) for the particular case of the preceding example, where $\omega'=0$ (cf. also Ex. 16).

27. For perpendicular incidence the formulae of Ex. 25 for the velocities of propagation of the refracted waves assume the form

$$v_0' = B' \quad \text{and} \quad v_e' = C'$$

(cf. text to formulae (B), Ex. 17).

28. Show for the particular case, where the normal to the reflecting surface coincides with the principal axis x' of the uniaxal crystal $B'=C'$, that the velocities of propagation and the angles of refraction of the refracted waves are determined by the expressions

$$v_0' = C', \qquad v_e'^2 = \frac{v^2C'^2}{v^2 - (A'^2 - C'^2)\sin^2\phi},$$

and

$$\sin^2\phi_0' = \frac{C'\sin\phi}{v}, \qquad \sin^2\phi_e' = \frac{C'^2\sin^2\phi}{v^2 - (A'^2 - C'^2)\sin^2\phi}.$$

Observe that these expressions do not contain the angle ω', a result we could have anticipated, since for $B'=C'$ the crystal has no principal axes in the yz plane.

Show also that

$$\theta_0' = \pi/2, \quad \theta_e' = 0,$$

$$\epsilon_0' = 0 \quad \text{and} \quad \tan\epsilon_e' = \frac{(A'^2 - C'^2)\sin\phi\sqrt{v^2 - A'^2\sin^2\phi}}{v^2C'}.$$

We could also obtain these formulae directly, on putting $B'=C'$ in those of Ex. 16, where $\omega'=0$. The formulae for the determination of the four unknown quantities A_0', A_e', a_1 and θ_1 will thus follow directly from those of Ex. 16, where $\omega'=0$, if we put there $B'=C'$. Observe that the expression for A_0' undergoes thereby no change, that is, it is immaterial here, as far as the amplitude of the ordinary refracted wave is concerned, whether the crystal employed be a biaxal or an uniaxal one.

29. To obtain the formulae for the particular case of the preceding example, where the waves are incident at right angles to the surface of the crystal, we put $B'=C'$ and $\omega'=0$ (cf. Ex. 28) in the formulae of Ex. 17, the corresponding case in biaxal crystals, and we have

$$A_0'=2a\sin\theta\frac{v^2}{(v+C')C'},$$

$$A_e'=2a\cos\theta\frac{v^2}{(v+C')C'},$$

$$a_1=a\frac{v-C'}{v+C'}, \text{ and } \tan\theta_1=-\tan\theta.$$

Observe that these formulae are identical to those that hold on the surface of an isotropic insulator, through which electromagnetic waves are propagated in all directions with one and the same velocity $v'=C'$ (cf. Chapter VII.); it thus follows that waves incident at right angles on the given surface of this uniaxal crystal will be reflected and refracted as on the surface of an isotropic insulator ($v'=C'$).

30. Show that the formulae of Ex. 20 for the determination of the four unknown quantities A_0', A_e', a_1 and θ_1 assume the following form on the surface of the uniaxal crystal $A'=B'$:

$$A_0'=a\sin\theta\frac{2v^2}{v^2+C'^2},$$

$$A_e'=a\cos\theta\frac{2v^2C'}{B'[vC'+B'\sqrt{v^2-B'^2+C'^2}]}$$

$$a_1^2=a^2\left\{\sin^2\theta\left(\frac{v-C'}{v+C'}\right)^2+\cos^2\theta\left[\frac{vC'-B'\sqrt{v^2-B'^2+C'^2}}{vC'+B'\sqrt{v^2+B'^2+C'^2}}\right]^2\right\}$$

and $$\tan\theta_1=-\tan\theta\frac{v^2-C'^2}{v^2+C'^2}\cdot\frac{vC'+B'\sqrt{v^2-B'^2+C'^2}}{vC'-B'\sqrt{v^2-B'^2+C'^2}}.$$

31. Examine the problem on reflection and refraction on the surface of the uniaxal crystal $A'=B'$ for the particular case, where the principal axes of the crystal coincide with coordinate-axes x, y, z and the angle of incidence ϕ is so chosen that $\phi+\phi_e'=\pi/2$ (cf. Ex. 21).

For $A'=B'$ equation (A), Ex. 21, gives the following value for ϕ:

$$\sin^2\phi=\frac{v^2}{v^2+B'^2} \quad \text{...(A)}$$

(cf. also Ex. 33).

Replace ϕ by this value and put $A'=B'$ in the formulae of Ex. 16, which hold for the corresponding case in biaxal crystals where ϕ was arbitrary, and we find the following values for the four unknown quantities sought:

$$A_0'=a\sin\theta\frac{2v^2B'}{C'[v\sqrt{v^2+B'^2-C'^2}+B'C']},$$

$$A_e'=a\cos\theta\frac{v}{B'},$$

$$a_1=a\sin\theta\frac{v\sqrt{v^2+B'^2-C'^2}-B'C'}{v\sqrt{v^2+B'^2-C'^2}+B'C'}$$

and $$\tan\theta_1=\infty \text{ hence } \theta_1=\pi/2.$$

32. Determine the quantities A_0', A_e', a_1 and θ_1 for the particular case of reflection and refraction on the surface of the uniaxal crystal $B' = C'$, where the normal to the reflecting surface coincides with the principal axis x' of the crystal and the angle of incidence ϕ is so chosen that $\phi + \phi_0' = \pi/2$.

For the complemental case, where $\phi + \phi_e' = \pi/2$, ϕ was determined by the quadratic equation (A), Ex. 21.

33. For the uniaxal crystal $A' = B'$ equation (D), Ex. 22, for the determination of ϕ corresponding to the condition $\Theta_{10} = \Theta_{1e}$ reduces to

$$-(v^4 - B'^4)\sin^2\phi + v^2(v^2 - B'^2) = 0,$$

hence

$$\sin^2\phi = \frac{v^2}{v^2 + B'^2};$$

that is, here there is only one value of ϕ for which $\Theta_{10} = \Theta_{1e}$. Observe that this value is that (A), Ex. 31, already found for the corresponding case, where ϕ was thereby determined that $\phi + \phi_e' = \pi/2$. It thus follows that there is one and only one angle of incidence ϕ, for which ordinary light will be reflected as linearly polarized from the surface of the uniaxal crystal $A' = B'$ cut parallel to its (one) optical axis (cf. p. 344), and that angle of incidence ϕ is thereby determined that $\phi + \phi_e' = \pi/2$, that is, the angle of incidence and the angle of refraction of the extraordinary refracted wave must make a right angle with each other.

The unknown quantities A_0', A_e', a_1 and θ_1 will evidently be given here by the same formulae as those that hold for the corresponding case, where $\phi + \phi_e' = \pi/2$ (cf. Ex. 31).

Observe that for the uniaxal crystal $B' = C'$ the equation for the determination of ϕ retains here its general form (A), Ex. 21.

34. For $A' = B' = C' = v'$ confirm that the formulae of the above examples all reduce to those (cf. Chapter VII.) that hold on the surface of the isotropic insulator, through which electromagnetic disturbances are propagated in all directions with one and the same velocity v'.

35. Examine the problem on total reflection on the surface of uniaxal crystals.

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